ON SPLITTING GRAPHS

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ABSTRACT

In this paper, self-centered, bi-eccentric splitting graphs are characterized. Further various bounds for domination number, global domination number and the neighborhood number of these graphs are obtained.

1. Introduction

Graphs discussed in this paper are connected, undirected and simple. Splitting graphs were first studied by Sampathkumar and Walikar [12] and were further developed by Patil and Thangamari [7]. Swaminathan and Subramanian [11] studied the domination number of these graphs. The line splitting graph of a graph was introduced by Kulli and Biradar [4]. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $\deg_G(v)$ or deg(v). The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance* between u and v and is denoted by d(u, v) or $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. The *eccentricity* of a vertex $u \in V(G)$ is defined as $e_G(u) = \max\{d_G(u, v): v \in V(G)\}$. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v). The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called *self-centered* graph with radius r, equivalently is *r-self-centered*. A vertex u is said to be an *eccentric vertex* of v in a graph G, if d(u, v) = e(v). In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. We also denote the i^{th} *neighborhood* of v as $N_i(v) = \{u \in V(G) : d_G(u, v) = i\}$. If $|N_{e(v)}(v)| = m$, for each vertex $v \in V(G)$, then G is called an *m-eccentric* *vertex graph.* If m = 2, we call the graph G as *bi-eccentric vertex graph.* For v∈V(G), the *neighborhood* N(v) of v is the set of all vertices adjacent to v in G. The set N[v] = N(v)∪{v} is called the closed neighborhood of v. For any set D of vertices of G, the subgraph of G induced by D is denoted by G[D].

The concept of domination in graphs was introduced by Ore [6]. A set D of vertices in a graph G is a *dominating set*, if every vertex in V(G)- D is adjacent to some vertex in D. The *domination number* γ(G) of G is the minimum cardinality of a dominating set of G. Sampathkumar and Neeralagi [9] introduced the concept of neighborhood number of graphs and they obtained many bounds and relationship with some other known parameters.These numbers were further studied by Brigham et al [1]. Algorithmic aspects of these numbers were obtained by Chang et al [2]. The line neighborhood number of a graph was introduced by Sampathkumar and Neeralagi [10]. A subset S of V(G) is a *neighborhood set* (written n-set) of G if G = $\cup_{v\in D}$E(G[N[v]]), where G[N[v]] is the subgraph of G induced by N[v]. The *neighborhood number* n_0(G) of G is the minimum cardinality of an n-set of G. For a graph G, let V′(G) = {v′: v∈V(G)} be a copy of V(G). Then the *Splitting graph* S(G) of G is the graph with the vertex set V(G)∪V′(G)and edge set {uv, u′v, uv′ : uv∈E(G)}.

In this paper, we characterize self-centered and bi-eccentric splitting graphs and determine bounds for their global domination number and neighborhood number. For simplicity we use deg*(v), e*(v), d*(u, v) to denote the degree of a vertex v, the eccentricity of a vertex v and the distance between u and v in S(G) respectively.

2. Prior Results

We use the following results.

Proposition 2.1. [12]: (i) If G is a (p, q) - graph, then S(G) is a (2p, 3q) - graph and
(ii) For any vertex v in G, deg*(v) = 2deg(v) and deg*(v′) = deg(v).

Proposition 2.2.[12]: S(G) - E(G)= G⊕K_2, where G⊕K_2 is the tensor product of G with K_2.

Theorem 2.1.[12]: A graph G is a splitting graph if and only if V(G) can be partitioned into two sets V_1, V_2 such that there exists a bijection $f: V_1 \to V_2$ and $N(f(v)) = N(v) \cap V_1$, for all $v \in V_1$.

Theorem 2.2.[7]: For any graph G of order $p \geq 2$, $\xi(S(G)) = \min\{p, 2\xi(G)\}$, where ξ is either domination number (or) neighborhood number.

Observation 2.1.[7]: S(G) is connected if and only if G is a non-trivial connected graph.

3. Main Results

In the following, we characterize the graphs G with $r(G) = 1$ for which S(G) is self-centered with radius 2.

Theorem 3.1: Let G be any graph with at least three vertices and $r(G) = 1$. Then S(G) is self-centered with radius 2 if and only if G has no pendant vertices.

Proof: Let G be any graph with at least three vertices and $r(G) = 1$. Assume G has no pendant vertices. Since $r(G) = 1$, each vertex of G lies on a triangle. Then for every v_i and v_j in V(G),

$$\begin{aligned}
&(a)\ d^*(v_i, v_j) = d^*(v_i, v_j') = 1, && \text{if } v_iv_j \in E(G) \\
&\qquad\qquad\quad\ = 2, && \text{if } v_iv_j \notin E(G) \\
&(b)\ d^*(v_i, v_i') = d^*(v_i', v_j') = 2. &&
\end{aligned}$$

Thus, it follows that, $e^*(v_i) = 2$ and $e(v_i') = 2$, for all $v_i \in V(G)$. Hence S(G) is self-centered with radius 2.

Conversely, assume S(G) is self-centered with radius 2 and $\delta(G) = 1$. Let $v \in V(G)$ be such that $deg(v) = 1$. Since $r(G) = 1$, there exists a vertex $u \in V(G)$ such that $deg(u) = |V(G)| - 1$. Then $d^*(u', v') = 3$, since $u'\ v\ u\ v'$ is a geodesic in S(G), which is a contradiction and hence $\delta(G) \geq 2$. □

Corollary 3.1.1: If $r(G) = 1$ and G has pendant vertices, then S(G) is bi-eccentric with radius 2. □

Next, we characterize the self-centered graphs G with radius 2 for which S(G) is also self-centered.

Theorem 3.2: Let G be a self-centered graph with radius 2.

Then S(G) is also self-centered with radius 2 if and only if for every pair of adjacent vertices u, v in G, $N_G(u) \cap N_G(v) \neq \phi$.

Proof: Let G be self-centered with radius 2. Assume for every pair of adjacent vertices u, v in G, $N_G(u) \cap N_G(v) \neq \phi$. Let v_i, $v_j \in V(G)$.

$$
\begin{aligned}
\text{Then } d^*(v_i, v_j) = d^*(v'_i, v'_j) &= 1, \quad \text{if } v_iv_j \in E(G); \\
&= 2, \quad \text{if } v_iv_j \notin E(G); \text{ and}
\end{aligned}
$$

$d^*(v_i, v'_i) = 2$, for all $v_i \in V(G)$. Hence, $e^*(v_i) = 2$. Also by the assumption, for every pair of adjacent vertices v_i and v_j in V(G), there exists a vertex v_k in V(G) adjacent to both v_i and v_j and hence $d^*(v'_i, v'_j) = 2$. Thus, $e^*(v'_i) = 2$, for all $v_i \in V(G)$. Hence S(G) is self-centered with radius 2.

Conversely, assume that there exists a pair of adjacent vertices v_i, v_j in G for which $N_G(v_i) \cap N_G(v_j) = \phi$. Then $d^*(v'_i, v'_j) = 3$, since $v'_i\ v_j\ v_i\ v'_j$ is a geodesic in S(G). This is a contradiction, since S(G) is self-centered with radius 2. □

Corollary 3.2.1: Let G be self-centered with radius 2. If there exists a pair of adjacent vertices v_i, v_j in G with $N_G(v_i) \cap N_G(v_j) = \phi$, then, S(G) is bi-eccentric with radius 2.

Proof: This follows from the proof of the converse part of Theorem 3.2. □

Lemma 3.1: Let G be any connected graph. If $u \in V(G)$ is such that $e_G(u) = m$, for $m \geq 3$, then $e^*(u) = e^*(u') = m$.

Proof: Let $u \in V(G)$ be such that $e_G(u) = m$, $m \geq 3$. Then there exists a vertex $v \in V(G)$ with $d_G(u, v) = m$. Let the shortest path joining u and v be

$P(u, v)$: $u\ u_1\ u_2 \cdots u_{m-1}\ u_m$ (= v), where $u_i \in V(G)$, i = 1, 2, $\cdots$, m-1. Then,

(a) $d^*(u, u_i) = d^*(u, u'_i) = d^*(u', u_i) = i$, $i = 1, 2, \cdots, m$.
(b) $d^*(u, u') = 2$;
(c) $d^*(u', u'_1) = 2$ (or) 3; and
(d) $d^*(u', u'_i) = i$, $i = 2, 3, \cdots, m$.

From (a), (b), (c) and (d), it follows that, $e^*(u) = e^*(u') = m$, $m \geq 3$. □

Next, we characterize graphs G for which S(G) is bi-eccentric with radius 2.

Theorem 3.3: For any connected graph G, S(G) is bi-eccentric with radius 2 if and only if G is one of the following graphs.

(1) r(G) = 1 and G has pendant vertices.

(2) G is self-centered with radius 2 and there exists at least one pair of vertices u, v in G such that $N_G(u) \cap N_G(v) = \phi$.

(3) G is bi-eccentric with radius 2.

Proof: Let G be any connected graph. Assume G is one of the graphs given above. If r(G) = 1 and G has pendant vertices, then by Corollary 3.1.1., it follows that S(G) is bi-eccentric with radius 2. If G is as in (2), then by Corollary 3.2.1., S(G) is bi-eccentric with radius 2. Let G be bi-eccentric with radius 2.

(i) If $v_i \in V(G)$ is such that $e(v_i) = 3$, then by Lemma 3.1, $e^*(v_i) = e^*(v_i') = 3$.

(ii) If $e(v_i) = 2$, then $e^*(v_i) = 2$ and $e^*(v_i') = 2$ (or) 3. Hence, S(G) is bi-eccentric with radius 2.

Conversely, for a connected graph G, assume S(G) is bi-eccentric with radius 2.

(a) If $r(G) \geq 3$, then by Lemma 3.1, $r(S(G)) \geq 3$, which is not possible and hence $r(G) \leq 2$.

(b) If r(G) = 1 and G has no pendant vertices then by Theorem 3.1, S(G) is self-centered with radius 2, which is a contradiction. Thus, if r(G) = 1, then G has pendant vertices.

(c) If r(G) = 2 and diam(G) = 4, then S(G) also has diameter 4, by Lemma 3.1.

(d) If G is self-centered with radius 2 and for every pair of adjacent vertices u, v in G, $N_G(u) \cap N_G(v) \neq \phi$, then S(G) is self-centered with radius 2.

Hence G is one of the graphs given in (1), (2) and (3). □

Remark 3.1: If G is any graph with radius r ($r \geq 2$) and diameter d, then S(G) is G-eccentricity preserving.

Theorem 3.4: $r(S(G)) = \max\{2, r(G)\}$

Proof: Follows from Theorem 3.1., 3.2., Lemma 3.1 and Theorem 3.3. □

Theorem 3.5:

$$\text{diam}(S(G)) = \begin{cases} \text{diam}(G), & \text{if } \text{diam}(G) \geq 3 \\ 2 \text{ (or) } 3, & \text{if } \text{diam}(G) = 2. \end{cases}$$

Proof: Follows from Lemma 3.1. and Theorem 3.3. □

Remark 3.2: Let G be any connected graph having at least three vertices. Then the complement $\overline{S}(G)$ of S(G) is bi-eccentric with radius 2 if and only if $r(G) = \delta(G) = 1$ and in all other cases $\overline{S}(G)$ is self-centered with radius 2.

For a graph G, define $S^2(G) = S(S(G))$, $S^n(G) = S(S^{n-1}(G))$, $n \geq 2$. In the following, we see the eccentricity properties of $S^n(G)$, $n \geq 2$.

Theorem 3.6: For any connected graph G with diameter 2, $S^n(G)$ ($n \geq 2$) is either bi-eccentric with radius 2 or self-centered with radius 2.

Proof: If $\text{diam}(G) = 2$, then $r(G) = 1$ (or) 2.

Case (i): $r(G) = 1$

(a) If $\delta(G) = 1$, then S(G) is bi-eccentric with radius 2 and by Corollary 3.1.1., $S^n(G)$ ($n \geq 2$) is bi-eccentric with radius 2.

(b) If $\delta(G) \geq 2$, then S(G) is self-centered with radius 2 by Theorem 3.1. Also each edge in $S^n(G)$ ($n \geq 2$) lies on a triangle and hence $S^n(G)$ ($n \geq 2$) is self-centered with radius 2.

Case (ii): $r(G) = 2$.

Then G is self-centered with radius 2 and hence S(G) is either self-centered with radius 2 (or) bi-eccentric with radius 2, by Theorem 3.2 and Corollary 3.2.1. Therefore, $S^n(G)$, ($n \geq 2$) is also either 2-self centered (or) bi-eccentric with radius 2. □

Remark 3.3: $S^n(G)$ ($n \geq 1$) is G-radius, diameter preserving graph, if G has radius r, where $r \geq 3$.

Remark 3.4: $S^n(K_m)$, ($n \geq 2$, $m \geq 3$) is self-centered with radius 2.

A total dominating set T of G is a dominating set such that the induced subgraph G[T] has no isolated vertices. The total dominating number γ_t of G is the minimum cardinality of a total dominating set. This concept was introduced in Cockayne, Dawes and Hedetniemi [3].

In the following, we give bounds on $\gamma(S(G))$.

Proposition 3.1: For any (p, q) graph G having no isolated vertices,
$2 \leq \gamma(S(G)) \leq p$.

Proof: Since the radius of $S(G) \geq 2$, $\gamma(S(G)) \geq 2$. Also the set $V'(G) \subseteq V(S(G))$ is a dominating set of S(G) and hence $\gamma(S(G)) \leq |V(G)| = p$. □

The bounds are sharp, since $\gamma(S(G))= 2$, if $G \cong K_{1,n}$; K_n, $n \geq 2$ and $\gamma(S(G)) = p$, if $G \cong mK_2$, $m \geq 2$.

In the following, we give an upper bound of $\gamma(S(G))$in terms of total domination number of G.

Theorem 3.7: Let G be any graph having no isolates. Then $\gamma(S(G)) \leq \gamma_t(G)$.

Proof: Let T be a total dominating set of G with $|T| = \gamma_t(G)$. For any $v \in (V(S(G) - D) \cap V(G)$, T dominates both the vertices v and v′ in S(G), since T is a dominating set of G. Also for any $w \in T$, the corresponding vertex w′ in V′(G) is adjacent to at least one vertex in T, since T is a total dominating set G. Hence T is a dominating set of S(G). Thus, $\gamma(S(G)) \leq \gamma_t(G)$. □

This bound is sharp since, when $G \cong C_3$ (or) C_6, $\gamma(S(G)) = \gamma_t(G)$.

The concept of global domination was introduced by Sampathkumar [8]. A set $S \subseteq V$ is called a global dominating set of G, if S is a dominating set in both G and its complement $\overline{G}$. The global domination number γ_g of G is the minimum cardinality of a global dominating set of G. In the following, we find a necessary and sufficient condition for a global dominating set of G to be a global dominating set of S(G).

Theorem 3.8: For any graph G, $\gamma_g(S(G)) \leq \gamma_g(G)$ if and only if there exists a global dominating set D of G with $|D| = \gamma_g(G)$ such that G[D] has no isolated vertices.

Proof: Let D be a global dominating set of G with $|D| = \gamma_g(G)$ such that G[D] has no isolated vertices. Then obviously D is a dominating set of S(G). Since D is a global dominating set of G, it is enough to prove D dominates v′ in $\overline{S}(G)$, for all $v \in V(G)$. For

any vertex v∈D, the vertex v′ is adjacent to v in $\overline{S}$(G). Similarly, for any vertex u∈V(G) - D, the vertex u′ in $\overline{S}$(G) is adjacent to at least one vertex in D, since D is a dominating set of $\overline{G}$. Hence D is a dominating set of $\overline{S}$(G) and is a global dominating set of S(G). Thus γ_g(S(G)) ≤ γ_g(G).
Conversely, assume a global dominating set D of G is also a global dominating set of S(G) with |D| = γ_g(G). If G[D] has isolated vertices, then D is not a dominating set of S(G), which is a contradiction. □

In the following, we find an upper bound for $\gamma_g(S(G))$.

Theorem 3.9: γ_g(S(G)) ≤ δ(G) + 2 if and only if the closed neighborhood of the vertex of minimum degree is a dominating set of G.

Proof: Let v∈V(G) be such that deg(v) = δ(G) and D = N[v]. Assume D is a dominating set of G. Since D = N[v], it is a total dominating set of G and hence is a dominating set of S(G). Let v′ be the vertex in S(G), corresponding to v in G. If D′ = D∪{v′}, then D′ is a dominating set of S(G) . Hence γ_g(S(G)) ≤ δ(G) + 2.
Conversely, assume γ_g(S(G)) ≤ δ(G) + 2. If D = N[v] is not a dominating set of G, then there exists a vertex w∈V(G) - D, not adjacent to any of the vertices in D and hence D is not a dominating set of S(G). Hence the theorem follows. □

This bound is attained, if G is a path on four vertices.

Remark 3.5: If G[D - {v}] = G[N(v)] contains isolated vertices, then
γ_g(S(G)) ≤ δ(G) + 1 and γ_g(S(G)) = δ(G) +1, if G ≅ $K_{1,n}$, n ≥ 2.

The global domination numbers of S(G) for some standard graphs G can be easily found and are given as follows.

(a) $\gamma_g(S(K_n))$ = n, $n \geq 3$
(b) $\gamma_g(S(K_{1,n}))$ = 2, $n \geq 2$
(c) $\gamma_g(S(P_k))$ = 2n+2, if k = 3n+2, 3n+3, 3n+4, n = 1, 2, $\cdots$
= 3, if k = 3, 4.
(d) $\gamma_g(S(C_n))$ = 2n + 2, if k = 4n+2, 4n+3, 4n+4
= 2n+3, if k = 4n+5, n = 1, 2, $\cdots$
= 2, if k = 4.
= 3, if k = 5.

In the following, we obtain bounds for $n_0(S(G))$.

Proposition 3.2: For a graph G without isolated vertices, $2 \leq n_0(S(G)) \leq p$.

Proof: Since radius of S(G) is at least 2 , $n_0(S(G)) \geq 2$. Also, the set V(G) is an n-set of S(G) and hence $n_0(S(G)) \leq p$. □

The bounds are sharp since, $n_0(S(G)) = 2$, if radius of G is 1 and $n_0(S(G)) = p$, if G is a regular bipartite graph.

Next, we find a necessary and sufficient condition for an n-set of G to be an n-set of S(G).

Theorem 3.10: For any graph G without isolated vertices, $n_0(S(G)) \leq n_0(G)$ if and only if there exists an neighborhood set D of G with $|D| = n_0(G)$ such that for every pair of adjacent vertices u, v in G, where $u \in D$, $v \in V(G) - D$, $[N_G(u) \cap N_G(v)] \cap D \neq \phi$.

Proof: Let there exist a neighborhood set D of G with $|D| = n_0(G)$ such that for every pair of adjacent vertices u, v in G, where $u \in D$, $v \in V(G) - D$, $[N_G(u) \cap N_G(v)] \cap D \neq \phi$. Since D is a neighborhood set of G, the edges xy in S(G), where x, $y \in V(G)$, xy', where $x \in D$ and $y \in V(G) - D$ and xy', $x'y$, where x, $y \in V(G) - D$ belong to $\cup_{u \in D} E(G[N[u]])$. Consider the edge $x'y$ in S(G), where $x \in D$ and $y \in V(G) - D$. Then $xy \in E(G)$. By the assumption, there exists a vertex $z \in D$ adjacent to both x, y in G and the edges $x'z$, yz are in S(G). Hence, $x'y \in E(G[N[z]])$. Hence, D is a neighborhood set of S(G) and so $n_0(S(G)) \leq n_0(G)$.
Conversely, assume $n_0(S(G)) \leq n_0(G)$. Then any neighborhood set of G is also a neighborhood set of S(G). Let D be a neighbor-

hood set of G with $|D| = n_0(G)$ and there exist a pair of adjacent vertices u, v in G such that $[N_G(u) \cap N_G(v)] \cap D = \phi$, where $u \in D$, $v \in V(G) - D$. Then the edge u'v in S(G) does not belong to $\cup_{w \in D} E(G[N[w]])$ which contradicts the fact that D is a neighborhood set of S(G). □

Now, we list the exact values of $n_0(S(G))$ for some standard graphs.

(a) $n_0(S(C_n)) = n, n \geq 4.$
(b) $n_0(S(P_n)) = n,$ if n is even
$= n-1,$ if n is odd, $n \geq 2.$
(c) $n_0(S(K_{m,n})) = 2\min(m, n), m, n \geq 1.$

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