Embedding graphs without edge accumulation points in tubular surfaces *

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Abstract

In this work, we study which *tubular surfaces* verify that the embeddings of infinite, locally finite connected graphs without vertex accumulation points are embeddings without edge accumulation points. Furthermore, we characterize the graphs which admit embeddings with no edge accumulation points in the *sphere* with n ends in terms of forbidden subgraphs.

1 Introduction

Dirac and Schuster [6] proved that a countable graph is planar if and only if each finite subgraph is planar, and Wagner [15] characterized all planar graphs. Thus, the Kuratowski Theorem on planarity of graphs holds for infinite graphs. However, as many authors such as Halin and Thomassen (see [8, 14]) have pointed out, some additional properties can be, and must be, added to planarity in the case of infinite, locally finite graphs. In particular, from a practical point of view, accumulation points must be avoided. Thus, a graph is $VAP-free$ planar if it admits a planar embedding such that the vertex set has no accumulation point in the plane. Halin [8] gave the characterization of graphs with a $VAP-free$ planar embedding in terms of forbidden subgraphs. Graphs which admit embeddings with no vertex accumulation points in generalized cylinders are characterized by Boza et al. [5, 12]. Thomassen [14] shows that all connected $VAP-free$ plane graphs admit locally finite plane embeddings without accumulation points ($EAP-free$ plane graphs in the sense of [14]), therefore when a graph is connected, there is a $VAP-free$ planar embedding if and only if there is an $EAP-free$ planar embedding. In 1994 Ayala et al. [4] characterized $EAP-free$ planarity by adding two new graphs to Halin's forbidden graphs. In section 3 of this paper we study in which *tubular surfaces* is verified that there is a $VAP-free$ embedding of infinite, locally finite,

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connected graph if and only if there is an $EAP-free$ embedding of that graph, extending the result given by Thomassen [14] for the sphere with n *Freudenthal Ends* and we find counter-examples to assure that the result cannot be extended further. In section 4 we characterize the graphs which admit embeddings with no accumulation points in the *sphere* with n *Freudenthal Ends.*

2 Preliminary

Basic notions are now explained to help in the understanding of the development of this paper. We will consider all graphs to be undirected and without loops nor multiple edges. We follow the standard terminology of graph theory, as presented in [9]. When we consider infinite graphs, we use the terminology followed by Köning [10] and the surveys of Thomassen [13] and Nash Williams [11].

We consider infinite, locally finite graphs to be those graphs with a countable vertex set, such that the degree of any vertex is finite.

Throughout this paper, we will talk about *tubular surfaces*. The first question regards the actual meaning of *the tubular surfaces of finite genus*. We will use an invariant of non-compact space, namely, *Freudenthal End*. Let X be a non-compact space. *Freudenthal Ends* of X are the elements of the inverse limit $F(X) = \varprojlim \pi_0(X-K)$ where K varies throughout the family of compact sets of X and where $\pi_0(X-K)$ stands for the set of connected components. These surfaces are built from a compact surface S of a finite genus, where n open balls are replaced by n half cylinders. $S(n)$ represents a non-compact surface of finite genus with n Freudenthal ends. For example, if S^2 is the sphere and $\mathcal{P}_2$ is the projective plane, then $S^2(1)$ is homeomorphic to a plane, $S^2(2)$ is homeomorphic to a cylinder and $\mathcal{P}_2(1)$ is homeomorphic to the Möbius band.

In addition, when G is a graph we can use a countable sequence $G_1 \subseteq G_2 \subseteq ...$ of finite subgraphs to define $F(G)$. An infinite ray in a graph G is a morphism $\varphi : J \rightarrow G$ inducing an injection on both the vertex set and the edge set, where J represents a graph such that its underlying topological space is homeomorphic to the positive half-line R^+. Two rays in G define the same Freudenthal end if for any finite subgraph H of G, there exist vertices of both rays in the same component of $G - H$.

For example, the Euclidean halfline, $R^+ = [0, +\infty)$ has one Freudenthal end and the Euclidean line has two. All Euclidean spaces R^n with $n \geq 2$ have, exactly, one Freudenthal end (see Figure 1).

We say that an infinite, locally finite graph G is *strongly stable* if there exists a finite subgraph H such that every component of $G - H$ is a ray. If G is a finite graph and W is a set of vertices of G, we denote by G_W to the strongly stable graph built from G, each ray starting from every vertex of W (see Figure 2). Thus, a graph G' is strongly stable if and only if there exists G such that G' is isomorphic to G_W.

This construction plays an important role in previous works [5, 12] as well as in the works of Archdeacon et al. [1, 2] and Bonnington and Richter [3].

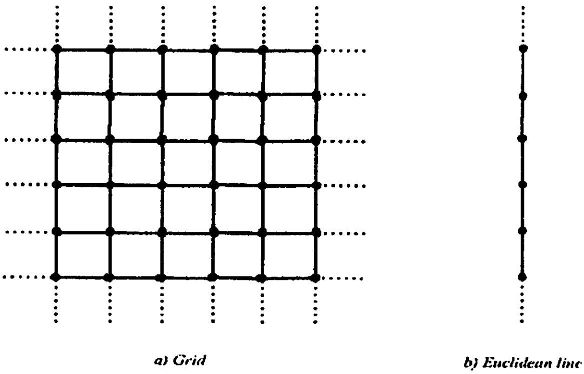

Figure 1: a) The Grid has 1 end. b) The Euclidean line has 2 ends.

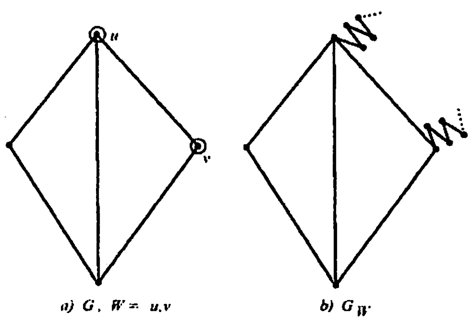

Figure 2: G_W is the ***strongly stable*** graph built from G

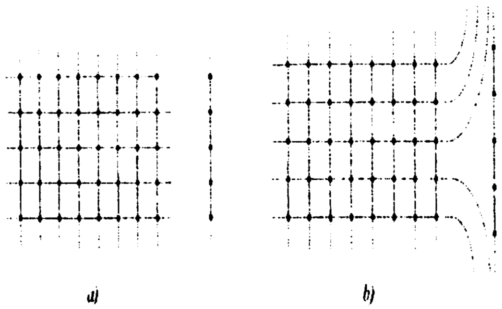

Figure 3: a) A non-connected graph with two components b) An embedding $VAP - free$ planar and non-$EAP - free$ planar

The above-mentioned works are mainly about embeddings of infinite connected graphs without vertex accumulation points.

The current work is about embeddings of infinite connected and not connected graphs without any type of accumulation points (neither vertices neither interior points of edges). As we have already mentioned previously, Thomassen [14] proves that in the case of an infinite, locally finite, connected planar graph, there exists an embedding without vertex accumulation points ($VAP - free$) if and only if there exists an embedding without accumulation points (neither vertices neither interior points of edges)($EAP - free$).

For instance, we consider the following infinite, locally finite planar graph with two connected components: an infinite grid and a subgraph homeomorphic to Euclidean line. It is possible to give an embedding without vertex accumulation points of this graph however this embedding has accumulation in the interior points of edges (see Figure 3).

In this sense, Ayala et al. [4] find the characterization of the infinite, locally finite planar graphs without accumulation points in terms of forbidden subgraphs, adding two new graphs (see Figure 5) to the Halin's forbidden graphs [8].

In this work, we extend the result given by Thomassen [14] to the sphere with n ends, we see that the result does not hold in general case, showing counterexamples for the Torus with one end and Klein Bottle with one end (see Figure 7) and finally we prove a characterization of the embeddings without accumulation points (neither vertices neither interior points of edges) in the sphere with n ends in terms of forbidden subgraphs.

Given an infinite, locally finite graph G, an $EAP - free$ planar embedding $\varphi : G \rightarrow R^2$ is said to be a *spanning embedding* if all the components of $R^2 - \varphi(G)$ are bounded. There exist obvious examples of infinite, locally finite graphs which admit both spanning and non-spanning embeddings (see Figure 4).

Graph G is a *tiling planar graph* if all of its $EAP - free$ planar embeddings

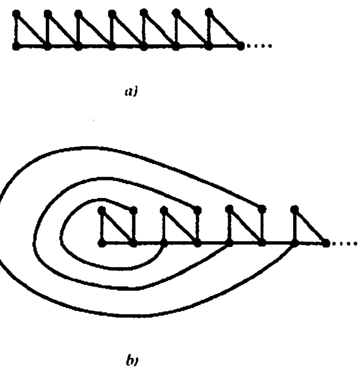

Figure 4: a) A non-spanning embedding. b) A spanning embedding.

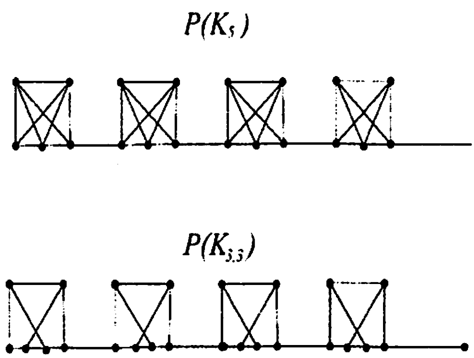

Figure 5: $P(K_5)$ and $P(K_{3,3})$

are spanning embeddings. In 1994 Ayala et al. [4] characterized these graphs with the following result:

Theorem 1 *[4] G is a tiling planar graph if and only if G has an $EAP-free-$planar embedding and it contains a subgraph homeomorphic to $P(K_5)$ or $P(K_{3,3})$ (see Figure 5).*

As a consequence of this result, the characterization of $EAP-free$ planarity was obtained [4] by adding two new graphs to Halin's forbidden graphs.

In this context, the following definitions arise naturally.

A graph is said to be a VAP (or EAP)$-free-S(n)$ graph if it has an embedding without vertex accumulation points (or an embedding without accumulation points (neither vertices neither interior point of edges)) in $S(n)$. An embedding without accumulation points in $S(n)$, $\varphi : G \to S(n)$ is said to be a *spanning embedding* if all the components of $S(n) - \varphi(G)$ are bounded. A graph is a *tiling graph in* $S(n)$ if all its $EAP - free - S(n)$ embeddings are spanning embeddings in $S(n)$. We show the characterization of the tiling graphs in $S^2(n)$.

Theorem 2 *G is a tiling graph in $S^2(n)$ if and only if G is EAP-free-$S^2(n)$ and it contains a subgraph homeomorphic to $\bigsqcup_{k=1}^{n} P^*(A_k)$, $P^*(A_k) = P(K_5)$ and/or $P(K_{3,3})$.*

Proof. By induction in n :

For $n = 1$, the result is proved in [4], due to $S^2(1)$ is homeomorphic to $\mathcal{R}^2$.

We suppose the result true for n and we show for $n + 1$.

Sufficient condition

$S^2(n+1)$ is homeomorphic to plane minus n points. Let $(0,0)$ be one of this points. We consider a compact K such that $(0,0)$ is in the interior of $(\mathcal{R}^2 - \{n$ points$\}) - K$.

Let G be a graph and $G_k = G \cap K$, then $G - G_k = G_1 \sqcup G_2$ where, $G_1 \subset Int((\mathcal{R}^2 - \{n$ points$\}) - K)$ y G_2 in the exterior. The interior face of $(\mathcal{R}^2 - \{n$ points$\}) - K$ is homeomorphic to $S^2(1)$. As the result is true for $n = 1$, there exist $P(A_k) \subset G_1$ with $A_k = K_5^*$ or $K_{3,3}^*$. The exterior face is homeomorphic to $S^2(n)$. Applying induction hypothesis, there exist $P(A_{1k}), P(A_{2k}), ..., P(A_{nk})$ such that $\bigsqcup_{i=1}^{n} P(A_{ik}) \subset G_2$ with $A_{ik} = K_5^*$ or $K_{3,3}^*$. As G_1 and G_2 are disjoints, we have $P(A_k) \cap P(A_{ik}) = \emptyset$, for each $i = 1, ..., n$. Therefore, we have found $P(A_k) \sqcup P(A_{1k}) \sqcup P(A_{2k}) \sqcup ... \sqcup P(A_{nk}) \subset G$.

Necessary condition

By hypothesis $\bigsqcup_{i=1}^{n+1} P(A_{ik}) \subset G$. As G has an embedding without accumulation points, we have that $G - \bigsqcup_{i=1}^{n+1} P(A_{ik})$ is compact set. We consider $\bigsqcup_{i=1}^{n+1} P(A_{ik}) = \bigsqcup_{i=1}^{n} P(A_{ik}) \sqcup P(A_{n+1,k})$.

$S^2(n+1)$ is homeomorphic to $\mathcal{R}^2$ minus n points, $(0,0)$ is one of them. Let d be distance from $(0,0)$ to another $n-1$ points. We denote by $K = \overline{B}(\frac{1}{2}d) - B(\frac{1}{3}d)$. Then, we have $(\mathcal{R}^2 - \{\, n$ points$\}) - K = R_1 \sqcup R_2$ where R_1 is homeomorphic to $S^2(1)$ and R_2 is homeomorphic to $S^2(n)$.

By induction hypothesis,

$\bigsqcup_{i=1}^{n} P(A_{ik})$ cover to R_2 and $P(A_{n+1,k})$ cover to R_1.

In addition, we have: $K \cap G \subset G - \bigsqcup_{i=1}^{n+1} P(A_{ik})$.

Therefore, $K \cap G$ is a compact and the faces contain in $K \cap G$ are bounded. As a consequence:

$\bigsqcup_{i=1}^{n} P(A_{ik}) \sqcup P(A_{n+1,k}) \cup (K \cap G)$ cover to $S^2(n+1)$.

Hence, G cover to $S^2(n+1)$. ■

3 Relation between $VAP-free$ and $EAP-free$ embeddings in $S(n)$

In this section, we establish that if an infinite, locally finite graph is connected, then there is a $VAP-free-S^2(n)$ embedding if and only if there is an $EAP-free-S^2(n)$ embedding, thereby obtaining an extension of the result given by Thomassen in [14]. In addition, we show that the result does not hold in the general case.

First of all, we prove that an infinite, locally finite, connected, $VAP-free-S^2(n)$ graph is isomorphic to a subgraph of a triangulation of $S^2(n)$. Since an infinite triangulation of $S^2(n)$ has no accumulation points, we obtain that there is an $VAP-free$ embedding if and only if there is an $EAP-free$ embedding in $S^2(n)$.

We know that $S^2(n)$ is homeomorphic to a sphere minus n points or to a plane minus $n-1$ points, therefore in a natural way, we can define a *triangulation* of $S^2(n)$. An embedding Λ of an infinite, locally finite connected graph in $S^2(n)$ is a *triangulation* if and only if Λ is $VAP-free-S^2(n)$ and for every vertex $u \in \Lambda$, there exits a cycle Φ_u of Λ such that u is the unique vertex of Λ in the interior of Φ_u where u is joined by an edge to every vertex of Φ_u. If, in addition, every point x is in $S^2(n) - \Lambda$ then we deduce x is contained in the interior of some cycle of Λ.

Theorem 3 *Let G be an infinite, locally finite, connected, $VAP-free-S^2(n)$ graph. There exists a triangulation Δ of $S^2(n)$ such that G is isomorphic to a subgraph of Δ.*

Proof.

We prove the result by induction in n:

- For $n=1$ the result is true by Thomassen [14].
- Supposing it is true for $n-1$, therefore for n:

  Let G be an infinite, locally finite, connected $VAP-free-S^2(n)$ graph, then G has an embedding in $S^2(n-1)$, Γ, with one vertex accumulation point. Let K be a compact which isolate the vertex accumulation point, and $\Gamma_k = \Gamma \cap K$, therefore

  $\Gamma - \Gamma_k = \Gamma_1 \sqcup \Gamma_2$ such that

  $\Gamma_1 \subset int(S^2(n-1) - K)$ and $\Gamma_2 \subset ext(S^2(n-1) - K)$.

Figure 6: G is homeomorphic to the wedge of $P(K_{3,3})$ and P_ω

The exterior face of $S^2(n-1) - K$ is homeomorphic to $S^2(n-1)$ and Γ_2 is $VAP - free - S^2(n-1)$ and connected, thereby applying induction hypothesis we have that Γ_2 is isomorphic to a subgraph of a triangulation, Δ_2.

The interior face of $S^2(n-1) - K$ is homeomorphic to $S^2(1)$ and Γ_1 is $VAP - free - S^2(1)$ and connected, so Γ_1 is isomorphic to a subgraph of a triangulation, Δ_1.

We have to see now in a compact K, it is a finite graph therefore it is a subgraph of a finite triangulation.

Let xy be an edge of K which is not contained in a separating 3-cycle of K. Let K' be a graph obtained to contract the edge xy to a vertex z. If Λ is an embedding of K, then an embedding of K' is obtained from Λ such that only the edges incident with x are affected. Reciprocally, if Λ' is an embedding of K', then an embedding of K can be obtained from Λ' such that only the incident edges with x are changed and x can be adjoint with y arbitrarily. Doing to coincide a cycle of Γ_1 and a cycle of Γ_2 with the boundary cycles of Λ, we have that Γ is isomorphic to a subgraph of a triangulation of $S^2(n)$.

In the case that a compact has separating 3-cycles, this edges are not contracted, deleting the interior of separating 3-cycles we obtain again a triangulation of $S^2(n)$. After that, we put in the interior of a separating 3-cycle what we had initially and adding edges until to obtain a triangulation.

■

As an infinite triangulation of $S^2(n)$ has not accumulation points, we obtain the following result:

Corollary 4 *Let G be an infinite, locally finite, connected and $VAP - free - S^2(n)$ graph. Therefore G is an $EAP - free - S^2(n)$ graph.*

We have just seen that in a sphere is true that there is a $VAP - free - S^2(n)$ embedding if and only if there is an $EAP - free - S^2(n)$ embedding.

We show now that in the general case this result does not hold. Hence, we consider the following counter-example: Let G be the graph in Figure 6 and let an embedding of G be on the Torus with one end and on the Klein Bottle with one end (Figure 7). These embeddings are $VAP - free - S(1)$ but are not $EAP - free - S(1)$, when S is a Torus or Klein Bottle.

This allows us to state the following:

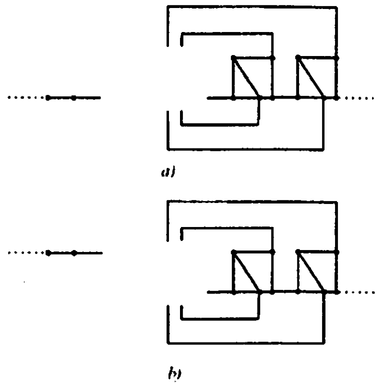

Figure 7: a)An embedding of G on the Torus with one end. b)An embedding of G on the Klein Bottle with one end.

Remark 5 *Let S be a compact surface of finite genus and the Euler characteristic $\chi(S) \geq 0$ (whereby the surface is different to a sphere and a projective plane). Let G be an infinite, locally finite graph homeomorphic to the wedge of P_ω with n ($P(K_5)$ or $P(K_{3,3})$). G is $VAP-free-S(n)$ graph and G has not $EAP-free-S(n)$ embeddings.*

Obviously G is $VAP-free-S(n)$ and its embedding is similar to that in Figure 7.

It remains to be proved that G has no $EAP-free-S(n)$ embedding. Therefore, the opposite is supposed. Let Γ be an embedding of G in $S(n)$ without accumulation points and let K be a compact. We consider $\overline{S(n)-K} = C_1 \sqcup C_2 \sqcup ... \sqcup C_n$

$|E(\Gamma \cap K)| < \infty$ since Γ is $EAP-free-S(n)$.

In $\overline{S(n)-K}$, we have n $P(K_5)$ or $P(K_{3,3})$ and P_ω and n half-cylinder C_i, $i = 1, ..., n$ hence, Γ is not an $EAP-free-S(n)$ embedding since n $P(K_5)$ or $P(K_{3,3})$ are tiling graphs in $S^2(n)$.

4 Characterization of EAP-free-$S^2(n)$ embeddings

The following is a characterization of embeddings without accumulation points in the sphere with n ends in terms of $VAP-free-S^2(n)$ forbidden graphs.

Assuming that G contains infinite planar components $C_1, ..., C_p, ...$, we define
$a(C_i) = \max\{ r \in N \ / \ C_i$ is not $VAP-free-S^2(n)\}$ and
$b(C_i) = \max\{ r \in N \ / \ C_i$ is not $EAP-free-S^2(n)\}$.
It is clear that $a(C_i) \leq b(C_i)$.

Lemma 6 *Let G be an infinite, locally finite graph, therefore there exists $G' \subseteq G$ whose ends are all strongly stable such that $a(G) = a(G')$.*

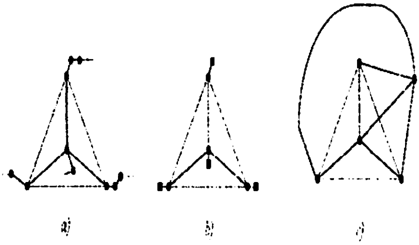

Figure 8: a) G with 4 ends b) G with 4 marked vertices c) G^c is K_5

Proof. We can build an infinite sequence of graphs $G_1 \subsetneq G_2 \subsetneq ... \subsetneq G_i \subsetneq ...$ such that $\cup_i G_i = G$.

Let us suppose the opposite is true, i.e. $\lim_{n\to\infty} a(G_n) < a(G)$ and $\forall k$, G_k is $VAP - free - S^2(a(G))$.

This implies that $G = \cup_i G_i$ is $VAP - free - S^2(a(G))$ which is clearly a contradiction. ■

By Lemma 6 we can suppose, without any loss of generality, that all the ends of G are strongly stable.

Let G be a strongly stable graph with n ends. We define an *n-compactification* of G as a finite graph G^c where the n ends of G are replaced by n vertices $v_1, ..., v_n$ and these vertices $v_i, i = 1, .., n$ are identified in one only vertex v in G^c (see Figure 8).

Lemma 7 *Let G be a graph whose ends are all strongly stable. Therefore G is a $VAP - free - S^2(k)$ graph if and only if there exists a planar $k - compactification$ of G.*

Lemma 8 *Suppose that G has two infinite components G_1 and G_2. If G_1 is not $VAP - free - S^2(n_1)$ and G_2 is not $VAP - free - S^2(n_2)$ then $G_1 \cup G_2$ is not $VAP - free - S^2(n_1 + n_2)$.*

Proof. If we suppose the opposite, i.e. $G_1 \cup G_2$ is $VAP - free - S^2(n_1 + n_2)$, then there exists an $(n_1 + n_2) - compactification$ planar of $G_1 \cup G_2$.

If this graph had only one compactification vertex shared by two components then we could give a $VAP - free - S^2(n_1)$ embedding of G_1 and a $VAP - free - S^2(n_2)$ embedding of G_2 which is clearly a contradiction.

If this graph had at least two vertices of compactification in such a way that $G_1 \cup G_2$ were incident therein, then both compactification vertices would be contained in one face. The embedding of $G_1 \cup G_2$ can be changed in such a way that only one compactification vertex exists: one of the compactification vertices is cut and the edges which leave from G_2 for example, to the other compactification vertex are taken, therefore G_1 is incident in a compactification vertex and G_2 is incident in the other vertex. ■

Lemma 9 *Let G be an infinite, locally finite graph with infinite planar components $C_1, ..., C_p, ...$. Therefore $a(G) = \sum a(C_i)$*

Proof. We prove the result for double inequality:

Let us first check that $a(G) \leq \sum a(C_i)$.

Let C_i be $VAP - free - S^2(a(C_i) + 1)$ components and $G = \cup C_i$.

We must have an embedding of each one of the components C_i whose $a(C_i)+1$ ends are strongly stable . We can give an embedding of G by identifying one of the ends of each C_i in a single end , through which we obtain $a(G) < \sum a(C_i)+1$, therefore $a(G) \leq \sum a(C_i)$, and we thereby deduce one of the inequalities.

Let us now check that $a(G) \geq \sum a(C_i)$.

We suppose that G has two components G_1 and G_2 such that $G = G_1 \cup G_2$. Hence $a(G) + 1 = \#\ of\ vertices\ of\ compactification$
$a(G) + 1 \geq (n_1 + 1) + (n_2 + 1) - 1 = n_1 + n_2 + 1$,
$a(G) \geq n_1 + n_2$.
It is easy to generalize since we have n components.

■

Lemma 10 *Let $G = C_1 \cup C_2$ be an infinite, locally finite graph and $b(G) = \max\{n \in N \ / \ G$ is not $EAP - free - S^2(n)\}$ such that C_i is not a tiling graph in $S^2(b_i + 1)$ where $b(C_i) = b_i$, $i = 1, 2$. Hence $b(G) \leq b(C_1) + b(C_2)$.*

Proof. We consider an embedding of G_1 in $S^2(b_1 + 1)$ and an embedding of G_2 in $S^2(b_2+1)$. By hypothesis, G_i is not a tiling graph of $S^2(b_i+1)$, hence we can make an end of G_1 coincide with an end of G_2. Therefore, we have $b_1 + b_2 + 1$ ends, hence we obtain an $EAP - free - S^2(b_1 + b_2 + 1)$ embedding of G and since $b(G) < b_1 + b_2 + 1$, therefore $b(G) \leq b(G_1) + b(G_2)$. ■

The main result is now considered:

Theorem 11 *Let G be an infinite, locally finite graph and let n be a positive integer. G is not an $EAP - free - S^2(n)$ graph if and only if for $k = 0, ..., n$, $\exists\ J_1, ..., J_k$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$, and $\exists\ H$ without finite planar components is not $VAP - free - S^2(n - k)$ such that $H \cup J_1 \cup ... \cup J_k$ is a subgraph of G.*

Proof. By induction in n:

We denote $b = b(G) = n$ and suppose by induction that if $b(G) < n$ then the result is verified.

Let G be a non-$EAP - free - S^2(n)$ graph and we consider $G = G_1 \cup G_2$ where $b(G_i) = b_i < b$, $i = 1, 2$.

Case 1: We suppose that G_i is not a tiling graph of $S^2(b_i + 1)$,$i = 1, 2$. By induction:

$\exists\ J_1, ..., J_{k1}$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$ and $\exists\ H_1$ without finite planar components is not $VAP - free - S^2(b_1 - k_1)$ which implies $a(H_1) \geq b_1 - k_1$

$\exists\ J_{k1+1},...,J_{k_1+k_2}$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$ and $\exists\ H_2$ without finite planar components is not $VAP-free-S^2(b_2-k_2)$ which further implies $a(H_2) \geq b_2 - k_2$.

Therefore $J_1,...,J_{k_1+k_2}$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$ and

$a(H_1 \cup H_2) = a(H_1) + a(H_2) \geq (b_1 + b_2) - (k_1 + k_2) \geq b - (k_1 + k_2)$

which implies $H_1 \cup H_2$ is not $VAP - free - S^2(b - (k_1 + k_2))$.

Case 2: We suppose that G_2 is a tiling graph of $S^2(b_2 + 1)$. By induction:

$\exists\ J_1,...,J_{k_1}$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$ and $\exists\ H_1$ without finite planar components is not $VAP - free - S^2(b_1 - k_1)$ and since G_2 is a tiling graph of $S^2(b_2 + 1)$, therefore $\exists\ J_{k_1+1},...,J_{k_1+b_2+1}$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$.

Claim 12 *We prove that* $b \leq b_1 + b_2 + 1$.

We have an embedding of G_1 *with* (b_1+1) *ends and an embedding of* G_2 *with* (b_2+1) *ends, therefore we have an embedding of* $G = G_1 \cup G_2$ *with* (b_1+b_2+2) *ends. Hence,* G *is* $EAP-free-S^2(b_1+b_2+2)$. *Therefore* $b(G) = b < b_1+b_2+2$ *and also* $b \leq b_1 + b_2 + 1$.

If, in addition, we knew that G_i *were not a tiling graph of* $S^2(b_i+1)$, $i = 1,2$ *we would deduce that* $b \leq b_1 + b_2$.

By applying this logic to Case 2, we conclude that $\exists\ J_1,...,J_{k_1+b_2+1}$ are homeomorphic to $P(K_5)$ or to $P(K_{3,3})$ and $\exists\ H_1$, without finite planar components is not $VAP - free - S^2(b_1 - k_1)$.

Hence, $b_1 - k_1 = (b_1 + b_2 + 1) - (k_1 + b_2 + 1) \geq b - (k_1 + b_2 + 1)$ proves that H_1 is not $VAP - free - S^2(b - (k_1 + b_2 + 1))$. This completes the proof. ∎

As a continuation of this work we propose extending this theorem to any compact surface S. We are presently working on this conjecture and hope to present this work in the near future.

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