Construction for *OGDD* of Type 4^4

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Abstract For a long time we had thought that there does not exist an *OGDD* of type 4^4. In this article, an *OGDD* of type 4^4 will be constructed.

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1 Introduction

A ***group-divisible design with block size 3***(briefly, 3-GDD) $(X, \mathcal{G}, \mathcal{A})$ is a set X and a partition $\mathcal{G}$ of X into classes (usually called *groups*), and a set $\mathcal{A}$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{A}$ if x and y are from different groups, and does not appear in a 3-subset of $\mathcal{A}$ if x and y are from the same groups.

An ***orthogonal group-divisible design*** (briefly, *OGDD*) $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is a pair of 3-GDDs $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ satisfying two orthogonality conditions:

(i) if $\{x, y, z\} \in \mathcal{A}$ and $\{x, y, w\} \in \mathcal{B}$, then z and w are in different groups; and

(ii) for two distinct intersecting triples $\{x, y, z\}$ and $\{u, v, z\}$ of $\mathcal{A}$, the triples $\{x, y, w\}$ and $\{u, v, t\}$ of $\mathcal{B}$ satisfy $w \neq t$.

A ***transversal design*** (briefly,TD) TD$(3, 4)$ is a 3-GDD of type 4^3.

For the existence of *OGDD* of type g^u, Colbourn and Gibbons [4] have done excellent work. The following were their concluding remarks:

The main question that remains open is whether there is any value of g for which an OGDD of type g^4 exists. On the basis of the nonexistence when $g = 2$ and $g = 4$, one might be tempted to conjecture that the answer *is negative.*

In this article, an $OGDD$ of type 4^4 will be constructed by hand.

2 Construction for an $OGDD$ of type 4^4

Let $\mathcal{G} = \{G_0, G_1, G_2, H\}$ and $X = G_0 \cup G_1 \cup G_2 \cup H$, where $G_0 = \{0,3,6,9\}, G_1 = \{1,4,7,10\}, G_2 = \{2,5,8,11\}, H = \{a,b,c,d\}$.

Assume $(X, \mathcal{G}, \mathcal{A}, \mathcal{B})$ is an $OGDD$ of type 4^4. $\mathcal{A}$ can be partitioned into two parts, namely, $\mathcal{C}$ and $\mathcal{D}$ such that the first part does not contain any point of H and the second part does.

Let

$$P_a = \{\{x,y\} : \{a,x,y\} \in \mathcal{D}\}$$

$$P_b = \{\{x,y\} : \{b,x,y\} \in \mathcal{D}\}$$

$$P_c = \{\{x,y\} : \{c,x,y\} \in \mathcal{D}\}$$

$$P_d = \{\{x,y\} : \{d,x,y\} \in \mathcal{D}\}$$

$$K = \{\{x,y\} : \{x,y,z\} \in \mathcal{C}\}$$

Similarly, $\mathcal{B}$ can be partitioned into two parts, namely, $\mathcal{E}$ and $\mathcal{F}$ such that the first part does not contain any point of H and the second part does.

Let

$$Q_a = \{\{x,y\} : \{a,x,y\} \in \mathcal{F}\}$$

$$Q_b = \{\{x,y\} : \{b,x,y\} \in \mathcal{F}\}$$

$$Q_c = \{\{x,y\} : \{c,x,y\} \in \mathcal{F}\}$$

$$Q_d = \{\{x,y\} : \{d,x,y\} \in \mathcal{F}\}$$

$$L = \{\{x,y\} : \{x,y,z\} \in \mathcal{E}\}$$

It follows from the definition of $OGDD$ that
(i) $L = P_a \cup P_b \cup P_c \cup P_d$ and $K = Q_a \cup Q_b \cup Q_c \cup Q_d$;
(ii) for $x \in H$, P_x is a partition of $X \setminus H$, so is Q_x;
(iii) $\mathcal{C} \cup \mathcal{E}$ is a $TD(3,4)$;
(iv) each point of $X \setminus H$ appears exactly twice in $\mathcal{C}$ and twice in $\mathcal{E}$.

We will now construct an $OGDD$ of type 4^4.

It is easy to see that there are only two non-isomorphic Latin squares of side 4, so there are only two non-isomorphic $TD(3,4)$.

We choose one $TD(3,4)$ as follows:

$$\{0,1,2\},\{0,4,5\},\{0,7,8\},\{0,10,11\},$$

$$\{3,1,5\},\{3,4,2\},\{3,7,11\},\{3,10,8\},$$

$$\{6,1,8\},\{6,4,11\},\{6,7,2\},\{6,10,5\},$$

$$\{9,1,11\},\{9,4,8\},\{9,7,5\},\{9,10,2\}.$$

When $\{0,1,2\}$ and $\{0,4,5\}$ are put into $\mathcal{C}$, the blocks of the $TD(3,4)$ can uniquely be partitioned into two parts with the condition (iv):

$\mathcal{C}= \{\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\},\{6,1,8\},\{6,4,11\},$

$\{9,7,5\},\{9,10,2\}\}$ and

$\mathcal{E}= \{\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\},\{6,7,2\},\{6,10,5\},$

$\{9,1,11\},\{9,4,8\}\}.$

Hence we have K and L as follows:

$$L = \{\{7,8\},\{10,11\},\{1,5\},\{4,2\},\{7,2\},\{10,5\},\{1,11\},\{4,8\}$$
$$\{0,7\},\{0,8\},\{0,10\},\{0,11\},\{3,1\},\{3,5\},\{3,4\},\{3,2\}$$
$$\{6,7\},\{6,2\},\{6,10\},\{6,5\},\{9,1\},\{9,11\},\{9,4\},\{9,8\}\}.$$
$$K = \{\{1,2\},\{4,5\},\{7,11\},\{10,8\},\{1,8\},\{4,11\},\{7,5\},\{10,2\}$$
$$\{0,1\},\{0,2\},\{0,4\},\{0,5\},\{3,7\},\{3,11\},\{3,10\},\{3,8\}$$
$$\{6,1\},\{6,8\},\{6,4\},\{6,11\},\{9,7\},\{9,5\},\{9,10\},\{9,2\}\}.$$

Now we want to arrange P_x and Q_x for $x \in H$ such that the conditions (i), (ii), (a) and (b) hold.

We first arrange the pairs of points of G_1 and G_2.

When $\{1,2\}$ is put into Q_a, $\{4,5\}$ can not be put into Q_a, and $\{7,11\}$ is put into Q_a; and this forces $\{10,5\}$ and $\{4,8\}$ in P_a. $\{4,5\}$ and $\{10,8\}$ are put into Q_b and this forces $\{1,11\}$ and $\{7,2\}$ in P_b. Thus we have

$$Q_a = \{\{1,2\},\{7,11\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$P_a = \{\{10,5\},\{4,8\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$Q_b = \{\{4,5\},\{10,8\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$P_b = \{\{1,11\},\{7,2\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$Q_c = \{\{1,8\},\{7,5\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$P_c = \{\{4,2\},\{10,11\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$Q_d = \{\{4,11\},\{10,2\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

$$P_d = \{\{1,5\},\{7,8\},\{0,-\},\{3,-\},\{6,-\},\{9,-\}\}$$

Note

$$\{1,2,7,11\} \cap \{4,5,8,10\} = \emptyset, \{1,8,7,5\} \cap \{4,11,10,2\} = \emptyset.$$

Based on this observation, it is easy to see the following arrangement: Q_a and Q_b contain $\{0,x\},\{3,y\},\{6,z\},\{9,t\}$ with $x,y \in G_1$ and $z,t \in G_2$; Q_c and Q_d contain $\{0,x\},\{3,y\},\{6,z\},\{9,t\}$ with $x,y \in G_2$ and $z,t \in G_1$; P_a and P_b contain $\{0,x\},\{3,y\},\{6,z\},\{9,t\}$ with $x,y \in G_2$ and $z,t \in G_1$; P_c and P_d contain $\{0,x\},\{3,y\},\{6,z\},\{9,t\}$ with $x,y \in G_1$ and $z,t \in G_2$.

Based on the above discussion, it is easy to obtain the following arrangement.

$$Q_a = \{\{1,2\},\{7,11\},\{0,4\},\{3,10\},\{6,8\},\{9,5\}\}$$

$$P_a = \{\{10,5\},\{4,8\},\{0,11\},\{3,2\},\{6,7\},\{9,1\}\}$$

$$Q_b = \{\{4,5\},\{10,8\},\{0,1\},\{3,7\},\{6,11\},\{9,2\}\}$$

$$P_b = \{\{1,11\},\{7,2\},\{0,8\},\{3,5\},\{6,10\},\{9,4\}\}$$

$$Q_c = \{\{1,8\},\{7,5\},\{0,2\},\{3,11\},\{6,4\},\{9,10\}\},$$

$$P_c = \{\{4,2\},\{10,11\},\{0,7\},\{3,1\},\{6,5\},\{9,8\}\}$$

$$Q_d = \{\{4,11\},\{10,2\},\{0,5\},\{3,8\},\{6,1\},\{9,7\}\}$$

$$P_d = \{\{1,5\},\{7,8\},\{0,10\},\{3,4\},\{6,2\},\{9,11\}\}$$

Hence an $OGDD$ of type 4^4 is constructed as follows:

$\mathcal{A} = \{$

$$\{0,1,2\},\{0,4,5\},\{3,7,11\},\{3,10,8\}$$

$$\{6,1,8\},\{6,4,11\},\{9,7,5\},\{9,10,2\}$$

$$\{a,10,5\},\{a,4,8\},\{a,0,11\},\{a,3,2\},\{a,6,7\},\{a,9,1\}$$

$$\{b,1,11\},\{b,7,2\},\{b,0,8\},\{b,3,5\},\{b,6,10\},\{b,9,4\}$$

$$\{c,4,2\},\{c,10,11\},\{c,0,7\},\{c,3,1\},\{c,6,5\},\{c,9,8\}$$

$$\{d,1,5\},\{d,7,8\},\{d,0,10\},\{d,3,4\},\{d,6,2\},\{d,9,11\}\}$$

$\mathcal{B} = \{$

$$\{0,7,8\},\{0,10,11\},\{3,1,5\},\{3,4,2\}$$

$$\{6,7,2\},\{6,10,5\},\{9,1,11\},\{9,4,8\}$$

$$\{a,1,2\},\{a,7,11\},\{a,0,4\},\{a,3,10\},\{a,6,8\},\{a,9,5\}$$

$$\{b,4,5\},\{b,10,8\},\{b,0,1\},\{b,3,7\},\{b,6,11\},\{b,9,2\}$$

$$\{c,1,8\},\{c,7,5\},\{c,0,2\},\{c,3,11\},\{c,6,4\},\{c,9,10\}$$

$$\{d,4,11\},\{d,10,2\},\{d,0,5\},\{d,3,8\},\{d,6,1\},\{d,9,7\}\}$$

For convenience to the reader, we check the orthogonality as follows:

$a:\{10,5\}-6;\{4,8\}-9;\{0,11\}-10;\{3,2\}-4;\{6,7\}-2;\{9,1\}-11;$

$b:\{1,11\}-9;\{7,2\}-6;\{0,8\}-7;\{3,5\}-1;\{6,10\}-5;\{9,4\}-8;$

$c:\{4,2\}-3;\{10,11\}-0;\{0,7\}-8;\{3,1\}-5;\{6,5\}-10;\{9,8\}-4;$

$d:\{1,5\}-3;\{7,8\}-0;\{0,10\}-11;\{3,4\}-2;\{6,2\}-7;\{9,11\}-1;$

$0:\{1,2\}-a;\{4,5\}-b;\{a,11\}-7;\{b,8\}-10;\{c,7\}-5;\{d,10\}-2;$

$3:\{7,11\}-a;\{10,8\}-b;\{a,2\}-1;\{b,5\}-4;\{c,1\}-8;\{d,4\}-11;$

$6:\{1,8\}-c;\{4,11\}-d;\{a,7\}-11;\{b,10\}-8;\{c,5\}-7;\{d,2\}-10;$

$9:\{7,5\}-c;\{10,2\}-d;\{a,1\}-2;\{b,4\}-5;\{c,8\}-1;\{d,11\}-4;$

$1:\{0,2\}-c;\{6,8\}-a;\{a,9\}-5;\{b,11\}-6;\{c,3\}-11;\{d,5\}-0;$

$4:\{0,5\}-d;\{6,11\}-b;\{a,8\}-6;\{b,9\}-2;\{c,2\}-0;\{d,3\}-8;$

$7:\{3,11\}-c;\{9,5\}-a;\{a,6\}-8;\{b,2\}-9;\{c,0\}-2;\{d,8\}-3;$

$10:\{3,8\}-d;\{9,2\}-b;\{a,5\}-9;\{b,6\}-11;\{c,11\}-3;\{d,0\}-5;$

$2:\{0,1\}-b;\{9,10\}-c;\{a,3\}-10;\{b,7\}-3;\{c,4\}-6;\{d,6\}-1;$

$5:\{0,4\}-a;\{9,7\}-d;\{a,10\}-3;\{b,3\}-7;\{c,6\}-4;\{d,1\}-6;$

$$8:\{6,1\}-d;\{3,10\}-a;\{a,4\}-0;\{b,0\}-1;\{c,9\}-10;\{d,7\}-9;$$

$$11:\{3,7\}-b;\{6,4\}-c;\{a,0\}-4;\{b,1\}-0;\{c,10\}-9;\{d,9\}-7.$$

Note that

$$a:\{10,5\}-6;\{4,8\}-9;\{0,11\}-10;\{3,2\}-4;\{6,7\}-2;\{9,1\}-11$$

means

$$\{a,10,5\},\{a,4,8\},\{a,0,11\},\{a,3,2\},\{a,6,7\},\{a,9,1\}\in\mathcal{A}$$

and

$$\{10,5,6\}\{4,8,9\}\{0,11,10\}\{3,2,4\}\{6,7,2\}\{9,1,11\}\in\mathcal{B}.$$

It follows from 6,9,10,4,2 and 11 are distinct and in different groups with a that the condition (i) holds with

$$\{x,y\}\in\{\{10,5\},\{4,8\},\{0,11\},\{3,2\},\{6,7\},\{9,1\}\}$$

and the condition (ii) holds with $z=a$.

By the way, OGDDs of type 8^4 and 12^4 were constructed by Dukes in [2], we have no other direct construction for the case with four groups.

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