# CLASSIFICATION OF LARGE SETS BY TRADES

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ABSTRACT. In this paper, an algorithm based on *trades* is presented to classify two classes of large sets of $t$-designs, namely LS[14](2, 5, 10) and LS[6](3, 5, 12).

## INTRODUCTION

A $t$-$(v, k, \lambda)$ *design*, or briefly a $t$-*design*, is a pair $(X, \mathcal{A})$ which satisfies the following properties:

- $X$ is a set of $v$ elements called *points*.
- $\mathcal{A}$ is a family of subsets of $X$, each of cardinality $k$ called *blocks*.
- every $t$-subset of distinct points occurs in exactly $\lambda$ blocks.

A $t$-$(v, k, \lambda)$ design is called *simple* if it contains no repeated blocks. By an elementary counting argument, it can be shown that if $s < t$, a $t$-$(v, k, \lambda)$ design is also an $s$-$(v, k, \mu)$ design, where $\mu = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since $\mu$ must be an integer, this relation yields a necessary condition for the existence of $t$-designs, for any $s < t$. Given $t, k$ and $v$, there is a smallest positive integer $\lambda^*(t, k, v)$ such that these conditions are satisfied for all $0 \leq s < t$.
Let $\binom{X}{k}$ denote the set of all $k$-subsets of a $v$-set $X$. Let $\lambda = \lambda^*(t, k, v)$. A *large set* of $t$-$(v, k, \lambda)$ designs is a partition of $\binom{X}{k}$ into $t$-$(v, k, \lambda)$ designs. The number of designs in the partition is $N = \binom{v-t}{k-t}/\lambda$. We shall denote a large set of $t$-$(v, k, \lambda)$ designs by LS$[N](t, k, v)$. Note that all the designs in a large set are simple and we use the term "large set" only when $\lambda = \lambda^*(t, k, v)$. A $t$-$(v, k)$ *trade* $\mathbf{T} = \{T_1, T_2\}$ consists of two disjoint collections of blocks $T_1$ and $T_2$ such that for every $A \in \binom{X}{t}$, the number of blocks containing $A$ is the same in both $T_1$ and $T_2$. $T$ is called *simple* if there are no repeated blocks in $T_1$ ($T_2$). The number of blocks in $T_1$ ($T_2$) is called the *volume* of $T$ and is denoted by *vol*$(T)$. Clearly, two disjoint $t$-$(v, k, \lambda)$ designs form a trade with *vol*$(T) = \lambda\binom{v}{t}/\binom{k}{t}$. Trades can be defined alternatively from an algebraic point of view in which they constitute the kernel of some well-known (*inclusion*) matrices. The *standard basis* [6] for trades has been used to classify and construct some large sets and designs [1, 2, 3, 4]. Here, we exploit this structure to enumerate some classes of large sets.
Let $(X, \mathcal{A})$ be a $t$-$(v, k, \lambda)$ design and let $\pi$ be a permutation of $X$. If we let $\pi$ act on $(X, \mathcal{A})$, then we obtain an isomorphic copy of the design, which we denote by $(X, \mathcal{A}^\pi)$, where $\mathcal{A}^\pi = \{A^\pi : A \in \mathcal{A}\}$ and $A^\pi = \{x^\pi : x \in A\}$ for all $A \in \mathcal{A}$). Suppose $\mathbf{A} = \{(X, \mathcal{A}_i)\}_{i=1}^N$ is an LS$[N](t, k, v)$. Then define $\mathbf{A}^\pi = \{(X, \mathcal{A}_i^\pi)\}_{i=1}^N$. It is clear

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that $\mathbf{A}^{\pi}$ is also an LS$[N](t,k,v)$, and $\mathbf{A}^{\pi}$ is isomorphic to $\mathbf{A}$. In order to define the same concepts for trades and large sets, consider the set $\mathbf{A} = \{(X, \mathcal{A}_i)\}_{i=1}^{N}$, where $\mathcal{A}_i \subseteq \binom{X}{k}$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for all $i \neq j$. A permutation $\sigma \in Sym(X)$, the symmetric group on $X$, is said to be an *automorphism* of $\mathbf{A}$ if $\mathbf{A}^{\sigma} = \mathbf{A}$, that is, if $\mathcal{A}_i^{\sigma} \in \mathbf{A}$ for each $\mathcal{A}_i \in \mathbf{A}$. The set of all automorphisms of $\mathbf{A}$ is, of course, a subgroup of $Sym(X)$ denoted by $Aut\mathbf{A}$. If $G$ is a subgroup of $Aut\mathbf{A}$, we say that $G$ is an automorphism group of $\mathbf{A}$ or that $\mathbf{A}$ is *$G$-invariant*. Note that a $G$-invariant large set (trade) may contain designs which are not $G$-invariant themselves. Furthermore, for an arbitrary permutation $\sigma$ of $X$, $\mathbf{A}^{\sigma}$ is not, in general $G$-invariant. However, if $\sigma \in N(G)$, i.e. the normalizer of $G$ in $Sym(X)$, then $\mathbf{A}^{\sigma}$ is $G$-invariant. This observation is important in determining the isomorphism of $G$-invariant large sets.

In [7], some classes of large sets of t-designs are considered. Certain classes are completely enumerated, while in some cases only the existence question is tackled. Here, we employ trades to complete the classification of two such parameter sets, namely $G$-invariant LS[14]$(2,5,10)$ and LS[6]$(3,5,12)$, where $G = <(1)(2)(3)(4,\ldots,10)>$ and $G = <(1)(2)(3,\ldots,7)(8,\ldots,12)>$, respectively.

## LARGE SETS OF 2-(10, 5, 4) DESIGNS

There are exactly 21 non-isomorphic 2-$(10,5,4)$ designs [10]. For completeness, we list the number of these designs:

| #Designs | \|Aut\| |
|---|---|
| 4 | 1 |
| 5 | 2 |
| 1 | 4 |
| 2 | 6 |
| 3 | 8 |
| 2 | 9 |
| 3 | 16 |
| 1 | 72 |
| total: 21 | |

Let $\mathbf{A} = \{(X, \mathcal{A}_i)\}_{i=1}^{14}$ be a $G$-invariant LS[14]$(2,5,10)$, where $G = <\sigma>$. Clearly, we can take a suitable power of $\sigma$ of prime order $p$. If $\sigma$ consists of $m$ cycles of length $p$ and $n$ fixed points, we say that $\sigma$ is of the type $1^n p^m$. Furthermore, if $\mathcal{A}_i^{\sigma} \neq \mathcal{A}_i$ for some $1 \leq i \leq 14$, then $\mathbf{A}$ contains also $\mathcal{A}_i^{\sigma^j}, j = 1, \ldots, (p-1)$. In what follows, we take $\sigma$ as $1^3 7^1$. The non-existence of a 2-$(10,5,4)$ design admitting an automorphism of order 7 shows that a large set is comprised of exactly two orbits of designs under $G$. In [7], an instance of such a large set is provided but it is noted that the complete classification would be quite time-consuming. Hence, we propose the following algorithm to construct all large sets of the form $(\mathcal{A}_1, \mathcal{A}_1^{\sigma}, \ldots, \mathcal{A}_1^{\sigma^6})(\mathcal{A}_2, \mathcal{A}_2^{\sigma}, \ldots, \mathcal{A}_2^{\sigma^6})$ : take $\mathcal{A}_1$ to be any of twenty-one 2-$(10,5,4)$ designs. Now determine all permutations $\pi$ of type $1^3 7^1$ such that $\mathcal{A}_1, \mathcal{A}_1^{\pi}, \ldots \mathcal{A}_1^{\pi^6}$ are disjoint. Relabel the points so that the designs are disjoint under the action of the same permutation, say $\sigma$. This procedure produces 9980 sets of 7 disjoint isomorphic designs. Removing orbits of these designs from consideration, we are left with 126 blocks from among which we need to construct 7 more disjoint, isomorphic under $\sigma$, 2-$(10,5,4)$ designs to complete the large set. To do so, we notice that $\mathcal{A}_2$ and $\mathcal{A}_2^\sigma$ form a 2-$(10,5)$ trade $\mathbf{T} = \{T_1, T_2\}$ with $vol(T) = 18$ such that $T_1$ is a design, $T_1^\sigma = T_2$, and for each block $A \in T_1$, we have $A^{\sigma^2}, \ldots, A^{\sigma^6} \notin \mathbf{T}$. Hence, we can implement backtracking on the standard basis of trades to produce such trades. Implementing this procedure on a 2000MHz Pentium IV PC running a C program, we produced (in a few hours) 3,022 large sets.

## Isomorphism test

The isomorphism test is carried out in two phases. In the first phase, we identify isomorphism classes as follows. Let $\mathbf{A} = \{(X, \mathcal{A}_i)\}_{i=1}^{14}$ and $\mathbf{A}' = \{(X, \mathcal{A}_i')\}_{i=1}^{14}$ be two of the above solutions with $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_1', \mathcal{A}_2'$ as starter designs, respectively. If $\mathcal{A}_1$ ($\mathcal{A}_2$) is not isomorphic to either of $\mathcal{A}_1'$ and $\mathcal{A}_2'$, then $\mathbf{A}$ and $\mathbf{A}'$ are in different isomorphism classes and no further design in them should be checked. Implementing this process using *nauty* [8], 109 isomorphism classes are obtained. Clearly, having isomorphic starter designs does not guarantee isomorphism of two large sets within each class and there might still be non-isomorphic copies present. Hence, in the second phase, we employ backtracking to check for isomorphisms between large sets of the same class. Let $X_i = \{1, \ldots, i\}$, for $k \le i \le v$. Let $\Pi$ be the set of all one-one functions from $X_i$ to $X_v$ for all $k \le i \le v$. We consider $\Pi$ to be lexicographically ordered. If $\pi \in \Pi$ is defined on $X_i$, we write $\pi_i$ to show its domain. We search $\Pi$ for possible permutation $\pi_v$ such that $\pi_v \mathbf{A}_1 = \mathbf{A}_2$: if there is a $k$-subset $B$ of $X_i$ containing $i$ such that $B \in \mathcal{A}_j$ and $\pi B \in \mathcal{A}_l'$, then in all subsequent extensions of $\pi_i$, the same relation should hold between this two designs of $\mathbf{A}$ and $\mathbf{A}'$ or else we backtrack and skip to the next element $\pi_m$ of $\Pi$ with $m \le i$. Note that the same procedure can be applied to determine the number of automorphisms of the final solutions.

**Theorem 1.** *Let* $G =< (1)(2)(3)(456789A) >$. *Then there are exactly* 410 *non-isomorphic* $G$*-invariant* $LS[14](2,5,10)$.

| #Large Sets | \|Aut\| |
|---|---|
| 366 | 7 |
| 30 | 14 |
| 12 | 21 |
| 2 | 56 |
| Total: 410 | |

The starter designs of the two large sets with 56 automorphisms can be obtained as follows. For the first large set, take the following design as $\mathcal{A}_1$:

| | | | | | |
|---|---|---|---|---|---|
| 12369 | 12457 | 12468 | 1249A | 13457 | 1356A |
| 13589 | 1678A | 1789A | 2348A | 2358A | 23679 |
| 25678 | 2579A | 3467A | 34789 | 45689 | 4569A |

Now let the permutation $< (185476A)(23) >$ act on its blocks to get $\mathcal{A}_2$. Applying $< (1968A7)(23) >$ and $< (1A)(23)(59768) >$ on the above design produce the starter designs of the second large set.

## LARGE SETS OF 3-(12, 5, 6) DESIGNS

In [7], an instance of an LS[6](3, 5, 12) is given, where one design is fixed by $\rho = (12345)(6789A)BC$ and the others cycle through an orbit of size five. The starter design for this orbit is chosen to be invariant under $\sigma = (1)(23456789ABC)$. We can have total enumeration for this class of large sets as well. We first obtain a catalogue of all 3-(12, 5, 6) designs invariant under the action of $\sigma$. For this, we employ the algorithm in [5, 9] and get the following result.

**Theorem 2.** *Up to isomorphism, the number of* 3-(12, 5, 6) *designs, invariant under* $G =< (1)(23456789ABC) >$ *is as follows:*

| #Designs | \|Aut\| |
|---|---|
| 162 | 11 |
| 1 | 22 |
| 2 | 55 |
| 1 | 110 |
| 1 | 1320 |
| 1 | 7920 |
| total: 168 | |

Now, to construct the orbit of five designs in the large set, we take the starter design $\mathcal{A}_1$ to be any of these 168 designs and backtrack on the standard basis of trades to produce all 3-(12, 5) trades of volume 132 having $\mathcal{A}_1$ as one part and the other part isomorphic to $\mathcal{A}_1$ under an isomorphism of order 5. There is also an alternative approach which can be used to check the results. First, for each starter design $\mathcal{A}_1$, determine all permutations $\pi$ of type $1^2 5^2$ such that $\mathcal{A}_1, \mathcal{A}_1^{\pi}, \ldots \mathcal{A}_1^{\pi^5}$ are disjoint. Only three of the 168 designs accept such permutations. Relabel the points so that the designs are disjoint under the action of $\sigma$. This procedure produces 33 orbits of designs. Clearly, the unused orbits form a design invariant under $\sigma$. Furthermore, the large sets arising from the three classes of starter designs can not be isomorphic. Hence, we determine isomorphism within each class as in the previous section. The results are as follows.

**Theorem 3.** *Let* $G =< (1)(2)(34567)(89ABC) >$. *Then there are exactly* 3 *non-isomorphic G-invariant* $LS[6](3, 5, 12)$ *whose starter design is invariant under the action of* $Z_{11}$. *The large sets admit* 55 *automorphisms each.*

The orbit representatives of the starter designs of these large sets are listed below. The point set is $X = \{1, \ldots, 9, A, \ldots, C\}$ and the automorphism group of the designs is $G =< (1)(2384769AC5B) >$.

**Design #1**

| | | | | | |
|---|---|---|---|---|---|
| 12348 | 1234A | 1236C | 1239C | 12456 | 23459 |
| 2346A | 2347B | 234BC | 23568 | 23569 | 23679 |

**Design #2**

| | | | | | |
|---|---|---|---|---|---|
| 12348 | 12379 | 1237A | 1239C | 1246A | 23467 |
| 2346C | 2347B | 2348A | 2349A | 2356A | 23579 |

**Design #3**

| | | | | | |
|---|---|---|---|---|---|
| 12348 | 12379 | 1237A | 1239C | 1246A | 2345C |
| 23467 | 2346C | 2348A | 2356A | 2357A | 2367B |

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