NESTED PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN

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ABSTRACT

In this note we construct nested partially balanced incomplete block designs based on NC_m-scheme. Secondly we construct NPBIB designs from a given PBIB design with $\lambda_1 = 1$ and $\lambda_2 = 0$ with same association scheme for both systems of PBIB designs. Finally, we give some results and examples where the two systems of PBIB designs in NPBIB designs have different association schemes.

Keywords & Phrases: Partially balanced incomplete block designs, Nested design, Association schemes.

INTRODUCTION:

In incomplete block designs there may exist natural grouping of blocks into sets of blocks, which can be used to reduce the inter-block variations. Preece [5] introduced nested balanced block design (NBIB) designs in this context first. Later Rees [7] and Robinson [8] proposed certain nested split plot designs. However these designs are special cases of nested PBIB (NPBIB) designs introduced by Homel and Robinson [3] with few exceptions. They have given some methods of constructions of NPBIB (m) design viz by Kronecker products, method of differences and geometric constructions along with its analysis. Recently, some results on NPBIB design are also due to Banerjee and Kageyama [1], Philip et. al.,[4] and Satpati and Parsad [10]. The design is defined as below:

The arrangement of v treatments in b_2 sub-blocks (second system of blocks) of size k_2, nested in $b_1 = b_2/\mu$ blocks (first system of blocks) each of size $k_1 = \mu k_2$, μ an integer such that

(i) each treatment occurs r times in the design and no treatment occurs more than once in a block

(ii) ignoring second system of blocking, the first system of b_1 blocks of size k_1 form a PBIB (m) design with any two treatments which are t-th associates appear together λ_{1t}, t = 1, 2, …, m times, assuming existence of an association scheme.

(iii) ignoring first system of blocking, the second system of b_2 blocks of size k_2 form a PBIB (m) design with any two treatments which are t-th associates appear together in λ_{2t} blocks, t = 1, 2, …, m, assuming existence of an association scheme.

(iv) the two association schemes both on same treatments may or may not be same.

Originally as defined by Homel and Robinson [3] the two association schemes were same. Hence the parameters of NPBIB (m) design are:

$$r;\ v;\ b_1, k_1 = \mu k_2,\ \lambda_{1t}; b_2 = \mu b_1, k_2, \lambda_{2t};\ \mu; n_t, P_t = \left(p_{jk}^t\right), t, j, k = 1, 2, \cdots, m$$

On para one page two, of Homel and Robinson [3], the authors admit "the authors have not been able to show that any PBIB design nested in another PBIB design must have an association scheme in common with it".

The authors present in this paper by means of some results and some examples that PBIB designs of two systems in NPBIB designs having different association schemes. Some such results are also due to Banerjee and Kageyama [1] and Philip et.al. [4].

Next, we have constructed NPBIB (m) designs based on NC_m-scheme of Saha et. al. [9] with $\mu = 2$ and a single initial block for the first system. These designs have smaller number of blocks and block size and therefore are more suitable. The designs of Theorem 2 practically, cannot be obtained by the methods given in Homel and Robinson [3].

We have also constructed NPBIB (2) designs from given PBIB (2) designs with $\lambda_1 = 1$ and $\lambda_2 = 0$. Such PBIB (2) designs based on Group Divisible, triangular association scheme, Cyclic and L_2-association schemes are available in the literature (Raghavarao [6]).

Homel and Robinson [3] (p 204, para 2) mention that one can construct nested PBIB design of m associate classes (denoted as NPBIB (m)) when $v = mk+1 \equiv p^n$, 'p' a prime and 'n' a positive integer. Referring Wilson [11], they say that when one considers any subset B of the multiplicative group of GF (v) then the initial sets $\{x^{mu} B: u = 0, 1, 2, \ldots, k-1\}$ forms a PBIB (m) design. If B can be spilt into sub-blocks-sets then these considered as initial blocks form a PBIB (m) design, which is nested in the previous one. However these NPBIB designs have large number of blocks or association scheme or large block size and hence become practically unsuitable.

MAIN RESULTS :

At first we are presenting below the NPBIB designs from single initial block for the first system.

A. NPBIB designs with same association scheme for both systems:

THEOREM 1: When $v = sm+1 = p^n$, p a prime, n is a positive integer, then we can always construct a NPBIB (m) design based on NC_m-scheme with parameters: $r = 2s$; v; $b_1 = v$, $k_1 = 2s$, $n_t = s$, $P_t = (p_{ij}^t)$, $\lambda_{1t} = p_{ii}^t + p_{jj}^t + p_{ij}^t$, i, t , j = 1,2, ...m; $b_2 = 2v$, $k_2 = s$, $\lambda_{2t} = p_{ii}^t + p_{jj}^t$, $i \neq j$, t = 1, 2, ..., m, any pair of cosets C_i and C_j, with same scheme for both the systems, by developing the initial block

$I = \{(C_i), (C_j)\}$, $i \neq j = 1,2, \ldots, m$, where cosets $C_i = x^{i-1} C_1 \equiv x^{i-1} \{ x^{qm} \mid 0 \leq q \leq s-1\}$ and x is a primitive element of GF(v).
The contents in parentheses represent blocks of second system.

EXAMPLE 1.1: Let s = 2 and m = 3. The cosets are:
$C_1 = \{1,6\}$, $C_2 = \{3,4\}$, $C_3 = \{2,5\}$.

The initial block I = {(1,6), (3,4)}, when developed mod 7 yields a NPBIB design with parameters: $r = 4$; $v = 7$; $b_1 = 7$, $k_1 = 4$, $\lambda_{11} = 3$, $\lambda_{12} = 2$, $\lambda_{13} = 3$; $b_2 = 14$, $k_2 = 2$; $\mu = 2$, $\lambda_{21} = 1$, $\lambda_{22} = 0$, $\lambda_{23} = 1$, $n_1 = n_2 = n_3 = 2$,

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

and blocks of the design are: [(1,6), (3,4)], [(2,0), (4,5)], [(3,1), (5,6)], [(4,2), (6,0)], [(5,3), (0,1)], [(6,4), (1,2)], [(0,5), (2,3)].

Next we construct NPBIB design from existing PBIB designs.

THEOREM 2: Existence of a PBIB (2) design D with parameters: v, b, r, k, n_1, n_2, $\lambda_1 = 1$, $\lambda_2 = 0$, p^i_{jk}, $1 \leq i, j, k, \leq 2$, implies the existence of NPBIB (2) design $D^\#$ with same association scheme and remaining parameters:

$$v^\# = b_1^\# = v,\ r^\# = r(k-1) = k_1^\#,\ \lambda_{11}^\# = p_{11}^1,\ \lambda_{12}^\# = p_{11}^2,$$
$$b_2^\# = vr,\ k_2^\# = k-1, \mu = r, \lambda_{21}^\# = k-2,\ \lambda_{22}^\# = 0.$$

PROOF: Construct blocks $B_{\theta t} = B_t - \{\theta\}$, $1 \leq t \leq r$ where B_t's are the blocks of D containing θ. Since $\lambda_1 = 1$, $B_{\theta t}$'s are all disjoint. Consider blocks $B_\theta = \bigcup_{t=1}^{r} B_{\theta t}$, $1 \leq \theta \leq v$. We take B_θ, $1 \leq \theta \leq v$ as first system of blocks and $B_{\theta t}$, $1 \leq t \leq r$ for each θ, as second system of blocks of $D^\#$. Thus $v^\# = b_1^\# = v, k_1^\# = r(k-1), b_2^\# = vr, k_2^\# = k-1, \mu = r$ in $D^\#$.

A block $B_{\theta t}$, $1 \leq t \leq r$, yields k-1 blocks B_ϕ, with respect to (k-1) treatments $\phi \varepsilon B_{\theta t}$. The treatment θ belongs to all these r(k-1) blocks B_ϕ. Hence $r^\# = r(k-1)$.

Now let us consider (θ, ϕ) two first associates in D i.e. θ and ϕ appear together in exactly one block of D. Obviously in the second system of blocks θ and ϕ shall be occurring together in the blocks $B_{\psi t}$, $1 \leq t \leq r$ where ψ too occurs in that one block in which θ and ϕ are occurring and in other r-1 blocks.

Since besides θ and ϕ there are k-2 other treatments ψ appearing in a block of D, so pair (θ, ϕ) appears in exactly k-2 blocks of $B_{\psi t}$ of second system of $D^\#$ as ψ changes. Thus $\lambda_{21}^\# = k-2$. Again any pair (θ, ξ) of treatments which are second associate in D do not occur together in any block of second system, since $\lambda_2 = 0$. Therefore $\lambda^{\#}_{22} = 0$. Thus parameters of second system are proved. Hence the association scheme of this system $D^{\#}$ is same as that of D.

Consider now the first system of blocks of $D^{\#}$ each of which is a collection of r blocks of second system. Consider two first associates θ and ϕ in D, they appear in exactly one block of D. The pair θ and ϕ occur together in a block B_ψ (may be in same or different sub-blocks $B_{\psi t}$) if ψ is a common first associate of θ and ϕ. Hence $\lambda^{\#}_{11} = p^1_{11}$. With similar arguments for any pair (θ, ξ) which are second associates in D occur together in the first system of blocks of $D^{\#}$ as many as $\lambda^{\#}_{12} = p^2_{11}$ blocks. This proves that the PBIB design with first system of blocks of $D^{\#}$ has same association scheme and also the other parameters of the PBIB design of first system are established.

This completes the proof.

NOTE 1: In case of Group Divisible-PBIB designs the resultant design will be disconnected.

EXAMPLE 2.1: Consider a PBIB design D based on triangular association scheme with parameters: $v = b = 10$, $r = k = 3$, $n_1 = 6$, $n_2 = 3$, $\lambda_1 = 1$, $\lambda_2 = 0$, $p^1_{11} = 3$, $p^2_{11} = 4$ and blocks as: (1, 2, 5), (1, 3, 6), (1, 4, 7), (2, 3, 8), (2, 4, 9), (3, 4, 10), (5, 6, 8), (5, 7, 9), (6, 7, 10) and (8, 9, 10).
The NPBIB design $D^{\#}$ by Theorem 3 has blocks:
{(2, 5), (3, 6), (4, 7)}, {(1, 5), (3, 8), (4, 9)}, {(1, 6), (2, 8), (4, 10)}, {(1, 7), (2, 9), (3, 10)}, {(1, 2), (6, 8), (7, 9)}, {(1, 3), (5, 8), (7, 10)}, {(1, 4), (5, 9), (6, 10)}, {(2, 3), (5, 6), (9, 10)}, {(2, 4), (5, 7), (8, 10)} and {(3, 4), (6, 7), (8, 9)}, with parameters: $r^{\#} = 6$; $v^{\#} = 10$; $b^{\#}_1 = 10$, $k^{\#}_1 = 6$, $\lambda^{\#}_{11} = 3$, $\lambda^{\#}_{12} = 4$, $\mu^{\#} = 3$; $b^{\#}_2 = 30$, $k^{\#}_2 = 2$, $\lambda^{\#}_{21} = 1$, $\lambda^{\#}_{22} = 0$, and same scheme for both the systems as that of D.

Existence of above L_2–type and PBIB design based on triangular association scheme are well known (cf. Clatworthy [1] p.20, 22).

B. NPBIB designs with different association schemes for two systems:

Finally, we are giving some NPBIB designs with different association schemes and also present some examples.

THEOREM 3: Existence of two BIB design with parameters: v', b', r', k', λ' and $v'', b'', r'', k'', \lambda''$ respectively, implies the existence of a series of nested PBIB design D with rectangular association scheme for treatments arranged in blocks and GD association scheme for treatments arranged in sub-blocks having parameters:

$r = r'r''$; $v = v'v''$; $b_1 = b'b''$, $k_1 = k'k''$, $n_{11} = v'-1, n_{12} = v''-1, n_{13} = (v''-1)(v'-1)$;

$n_{13}=(v'-1)(v''-1)$, $\lambda_{11}=r''\lambda'$, $\lambda_{12}=r'\lambda''$, $\lambda_{13}=\lambda'\lambda''$; $b_2=k''b'b''$, $k_2=k'$, $n_{21}=v'-1$, $n_{22}=0$, $\lambda_{21}=\lambda'r''$, $\lambda_{22}=0$.

$$P_{11}=\begin{bmatrix} v'-2 & 0 & 0 \\ 0 & 0 & v''-1 \\ 1 & v''-1 & (v''-1)(v'-2) \end{bmatrix},\ P_{12}=\begin{bmatrix} 0 & 0 & v'-1 \\ 0 & v''-1 & 0 \\ v'-1 & 0 & (v''-2)(v'-1) \end{bmatrix},\ P_{13}=\begin{bmatrix} 0 & 1 & v'-2 \\ 1 & 0 & v''-2 \\ v'-2 & v''-2 & (v'-2)(v''-2) \end{bmatrix}$$

$$P_{21}=\begin{bmatrix} v'-2 & 0 \\ 0 & v'(v''-1) \end{bmatrix},\ P_{22}=\begin{bmatrix} 0 & v'-1 \\ v'-1 & v'(v''-2) \end{bmatrix}.$$

PROOF: Let there be a BIB design D_1' with parameter v', b', r', k', λ' written by using the first v' natural numbers from 1 as treatments. Now, add v' to each treatment in all the blocks in the above to get another BIB design to be denoted by D_2'. Again v' is added to each treatment in all blocks of D_2' to get a design to be denoted by D_3'. Likewise number of designs $D_1', D_2', D_3' \ldots$, can be obtained.

Now, take another BIB design D″ with parameters: $v'', b'', r'', k'', \lambda''$. This design has the natural numbers from 1 to v'' as treatments. Next, the treatment in D″ is replaced by the design matrices D_1', D_2' etc., i.e. the i-th treatment in D″ is replaced by D_i'. The resulting design is a nested PBIB design D with $v'v''$ treatments (cf. Das and Giri [13]). Let the treatments of D_1' be $1, 2, \ldots, j, \ldots, v'$ then treatments of D_i' shall be $(i+1)v'+1, (i+1)v'+1, \ldots, (i+1)v'+j, \ldots, iv'$, $1 \le i \le v''$. Thus treatments of design D arrangement as shown below,

$$\begin{array}{llllll}
1 & D_1': & 1, & \ldots, & j, & \ldots, v' \\
2 & D_2': & v'+1, & \ldots, & v'+j, & \ldots, 2v' \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v'' & D_{v''}': & (v''-1)v'+1, & \ldots, & (v''-1)v'+j, & \ldots, v''v'
\end{array}$$

This is a rectangular arrangement of $v'v''$ treatments in v'' rows and v' columns. We represent a typical treatment by $\phi_{ij}=(i-1)v'+j$, $1 \le i \le v''$, $1 \le j \le v'$ of the design D. We define a rectangular association scheme for treatments of design D, which are arranged in blocks. Further, in this rectangle if we assume that every row is forming a group then we have a group divisible scheme for treatments arranged in b_2 sub-blocks of design D.

Now any treatment i, $1 \le i \le v''$ appears in BIB design D″, r'' times and is replaced by D_i' so it appears in design D″, r'' times too. But any treatment of $D_i^{'}$ in this BIB design appears r' times, so that a treatment and is replaced by $\phi_{ij} = (i-1)v'+j$, $1 \le i \le v''$, $1 \le j \le v'$ appears in design D, $r''r'$ times. The size of blocks of design D becomes $k_1 = k''k'$ and considering replacement of treatment i by $D_i^{'}$ in D'' as sub-blocks of design D we have in total $b_2 = k''b'b''$ sub-blocks of size $k_2 = k''$ and so $\mu = k''$.

Consider now a pair of elements, which are first associates arranged in blocks of D. They are belonging to same group/row say i-th i.e. in $D_i^{'}$ say viz. $\phi_{ij_1} = (i-1)v'+j_1$, and $\phi_{ij_2} = (i-1)v'+j_2$, where $1 \le i \le v''$ (kept fixed) and $1 \le j_1, j_2 \le v'$. This pair of treatments (first associates) appears together λ' times in BIB design D' which itself appears r'' times in D, hence in D_1 pair $(\phi_{ij_1}, \phi_{ij_2})$ appears in $\lambda_{11} = r''\lambda'$ blocks. In case we consider the sub-blocks of D by above argument this pair is a first associate in GD scheme and appears $\lambda_{21} = r''\lambda'$ times.

Further, since $D_i^{'}$ forms a sub-block of D for fixed i say i = 1 the treatments of other groups i = 2, 3,..., v'' do not belong in this sub-block. These treatment pair viz. $(j_1, (i-1)v' + j_2)$, $i \ne 1$, $1 \le j_1, j_2 \le v'$ are second associates in GD scheme, appear together in $\lambda_{22} = 0$ blocks of D'.

Next, consider a pair of second associates of rectangular association scheme arrangement of treatments in blocks of D, viz. $(\phi_{i_1 j}, \phi_{i_2 j}) = ((i_1-1)v'+j, (i_2-1)v'+j)$, $i_1 \ne i_2$, $1 \le i_1, i_2 \le v''$, j fixed, $1 \le i \le v'$. Since these elements belong, respectively to the group $D_{i_1}^{'}$ and $D_{i_2}^{'}$ which occur together in λ'' blocks in D'' corresponding to two treatments in BIB design D''. In $D_i^{'}$'s each treatment appears r' times, hence in D these second associate pairs appear together in $\lambda_{12} = r'\lambda''$ blocks.

Finally a pair of treatments not in the same row and same column i.e. third associates pair of type $((i_1-1)v'+j_1, (i_2-1)v'+j_2)$, $i_1 \ne i_2$, $j_1 \ne j_2$, $1 \le i_1, i_2 \le v''$, $1 \le j_1, j_2 \le v'$, appear in $\lambda_{13} = \lambda'\lambda''$ blocks of D by similar arguments as in preceding paragraph. This completes the proof of theorem.

EXAMPLE 3.1: Let us take a BIB design with parameters: $v' = 3$, $b' = 3$, $r' = 2$, $k' = 2$, $\lambda' = 1$ with the following blocks, (1,2), (1,3), (2,3). Then take another BIB design with parameters: $v'' = 5$, $b'' = 10$, $r'' = 4$, $k'' = 2$, $\lambda'' = 1$ having blocks (1,4), (2,5), (3,1), (4,2), (5,3), (2,3), (3,4), (4,5), (5,1), (1,2). From the above method we have following $D_i^{'}$ matrices

D_1	D_2	D_3	D_4	D_5
1 2	4 5	7 8	10 11	13 14
2 3	5 6	8 9	11 12	14 15
1 3	4 6	7 9	10 12	13 15

and we have this rectangle of size 5 x 3 for design D

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15

Then we have nested PBIB design with parameters: $r=12;\ v=15; b_1=30, k_1=4, n_{11}=2, n_{12}=4, n_{13}=8, \lambda_{11}=4, \lambda_{12}=2, \lambda_{13}=1, b_2=60,$ $k_2=2,\ n_{21}=2,\ n_{22}=12,\ \lambda_{21}=4, \lambda_{22}=0.$

The blocks of the design are:

$$\begin{bmatrix}(1 & 2) & (10 & 11)\\ (2 & 3)\vdots & (11 & 12)\\ (1 & 3) & (10 & 12)\end{bmatrix}\begin{bmatrix}(4 & 5) & (13 & 14)\\ (5 & 6)\vdots & (14 & 15)\\ (5 & 6) & (13 & 15)\end{bmatrix}\begin{bmatrix}(7 & 8) & (1 & 2)\\ (8 & 9)\vdots & (2 & 3)\\ (7 & 9) & (1 & 3)\end{bmatrix}$$

THEOREM 4: Let there exist a PBIB(2) design D_1 with block size k_1 and another PBIB (2) design D_2 with same number of treatments as in D_1 with block size $k_2 = k_1/m$. If the blocks of D_2 can be embedded in clocks of D_1 then get a NPBIB design.

NOTE 2: The designs of Theorem 5 may be with same or different association schemes for two systems of blocks.

However the examples given below have different association schemes for D_1 and D_2.

EXAMPLE 4.1: Consider L_2-type association scheme with n = 3. There exists a PBIBD D_1 with following parameters: v = 9, b = 18, r = 4, k = 2, $\lambda_1 = 1$, $n_1 = 4$, $\lambda_2 = 0$, $n_2 = 4$, $p_{12}^1 = p_{12}^2 = 2$. (Cf. Clathworthy W.H. [1]) with blocks: (1,2) (1,3) (2,5) (3,6) (2,3) (1,4) (1,7) (2,8) (3,9) (4,5) (4,6) (4,7) (5,6) (5,8) (6,9) (7,8) (7,9) (8,9).

Consider again a PBIB (2) design D_2 based on Cyclic association scheme with parameters: v = b = 9 (4t +1 and t = 2), r = k = 4, $\lambda_1 = 1$, $n_1 = 4$, $\lambda_2 = 2$, $n_2 = 4$, $p_{12}^1 = 4$, $p_{12}^2 = 4$ (Cf. Raghavarao [4]). Blocks of this design are: (2,4,6,8), (6,5,1,3), (9,2,8,5), (5,8,7,1), (7,3,4,2), (1,7,2,9), (4,9,5,6), (3,1,9,4), (8,6,3,7).

However parameters of the two schemes turn to be the same here. The blocks of design D_1 can be embedded in blocks of PBIB design D_2 as below:

{(1,3), (5,6)}, {(2,5), (8,9)}, {(1,7), (5,8)}, {(2,3), (4,7)}, {(1,2), (7,9)}, {(4,5), (6,9)}, {(1,4), (3,9)}, {(2,8), (4,6)}, and {(3,6), (7,8)}.

Hence this forms a nested PBIB (2) design with parameters: r = 4; v = 9, $b_1 = 9$, $k_1 = 4$, $n_{11} = 2$, $\lambda_{11} = 1$, $\lambda_{12} = 2$; $b_2 = 18$, $k_2 = 2$, $\lambda_{21} = 1$, $\lambda_{22} = 0$, with different association schemes contrary to the conjecture of Homel & Robinson [2].

EXAMPLE 4.2: Consider a PBIB design with blocks: (1,2) (1,3) (1,4) (1,7) (2,3) (2,5) (2,8) (4,7) (5,8) (4,5) (4,6) (5,6) (3.6) (3,9) (7,8) (7,9) (8,9) and (6,9). This is based on L_2 - association scheme with parameters: v = 9, $b_2 = 18$, r = 4, $k_2 = 2$, $\lambda_{21}= 1$, $n_{21} = 4$, $\lambda_{22} = 0$, $n_{22}= 4$, $P_{21} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, $P_{22} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$.

This design is nested in the following design with blocks: [(1,2), (3,6)], [(1,4), (2,3)], [(1,7), (8,9)], [(2,8), (7,9)], [(3,9), (7,8)], [(4,5), (6,9)], [(4,6), (5,8)] and [(4,7), (5,6)],)]. This design is a group divisible PBIB design with parameters: v = 9, $b_1 = 9$, r = 4, $k_1 = 4$, $\lambda_{11} = 3$, $n_{11} = 2$, $\lambda_{12} = 1$, $n_{12} = 6$, $P_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, $P_{12} = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$, $m = n = 3$.

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