Large Chordal Rings for Given Diameter and Uniqueness Property of Minima

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June 13, 2005

Abstract

This paper discusses the covering property and the Uniqueness Property of Minima (UPM) for linear forms in an arbitrary number of variables, with emphasis on the case of three variables (triple loop graph). It also studies the diameter of some families of undirected chordal ring graphs. We focus upon maximizing the number of vertices in the graph for given diameter and degree. We study the result in [2], we find that the family of triple loop graphs of the form $G(4k^2+2k+1;1;2k+1;2k^2)$ has a larger number of nodes for diameter k than the family $G(3k^2+3k+1;1;3k+1;3k+2)$ given in [2]. Moreover we show that both families have the Uniqueness Property of Minima.

Keywords: Chordal Ring , Linear Congruences, Shortest Path

1 Introduction

The communication performance between computing nodes of a parallel system is of essential importance because data exchange may become a bottleneck in parallel computation. The communication performance depends

on the underlying network topology, the chosen routing model, the communication pattern, the quality criteria, and the routing algorithm.
The topology is the physical interconnection structure of the network. It has an impact on several properties of the network, such as diameter, node degrees, bisection bandwidth, connectivity, and scalability. Since each of these properties bears an importance of its own, many theoretical and practical studies produced near optimal topologies in a global sense rather than optimized for a particular property.
Some of these topologies are mesh, hypercube, butterfly, shuffle-exchange, etc. . There is a common consensus that networks with simpler topologies will offer practical solutions to the problem of interconnecting a very large number of computing nodes. In addition to various kinds of meshes etc. as above, these topologies include $k-$loop networks (called triple-loop networks when $k = 3$). Informally, a $k-$loop network contains k interleaved rings. Among the properties of $k-$loop networks are scalability, fixed node degree, node symmetry (i.e., vertex transitivity), regularity,reasonable diameter, and reliability. For these reasons, loop networks have been studied in papers such as [2],[3],[4],[5],[6], where researchers studied loop-graphs intensively. In the present paper we feature on some triple loop networks.

2 The Uniqueness Property for Minima of Linear Diophantine Equations

Chordal rings are related to a distributed loop graph $G(n; c_1; c_2; \cdots ; c_d)$ which is a graph with a vertex set equal to $0, \cdots , n-1$, and the edge set equal to $(u, u \pm i)$ where $0 < u < n-1, i \in \{c_1; c_2; \cdots ; c_d\}$. We denote by $G(n; c_1; c_2)$, the chordal ring defined by n as the number of nodes and by c_1, c_2 the length of the chords. Assume that $c_1,, c_2, \ldots, c_d$ are d positive integers, with $gcd(c_1, c_2, \ldots, c_d) = 1$. Let $x_1, x_2, \ldots, x_d$ be d variables taking integer values, and let n be a fixed modulus. We consider the linear form

$$f = c_1 x_1 + c_2 x_2 + \cdots + c_d x_d \ . \tag{1}$$

(We also note as $\langle c_1, c_2, \ldots, c_d \rangle$) Now let $0 \leq w < n$ be any integer. We inquire about the solutions of the linear congruence

$$c_1 x_1 + c_2 x_2 + \cdots + c_d x_d \ \equiv \ w \mod n \tag{2}$$

and we ask whether among all such solutions the minimum of the L^1-norm

$$v \ = \ | x_1 | + | x_2 | + \cdots + | x_d | \tag{3}$$

is unique. If this is the case for all the residues modulo n, that is for all w with $0 \leq w \leq n-1$, then we say that f in (1) has the uniqueness property for minima (UPM) with respect to the modulus n.

In terms of multiple loop graphs, we may reinterpret UPM as the uniqueness of the shortest path between any two nodes in a topological sense.

We note that a necessary condition for the UPM is that all the c_i are distinct. Hence we may assume without loss of generality that $1 \leq c_1 < c_2 < \cdots < c_d$ holds. The UPM for binary linear forms $c_1x_1 + c_2x_2$ has been discussed in [1], and there it was shown that for given positive integers c_1, c_2 with $gcd(c_1, c_2) = 1$ there are only finitely many moduli n with UPM if $c_1 \equiv c_2 \mod 2$, while there are infinitely many moduli n with UPM if $c_1 \not\equiv c_2 \mod n$.

For the case of general d we first give an example that shows that for any d, there exist positive integers $c_1 < c_2 < \cdots < c_d$ which have the UPM for some modulus n.

Example 1 Consider the values $c_k = k$ for $k = 1, 2, \ldots, d$. The associated linear form

$$f = x_1 + 2x_2 + 3x_3 + \cdots + dx_d \tag{4}$$

has the UPM for the modulus $n = 2d + 1$.

This is easy since for the residue 0 we have that v in (3) is $v = 0$, and the residues $\pm 1, \pm 2, \ldots, \pm d$ are the only values for which we have $v = 1$.

This example is a trivial one in the sense that it only involves residues w with the property $v \leq 1$. From [2] we get examples in $d = 3$ variables of a less trivial kind.

Example 2 The linear form

$$f = 4x_1 + 6x_2 + 9x_3 \tag{5}$$

has the UPM for the modulus $n = 19$.

In order to verify examples of this kind it is convenient to use the following lemmas. It is trivial that the minimum is always unique for the residue $w = 0$.

Lemma 1 Assume that $c_1, c_2, \dots c_d$ (not necessarily distinct) and n is fixed. The residue $w \mod n$ has r distinct d–tuples attaining the minimum v iff the residue $-w \mod n$ has r distinct d–tuples attaining the minimum v.

This is clear by replacing each such d–tuple $(x_1, x_2, \dots, x_d)$ by its negative $(-x_1, -x_2, \dots, -x_d)$.

Lemma 2 Assume that $c_1, c_2, \dots c_d$ is given, and that n is even. The residue $\frac{n}{2} \mod n$ has at least two distinct d–tuples attaining the minimum v.

This is clear since for any solution $(x_1, x_2, \dots, x_d)$ attaining the minimum v the d–tuple $(-x_1, -x_2, \dots, -x_d)$ is another distinct solution.

We now give the verification of example 2. By lemma 1 we only need to verify that the minimum is unique for $1 \leq w \leq \frac{n-1}{2}$. Note that by lemma 2 n is necessarily an odd integer. We let $\rho(w)$ be the number of distinct representations, which in this case is always equal to 1, and we use the other parameters as above.

w	$\rho(w)$	v	(x_1, x_2, x_3)
1	1	2	(0,0,-2)
2	1	2	(-1,1,0)
3	1	2	(0,-1,1)
4	1	1	(1,0,0)
5	1	2	(-1,0,1)
6	1	1	(0,1,0)
7	1	2	(0,-2,0)
8	1	2	(2,0,0)
9	1	1	(0,0,1)

We can then recast the general construction of [2] in terms of the UPM as follows.

Proposition 1 For each positive integer k the following 3–variable linear form

$$f = k^2 x_1 + k(k+1)x_2 + (k+1)^2 x_3 \tag{6}$$

has the UPM for the modulus $n = 3k^2 + 3k + 1$.

The proof depends on a particular property of invariant subsets S in cubical lattices.

Lemma 3 Assume that $S \subset \mathbb{Z}^3$ is a non empty set that satisfies

$$(x, y, z) \in S \Leftrightarrow (x+1, y+1, z+1) \in S \quad . \tag{7}$$

The function $v = |x| + |y| + |z|$ takes its minimum on the set S in a point that has at least one coordinate equal to zero.

Consider a point $(x, y, z) \in S$ with all three coordinates not equal to zero. Then either at least two coordinates are positive, or two coordinates are negative. In the first case we have that $(x-1, y-1, z-1) \in S$ with

$$|x-1| + |y-1| + |z-1| \leq |x| + |y| + |z| - 1,$$

and in the second case we have that $(x+1, y+1, z+1) \in S$ with

$$|x+1| + |y+1| + |z+1| \leq |x| + |y| + |z| - 1.$$

In either case the minimum of $v = |x|+|y|+|z|$ is not taken on at $(x, y, z) \in S$. This proves lemma 3.

Lemma 4 Assume that $S \subset \mathbb{Z}^3$ is a non empty set that satisfies

$$(x, y, z) \in S \Leftrightarrow (x+1, y+1, z+1) \in S \quad . \tag{8}$$

If the function $v = |x| + |y| + |z|$ takes its minimum on the set S in a point that has precisely one coordinate equal to zero, then the two non zero coordinates of that point have an opposite sign.

Without loss of generality we may consider a point $(x, y, 0) \in S$ with $x, y \neq 0$. If both $x, y > 0$ are positive, then we have $(x-1, y-1, -1) \in S$ with $z = 0$ so that

$$|x-1| + |y-1| + |z-1| = |x| + |y| + |z| - 1,$$

and if both $x, y < 0$ are negative, then we have that $(x+1, y+1, 1) \in S$ so that (with $z = 0$)

$$|x+1| + |y+1| + |z+1| = |x| + |y| + |z| - 1.$$

In either case the minimum of $v = |x|+|y|+|z|$ is not taken on at $(x, y, 0) \in S$. This proves lemma 4.

The minima of the linear form 6 decompose as follows:
First into $6 \cdot k$ minima which are situated along the three coordinate axes, which are of the form $(x, 0, 0), (0, x, 0), (0, 0, x)$ with $1 \leq |x| \leq k$, and hence

they are unique by lemma 4. Further they decompose into $6 \cdot \frac{k^2-k}{2}$ minima which are situated in the three coordinate planes, of the form $(x,y,0),(x,0,y),(0,x,y)$ where x,y are non zero and of opposite signs. These minima are also unique by lemma 4. This completes the proof of proposition 1.

3 Equivalence of Linear Forms

We say that two linear forms $\langle c_1,c_2,c_3\rangle$ and $\langle c_1',c_2',c_3'\rangle$ with respect to the same modulus n are equivalent if there exists an integer c with $gcd(c,n)=1$ such that the three congruences

$$a_1' \equiv \pm c\cdot c_1 \mod n, \quad c_2' \equiv \pm c\cdot c_2 \mod n, \quad c_3' \equiv \pm c\cdot c_3 \mod n \tag{9}$$

are true for some choice of plus and minus signs. Note that it is not assumed that $c_1' < c_2' < c_3' < \frac{n}{2}$ should hold. After changing signs as necessary and rearranging according to size we obtain another (equivalent) form $\langle c_1'',c_2'',c_3''\rangle$ with $c_1'' < c_2'' < c_3'' < \frac{n}{2}$

Example 3 The two forms $\langle 1,7,8\rangle$ and $\langle 4,6,9\rangle$ for the modulus $n=19$ are equivalent. Indeed we choose $c=4$ and we get

$$4=4\cdot 1,\ 9\equiv(4\cdot 7) \mod 19,\ 6\equiv-(4\cdot 8) \mod 19 \quad .$$

Similarly it follows that $\langle 1,3k+1,3k+2\rangle$ and $\langle k^2,k(k+1),(k+1)^2\rangle$ are equivalent modulo $n=3k^2+3k+1$, see [2]

It is known [4] and not difficult to prove that if the linear forms are equivalent, then the corresponding triple loop graphs are isomorphic. In practice we first refer to the modulus, and then check additionally if two different forms are equivalent.

Lemma 5 The two 3−variable form

$$\langle 1,2k+1,2k^2\rangle \text{ modulo } n=4k^2+2k+1 \tag{10}$$

$$\langle 1,2k,2k^2+k\rangle \quad \text{modulo } n=4k^2+2k+1 \tag{11}$$

are equivalent.

This is easily seen by first observing that $2k$ is a unit modulo $4k^2+2k+1$, as $(2k)\cdot(-2k-1)\equiv 1 \mod 4k^2+2k+1$. Thus we may multiply the form (10) by $2k$ and obtain the equivalences modulo $4k^2+2k+1$ as follows:

$$\begin{aligned} \langle 1, 2k+1, 2k^2\rangle &\equiv \langle 2k, 4k^2+2k, 4k^3\rangle \\ &\equiv \langle 2k, -1, -2k^2-k\rangle \\ &\equiv \langle -1, 2k, -2k^2-k\rangle \\ &\equiv \langle 1, 2k, 2k^2+k\rangle \quad . \end{aligned} \tag{12}$$

4 Proof of the Covering Property of the form $\langle 1, 2k+1, 2k^2\rangle$ modulo $4k^2+2k+1$

In this and the next section we consider the linear form

$$x_1 + (2k+1)x_2 + 2k^2x_3 \quad \text{modulo} \quad 4k^2+2k+1.$$

We say that a form $f = c_1x_1 + c_2x_2 + c_3x_3$ has a *coveringproperty* if for all $0 < w < n$, $w \equiv c_1x_1 + c_2x_2 + c_3x_3$ *mod* n, $|x1| + |x2| + |x3| \leq k$. In other words the chordal ring $G(n; c_1; c_2; c_3)$ has diamter k.

Proposition 2 For each positive integer k the following 3−variable linear form $f = x_1 + (2k+1)x_2 + 2k^2x_3$ has the covering property for the modulus $n = 4k^2+2k+1$. Hence for a fixed diameter k the chordal ring $G(4k^2+2k+1; 1; 2k+1; 2k^2)$ has a larger number of nodes than the corresponding chordal ring of proposition 1.

By using lemma 5 in order to prove the covering property we only have to show that each residue w mod $4k^2+2k+1$ with $1 \leq w \leq 2k^2+k$ is represented in the form

$$w \equiv x_1 + x_2\cdot 2k + x_3\cdot(2k^2+k) \mod 4k^2+2k+1 \tag{13}$$

under the condition $|x_1| + |x_2| + |x_3| \leq k$. We first show that the multiples of k are represented in this way, distiguishing even (part (a)) and odd (part (b)) multiples in the following lemma.

Lemma 6 (a) The numbers $2ki$ with $1 \leq i \leq k$ are represented with $(x_1, x_2, x_3) = (0, i, 0)$ so that $|x_1| + |x_2| + |x_3| = i$.
(b) The numbers $(2j+1)k$ with $1 \leq j \leq k$ are represented with $(x_1, x_2, x_3) = (0, -k+j, 1)$ so that $|x_1| + |x_2| + |x_3| \leq k$.

Next we note that from part (a) of the lemma we get that each integer in the integer interval

$$[2ik - (k - i), 2ik + (k - i)] = [(2i - 1)k + i, (2i - 1)k - i] \tag{14}$$

which is centered at $2ki$ is represented in the form $w \equiv x_1 + x_2 \cdot 2k$ modulo $4k^2 + 2k + 1$ so that $|x_1| + |x_2| \le (k - i) + i = k$.
Next we note that from part (b) of the lemma,with the substitution $j' = k-j$ we get that each integer in the interval

$$[2k^2 + k - 2kj' - (k - j' - 1), 2k^2 + k - 2kj' + (k - j' - 1)] \tag{15}$$

which is centered at $2k^2 + k - 2kj'$ is also represented in the form $w \equiv x_1 + x_2 \cdot 2k + 1 \cdot 2k^2 + k$ modulo $4k^2 + 2k + 1$, $x_1 = j - 1, x_2 = -k + j$ and $x_3 = 1$ so that

$$|x_1| + |x_2| + |x_3| \le (j - 1) + (k - j) + 1 \le k.$$

By considering the union of the intervals of the type (14) and (15) we see that the only numbers not yet covered by a congruence of the required form are the $k - 1$ integers $(2i + 1)k + i$ with $1 \le i \le k - 1$. They are represented with $x_3 = -1$ in the following way.

$$w = (2i + 1)k + i \equiv (i - 1) - (k - 1) \cdot 2k - (2k^2 + k) \tag{16}$$

i.e. with $(x_1, x_2, x_3) = (i - 1, -(k - 1), -1)$ so that also $|x_1| + |x_2| + |x_3| = (i - 1) + 1 + 1 = i + 1 \le k$.
This completes the proof that each integer w with $1 \le w \le 2k^2 + k$ is represented in the required form

$$w \equiv x_1 + x_2 \cdot 2k + x_3 \cdot (2k^2 + k),$$

satisfying $|x_1| + |x_2| + |x_3| \le k$, and indeed the representation can be accomplished with $|x_3| \le 1$. Using lemma 1 we then complete the proof of proposition 2

5 The form $\langle 1, 2k + 1, 2k^2 \rangle$ modulo $4k^2 + 2k + 1$ has UPM.

Proposition 3 For each positive integer k the following 3−variable linear form $f = x_1 + (2k + 1)x_2 + 2k^2x_3$ has the UPM for the modulus $n = 4k^2 + 2k + 1$.

Lemma 7 Each integer z in the interval $[-2k^2 - k, +2k^2 + k]$ can be represented uniquely in the form

$$z = x_1 + x_2 \cdot (2k + 1) + x_3 \cdot 2k^2 \tag{17}$$

with

$$| x_1 | + | x_2 | + | x_3 | \leq k \quad \text{and} \quad | x_3 | \leq 1 \tag{18}$$

where for $x_3 = +1$ we have $x_2 \leq 0$ and for $x_3 = -1$ we have $x_2 \geq 0$.

We first note that there are $2k^2 + 2k + 1$ pairs of integers (x_1, x_2) with $| x_1 | + | x_2 | \leq k$. Hence there are $2k^2 - 2k + 1$ integers with $| x_1 | + | x_2 | \leq k - 1$. There are $2k - 1$ integers x_1 with $-k + 1 \leq x_1 \leq k - 1$. Thus if we assume (18) then we obtain a total of

$$2k^2 + 2k + 1 + 2k^2 - 2k + 1 + 2k - 1 = 4k^2 + 2k + 1$$

integers, which is the total number of integers in the following integer interval: $[-2k^2 - k, +2k^2 + k]$.
On the other hand, from proposition 2 it follows that each of the $4k^2 + 2k + 1$ residue classes can be represented by the form 17. Therefore none of the $4k^2 + 2k + 1$ integers occurs more than once with any given residue class, which means that the representation by these integers is unique. It is obvious that each class is represented by its minimum, and therefore UPM holds.

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