# (t,k)-Geodetic Set of a Graph

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**Abstract**

In this study we are going to give new (t,k)-geodetic set definition. This is a refinement of the geodetic set definition given in [11]. With this new definition we obtain more information about the graph. We also give a relationship between the (t,k)-geodetic set and the integrity of a graph. By using a (t,k)-geodetic set we give a new proof for the upper bound of integrity of trees and unicycle graphs.

## 1 Introduction

The distance $d(u,v)$ between two vertices $u$ and $v$ in a graph $G$ is the length of the shortest $u-v$ path. A $u-v$ path of length $d(u,v)$ is called a $u-v$ *geodesic.* Let $H(u,v)$ be the set of all vertices lying on some $u-v$ geodesic of $G$. Let $S$ be a subset of $V(G)$. We define

$$H(S) = \cup_{u,v\in S} H(u,v).$$

A set $S$ of vertices of $G$ is defined in [10] to be a *geodetic set* in $G$ if $H(S) = V(G)$, and a geodetic set of minimum cardinality is a *minimum geodetic set.* The cardinality of a minimum geodetic set in $G$ is the *geodetic number* $g(G)$. For more details about a geodetic set and the geodetic number of a graph, the interested reader may refer to [1, 2, 10, 11, 12, 13, 16, 17].
Obviously, for non-trivial graphs, any smallest geodetic set has at least two vertices. This is clear from the definition. The path $P_n$ in fact has minimum geodetic set of size 2. On the other hand the largest possible geodetic set can have size $n$, where $n$ is the number of vertices of the given graph. For the complete graph $K_n$, the geodetic set has size $n$. The following theorem gives an upper bound for the geodetic number of a nontrivial connected graphs.

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**Theorem 1** *[10] If $G$ is a nontrivial connected graph of order $n$ and diameter $d$, then*

$$g(G) \leq n - d + 1.$$

The following theorem says that the set of end vertices of a tree is not only a geodetic set but also has minimum size as well.

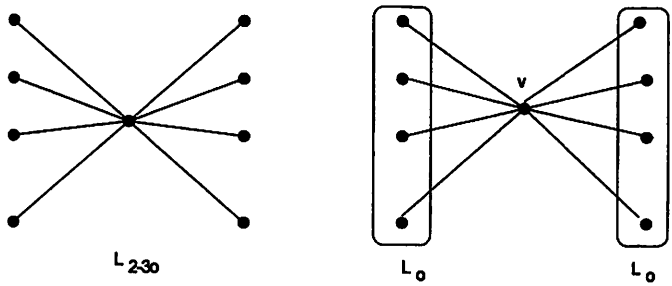
**Theorem 2** *[10]The geodetic number $g(T)$ of a tree $T$ is the number of end-vertices in $T$. In fact, the set of all end-vertices of $T$ is the unique minimum geodetic set of $T$.*

Harary, Chartrand, and Zhang gave the *minimum geodetic subgraph* definition in [11]. Graph $F$ is called a minimum geodetic subgraph if there exists a graph $G$ containing $F$ as an induced subgraph such that $V(F)$ is a minimum geodetic set in $G$. They have given the following theorem to characterize the minimum geodetic subgraph.

**Theorem 3** *A nontrivial graph $F$ is a minimum geodetic subgraph if and only if every vertex of $F$ has eccentricity 1 or no vertex of $F$ has eccentricity 1.*

The geodetic set and geodetic number of some well-known graphs are studied and computed in the literature that we have listed above. It has been shown [1] that geodetic set problem is NP-complete.
The following example shows that two different graphs may have the same geodetic number but be very different. If we look at their $(t, k)$-geodetic number, then we have a better understanding between the graphs. Hence we introduce a new geodetic set definition.

**Example:** Let $G_1$ be the star graph and $K_{1,2n}$ and $G_2$ the graph described as follows: Take 2 copies of $K_n$ the complete graph and add a new vertex $u$. Join $u$ to every vertices of both $K_n$. Let $S_i$ be geodetic set of $G_i$ for $i = 1, 2$. Assume here $S_i$ are minimum geodetic sets. By the theorem above minimum geodetic set of star graph is $|S_1| = 2n$. One can show that minimum geodetic set $S_2$ of $G_2$ has size $2n$. But $m(\langle S_1 \rangle) = 1$ where as $m(\langle S_2 \rangle) = n$, where $\langle U \rangle$ indicates *induced subgraph* and $m(\langle U \rangle)$ indicates the maximum order of the components of induced graph $\langle U \rangle$.




Graphs $G_1$ and $G_2$

In this study we assume that graph $G$ is a simple connected graph.

# 2 (t,k)-Geodetic Set

When we look at the definition of a geodetic set, all we know is that every vertex of graph is on a shortest path between two vertices which are in the geodetic set.
Let $G$ be a given graph and $S$ be a subset of $V(G)$. We say $S$ is a *(t,k)-geodetic* set if $S$ is a geodetic set of $G$ with $|S| = k$ and $m(\langle S \rangle) = t$. We say $(t, k)$-geodetic set is minimum if $S$ is a minimum geodetic set with $|S| = k$. Existing studies are about mainly minimum geodetic set and geodetic number of graphs. Here there are some interesting questions to investigate about a $(t, k)$-geodetic set:

1. Let $S$ be a minimum geodetic set of a given graph $G$. What can we say about $t = m(\langle S \rangle)$? Is it fixed for all minimum geodetic sets of given graph $G$?

2. Find a $(t, k)$-geodetic set such that $G(t, k) = min_{S \subseteq V(G)}\{\frac{t}{k} : m(\langle S \rangle) = t, |S| = k\}$.

3. Find a $(t, k)$-geodetic set such that $t + (n - k)$ is minimum, where n is the number of vertices of a given graph $G$.

It is obvious that a complete graph $K_n$ has a unique geodetic set $S$ which is vertex set of $K_n$. Hence $m(\langle S \rangle) = n$. Theorem 2 yields the following theorem.

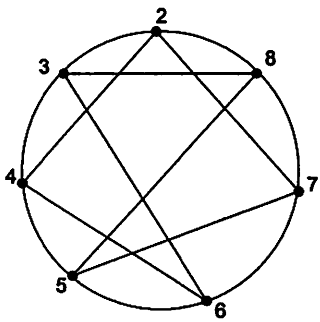
**Theorem 4** *If graph $G$ is a tree with $k \geq 3$ end vertices, then there exist unique(1,k)-geodetic set of $G$.*

Here are some observations:

a. There are some graphs that have a minimum $(1,k)$-geodetic set but $G$ is not a tree. For example, Peterson graph has a minimum $(1,4)$-geodetic set.

b. If a given graph $G$ has unique minimum geodetic set $S$, then it has a minimum $(t,|S|)$-geodetic set with fixed $t$. If geodetic set is not unique, then the value of $t$ of $(t,|S|)$-geodetic may not be unique. For example: If $G = C_7$, then $S_1 = \{1,4,5\}$ is minimum geodetic set. Hence it yields a minimum $(2,3)$-geodetic set. Also $S_2 = \{1,3,5\}$ is also a minimum geodetic set, which yields a $(1,3)$-geodetic set.

c. The following graph G has seven different minimum geodetic sets of size 3. These are $S_1 = \{1,2,4\}$, $S_2 = \{1,5,4\}$, $S_3 = \{1,5,7\}$, $S_4 = \{2,3,6\}$ $S_5 = \{2,4,6\}$, $S_6 = \{3,5,7\}$, and $S_7 = \{3,6,7\}$. We have $m(\langle S_i\rangle) = 2$ for $i = 1,2,...,7$.



Graph G

For the second question given above, we have an obvious upper and lower bounds. Specifically, $\frac{1}{n-1} \leq G(t,k) \leq 1$, where $n$ is the number of vertices of given graph. The star graph $K_n$ realizes the lower bound. The complete graph $K_n$ realizes the upper bound.

**Theorem 5** *Let $P_n$ be a path on $n$ vertices. Then*

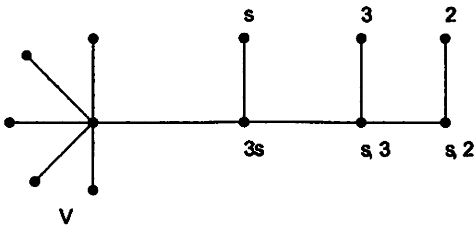
$$P_n(t,k) = \begin{cases} \frac{2}{n+1} & \text{if } n \text{ is odd} \\ \frac{2}{n} & \text{if } n \text{ is even.} \end{cases}$$

*Proof:* Suppose $n$ is odd. Let $S$ be set of every other vertices on the $P_n$. Obviously $|S| = \frac{n+1}{2}$ and $S$ is a geodetic set and $m(\langle S\rangle) = 1$. If $S'$ is a set of vertices and $S'$ contains $S$ but $|S'| > |S|$, then $m(\langle S'\rangle) \geq 2$ and $|S'| \leq n-1$. □

One can easily observe that for any given tree $T$ with $k \geq 3$ end vertices $T(t,k) \leq \frac{1}{k}$. The following is a realization theorem.

**Theorem 6** *Let $n \geq 3$ be and integer. For any $\frac{n}{2} \leq k \leq n-1$ there exist a tree with $n$ vertices such that $T(t,k) = \frac{1}{k}$.*

*Proof:* Let $k$ be given number in the stated range. There exists an integer $r$ such that $k + r = n - 1$. We construct the following tree with $n$ vertices.




Tree T

In $T$, the indicated subgraph $U$ has $n - 2r$ vertices. Let $S$ be set of all end vertices. $\frac{m(\langle S \rangle)}{|S|} = \frac{1}{k}$. Any other geodetic set $S'$ that contains $S$ has $m(\langle S' \rangle) \geq 2$ □

Similar to the path, we can give the following theorem for the cycle $C_n$.

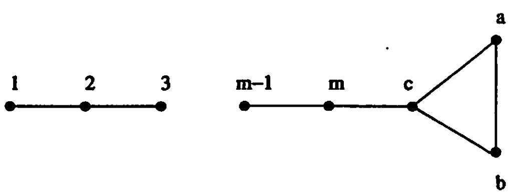
**Theorem 7** *Let $C_n$ be a cycle with $n$ vertices. Then*

$$C_n(t,k) = \begin{cases} \frac{2}{n-1} & \textit{if } n \textit{ is odd} \\ \frac{2}{n} & \textit{if } n \textit{ is even.} \end{cases}$$

For any tree $T$ with $n$ vertices, there exists a geodetic set $S$ with $|S| \geq \frac{n}{2}$. Hence we can give the following bound.

**Theorem 8** *Of all trees $T$ with $n \geq 3$ vertices, the path $P_n$ has maximum $G(t,k)$. That is $T(t,k) \leq P_n(t,k)$.*

It would be very nice to give an upper bound for any graph family. In general this is not an easy question to answer. Here we study graphs with a unique cycle. These graphs are known to be *unicycle graphs.* We fist give the following theorem.




Graph $G^*$

**Theorem 9** *The graph $G^*$ has* $G^*(t,k) = \begin{cases} \frac{4}{n+2} & \text{if } n{=}m{+}3 \text{ is even} \\ \frac{4}{n+1} & \text{if } n{=}m{+}3 \text{ is odd.} \end{cases}$

*Proof:* Subgraph $P_m$ has the geodetic set $S$ which gives the bound in Theorem 5. So $|S| = \frac{m+1}{2}$ if $m$ is odd or $|S| = \frac{m}{2}$ if $m$ is even. Now in the given graph vertices $a$ and $b$ have to be in its geodetic set, say $S'$. Hence $S' = S \cup \{a, b\}$. □

**Theorem 10** *Let $G$ be a any unicycle graph with $|V(G)| = |V(G^*)|$. Then $G(t,k) \leq G^*(t,k)$.*

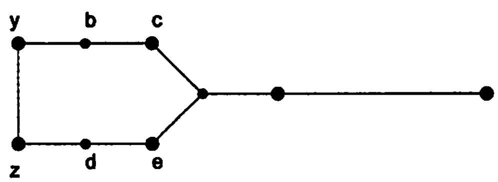
*Proof:* Let $G$ be a unicycle graph with $n$ vertices. Then $G$ has a cycle $C$. There must be an edge $e = (x, y)$ on the cycle $C$ such that $T = G - e$ is a tree. $T$ has a geodetic set $S$ such that $T(t,k) = \frac{1}{|S|} \leq \frac{2}{n+1}$ or $\frac{2}{n}$ by Theorem 5. We will show that graph $G$ has a geodetic set $S'$ such that $|S'| \leq |S|$.

*Case 1:* Both $x$ and $y$ are not $S$, then $S = S'$ is a geodetic set of $G$ and $m(\langle S \rangle) = m(\langle S' \rangle) = 1$.

*Case 2:* Only one of the $x$ or $y$ in the $S$. Then $S' = S$ is a geodetic set of $G$ and $m(\langle S \rangle) = m(\langle S' \rangle) = 1$.

*Case 3:* Both $x$ and $y$ are in $S$. If there is a vertex $w \in S$ such that $d(x,w) > d(y,w)$, then $y$ will be on $x - w$ geodetic on the $G$. So $S' = S - y$ is a geodetic set of the $G$. If $d(x,u) = d(y,u)$ for every $u \in S$, then we have following cases:

*Case 3a:* If the cycle $C$ has at leat 5 edges as follows:




Define $S' = S - \{x, b\} \cup \{a\}$. In the above figure larger vertices indicates vertices that are in a geodetic set. Obviously $x$ is on $y - a$ geodetic.

*Case 3b:* If $C$ is a triangle, then $G = G^*$.

Now suppose $S = S'$. If $n$ is odd, then $\frac{1}{|S'|} \leq \frac{2}{n+1} \leq \frac{4}{n+1}$. If $n$ is even, then $\frac{1}{|S'|} \leq \frac{2}{n} \leq \frac{4}{n+2}$. Suppose $|S'| = |S| - 1$. If $n$ is odd, then $\frac{1}{|S'|} \leq \frac{2}{n} \leq \frac{4}{n+1}$. If

$n$ is even, then $\frac{1}{|S'|} \leq \frac{2}{n-2} \leq \frac{4}{n+2}$ when $n \geq 5$. If $n = 4$, then $G^*(t,k) = \frac{2}{3}$. On the other hand $P_4(t,k) = \frac{1}{2}$ and $K_{1,3}(t,k) = \frac{1}{3}$ □

If we can find a minimum geodetic set $S$ of a graph $G$. Then we we have the following upper bound in general.

**Theorem 11** *Let $G$ be graph and $S$ be its minimum geodetic set. Then* $G(t,k) \leq \frac{m(\langle S \rangle)}{|S|}$.

Let $P_n$ be a path with $n$ vertices. If we remove $r$ vertices, then maximum number of components will be $r+1$. Maximum number of vertices in each component is bounded by $\frac{n-r}{r+1}$. We take the following function

$$f(r) = r + \frac{n-r}{r+1}$$

This function gets its minimum value $\sqrt{n+1}-1$ at $r = \sqrt{n+1}-1$. Therefore we can state the following theorem.

**Theorem 12** *Let $P_n$ be a path with $n$ vertices. There exists a $(t,k)$-geodetic set of $P_n$ such that minimum value of $t+n-k$ is bounded by* $2\sqrt{n+1}-2$.

Similarly we can prove the following.

**Theorem 13** *Let $C_n$ be a cycle with $n$ vertices. There exist a $(t,k)$-geodetic set of $C_n$ such that minimum value of $t+n-k$ is bounded by* $2\sqrt{n}-1$.

The concept of integrity in a graph theory was introduced by Barefoot, Entringer and Swart [7] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. Formally, the integrity is

$$I(G) = min_{\ L \subseteq V(G)} \{|L| + m(G-L)\}$$

where $m(G-L)$ is the maximum order of the components of $G-L$.

Let $G$ be a graph and $L$ be subset of $V(G)$. Then $L$ is called $I-set$ if $I(G) = |L| + m(G-L)$. It is obvious that $S = G-L$ is a geodetic set of the graph $G$. So if we can find $(t,k)$-geodetic set $S$ such that $t+(n-k)$ is minimum, where $n$ is the number of vertices of the graph $G$. Then $G-S$ is $I$-set of the graph $G$.

We also refer to the following papers for the integrity and related studies [3, 4, 5, 6, 7, 14, 15, 18].

## 3 Integrity and Geodetic Set

Barefoot, Entringer, and Swart computed integrity of trees [6]. They have showed that among all the trees with $n$ vertices, the path $P_n$ has the maximum integrity. Vince also proved the same theorem in [18] by using different argument. Atici has proved in [4] that among all unicycle graphs with $n$ vertices, the cycle $C_n$ has the maximum integrity. Here is the integrities of path $P_n$ and cycle $C_n$:

$$I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$$
$$I(C_n) = \lceil 2\sqrt{n} \rceil - 1$$

We give new proofs for those two integrity by using $(t,k)$-geodetic set of graph $G$. First let us prove the following lemmas.

**Lemma 14** *Let $T_1$ and $T_2$ be two trees with $n$ vertices. Suppose $T_i$ has $(t_i,k)$-geodetic sets for $i = 1,2$. If the number of end vertices of $T_1$ is less than the number of end-vertices of $T_2$, then $t_1 \geq t_2$.*

*Proof:* Let $S_i$ be $(t_i,k)$-geodetic set of $T_i$ for $i = 1,2$. $S_2$ contains all the end-vertices of $T_2$ and $T_2$ has more end-vertices that $T_1$. Hence $S_1$ contains at least one or more vertices that are not end-vertices of $T_1$. Hence $t_1 \geq t_2$.
□
Similarly we can show the following for unicycle graphs.

**Lemma 15** *Let $G$ be a unicycle graph with $n$ vertices. Suppose $G$ and $C_n$ have $(t_1,k)$-geodetic and $(t_2,k)$-geodetic sets, respectively. Then $t_1 \leq t_2$.*

**Theorem 16** *Among all trees with $n$ vertices, the path $P_n$ has maximum integrity.*

*Proof:* Let $T$ be a tree with $n$ vertices. $T$ has a $(t,k)$-geodetic set $S$ such that $T - S$ is $I$-set of $T$. Suppose the path $P_n$ has $(t',k)$-geodetic set. Then Lemma 14, $t' \geq t$. Hence $I(T) = t + n - k \leq t' + n - k$. Let $S$ be $(t,k)$-geodetic set of $P_n$ such that $P_n - S$ is an $I$–set of $P_n$. By Theorem 12, $t = \sqrt{n+1} - 1$ and $k = n - \sqrt{n+1} + 1$. We will show that $T$ has $(t_1,k_1)$-geodetic set such that $t_1 \leq t$ and $k_1 \geq k$.

*Case 1:* If $T$ has more than $k$ or equal end-vertices, then take $S$ as all end-vertices of $T$. Hence $k_1 > k$ and $m(\langle S \rangle) = 1$. Therefore $1+n-k_1 \leq I(P_n)$.

*Case 2:* If $T$ has less than $k = n - \sqrt{n+1} + 1$ end-vertices. Then take $S$ set of all end-vertices and some other vertices such that $|S| = n - \sqrt{n+1} + 1$. So $S$ is a $(t_1, n - \sqrt{n+1} + 1)$-geodetic set of $T$. By Lemma 14, $t_1 \leq t$. Hence $I(T) \leq t_1 + \sqrt{n+1} - 1 \leq I(P_n) = t + n - k$. □
Similarly one can show the following:

**Theorem 17** *Among all unicycle graph with n vertices, the cycle $C_n$ has the maximal integrity.*

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