A characterization of $(\alpha, s+1)$-geometries, $1 < \alpha < s+1$, satisfying the axiom of Pasch

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Abstract. In this paper, a characterization of two classes of $(q, q+1)$-geometries, that are fully embedded in a projective space $\mathrm{PG}(n,q)$, is obtained. The first class is the one of the $(q, q+1)$-geometry $\mathrm{H}_q^{n,m}$, having points the points of $\mathrm{PG}(n,q)$ that are not contained in an m-dimensional subspace $\Pi[m]$ of $\mathrm{PG}(n,q)$, for $0 \leq m \leq n-3$, and lines the lines of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$. The second class is the one of the $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$ having the same point set as $\mathrm{H}_q^{n,m}$, but with $-1 \leq m \leq n-3$, and lines the lines skew to $\Pi[m]$ that are not contained in a certain partition of the point set of $\mathrm{SH}_q^{n,m}$. Our characterization uses the axiom of Pasch, which is also known as axiom of Veblen-Young. It is a generalization of the characterization for partial geometries satisfying the axiom of Pasch by J. A. Thas and F. De Clerck. A characterization for $\mathrm{H}_q^{n,m}$ was already proved by H. Cuypers. His result however does not include $\mathrm{SH}_q^{n,m}$.

Keywords: (α, β)-geometries, characterization, projective embedding

AMS subject classification: 51E30, 51A45

1 Introduction

A *partial linear space* of order (s,t) is a connected incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ a finite non-empty set of elements called *points*, $\mathcal{L}$ a family of subsets of $\mathcal{P}$ called *lines* and I a symmetric incidence relation satisfying the following axioms. (i) Any two distinct points are incident with at most one line. (ii) Each line is incident with exactly $s+1$ points, $s \geq 1$. (iii) Each point is incident with exactly $t+1$ lines, $t \geq 1$.

An *antiflag* of a partial linear space $\mathcal{S}$ is a pair (x, L), with x a point of $\mathcal{S}$, L a line of $\mathcal{S}$ and such that x is not incident with L. Two points p_1 and p_2 are *collinear* if there is a line L of $\mathcal{S}$ such that $p_1 \mathrm{~I~} L \mathrm{~I~} p_2$; we denote $p_1 \sim p_2$. Two lines L_1 and L_2 are concurrent if there is a point p of $\mathcal{S}$ such that $L_1 \mathrm{~I~} p \mathrm{~I~} L_2$; we denote $L_1 \sim L_2$. The *incidence number* of an antiflag (x, L) of $\mathcal{S}$ is the number, denoted by $i(x, L)$, of points collinear with the point $x \in \mathcal{P}$ and incident with the line $L \in \mathcal{L}$.

An (α,β)*-geometry* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t), for some s and t, such that for each antiflag (x, L) of $\mathcal{S}$, $i(x, L) = \alpha$ or $i(x, L) = \beta$.

An (α,β)-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is *fully embedded* in a projective space $\mathrm{PG}(n,q)$ if $\mathcal{P}$ is a subset of the point set of $\mathrm{PG}(n,q)$, $\mathcal{L}$ is a subset of the line set of $\mathrm{PG}(n,q)$, I is the incidence inherited from $\mathrm{PG}(n,q)$ and $s = q$. In what follows, we always assume that the points of $\mathcal{S}$ span $\mathrm{PG}(n,q)$.

An (α,β)-geometry that satisfies $\alpha = \beta$, is called a *partial geometry*, and it is denoted by $\mathrm{pg}(s,t,\alpha)$. Partial geometries fully embedded in a projective space $\mathrm{PG}(n,q)$ have been studied by F. De Clerck and J. A. Thas [4]. They have proved that the only partial geometry fully embeddable in $\mathrm{PG}(n,q)$, for which $1 < \alpha < q+1$ and $\alpha < t+1$, is the partial geometry H_q^n, defined as follows. Let H be an $(n-2)$-dimensional subspace of $\mathrm{PG}(n,q)$. Points of H_q^n are the points of $\mathrm{PG}(n,q) \setminus \mathrm{H}$, lines of H_q^n are the lines of $\mathrm{PG}(n,q)$ that have no point in common with H, incidence is the incidence of $\mathrm{PG}(n,q)$ restricted to the points and lines of H_q^n. It is easy to prove that H_q^n is indeed a partial geometry and that it has parameters $s = q$, $t = q^{n-1} - 1$ and $\alpha = q$.

In [1, 2] we have introduced the $(q, q+1)$-geometries $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$. Both of them are fully embeddable in $\mathrm{PG}(n,q)$. $\mathrm{H}_q^{n,m}$ has points the points of $\mathrm{PG}(n,q) \setminus \Pi[m]$, where $\Pi[m]$ is an m-dimensional subspace of $\mathrm{PG}(n,q)$, $0 \le m \le n-3$, and lines the lines of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$. Note that for $m = n-2$, this construction gives the partial geometry H_q^n defined in the previous paragraph. For $m = -1$, it gives the partial geometry of points and lines of the projective space $\mathrm{PG}(n,q)$. $\mathrm{SH}_q^{n,m}$ has the same point set as $\mathrm{H}_q^{n,m}$, but now with $-1 \le m \le n-3$, and its lines are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathrm{PG}(n,q) \setminus \Pi[m]$, where $l = (q^{n-m} - 1)/(q^{m'-m} - 1)$, such that for $i = 1, \ldots, l$, $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, with $\Omega_i[m']$ an m'-dimensional subspace of $\mathrm{PG}(n,q)$ that contains $\Pi[m]$, and with $m+2 \le m' \le n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for this partition and the $(q, q+1)$-geometry to exist is that $(m' - m) \mid (n - m)$.

In this paper a characterization of the $(q, q+1)$-geometries $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$ will be obtained. This characterization is an extension of the existing characterization for the partial geometry H_q^n, which we will shortly describe in the next section. In section 3 we will generalize the definitions of section 2 to (α,β)-geometries. In section 4 an important lemma will be proved, while in section 5 some new terminology will be defined. In section 6 a characterization for $\mathrm{H}_q^{n,m}$ obtained in 1995 by H. Cuypers [3] is recalled. In section 7 our characterization theorem is stated and proved.

2 A characterization of the partial geometry H_q^n

In [5], J. A. Thas and F. De Clerck characterized the partial geometry H_q^n. Before stating their characterization theorem, we will explain the terminology that is used.

An (α,β)-geometry $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ satisfies *the axiom of Pasch* (also called *axiom of Veblen* or *axiom of Veblen-Young*) if $\forall L_1, L_2, M_1, M_2 \in \mathcal{L}$, $L_1 \neq L_2$, $L_1 \mathrm{~I~} x \mathrm{~I~} L_2$, $x \notin M_1$, $x \notin M_2$, $L_i \sim M_j$ for all $i,j \in \{1,2\} \Rightarrow M_1 \sim M_2$. Note that for $\alpha = \beta = 1$ and for $\alpha = \beta = t+1$, the axiom of Pasch is trivially satisfied.

Let $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a partial geometry, for which $\alpha \notin \{1, s+1, t+1\}$, that satisfies the axiom of Pasch. Let L and M, $L \neq M$, be two concurrent lines of $\mathcal{S}$ with intersection point x. Then the *substructure* $\mathcal{S}(L,M) = (\mathcal{P}^*,\mathcal{L}^*,\mathrm{I}^*)$ of $\mathcal{S}$ is defined as follows: $\mathcal{L}^*$ is the set of the $s(\alpha-1)$ lines N, such that $x \notin N$ and $L \sim N \sim M$, together with the set of the α lines through x that are concurrent with at least one of these $s(\alpha-1)$ lines; $\mathcal{P}^*$ is the set of points of $\mathcal{S}$ that lie on the lines of $\mathcal{L}^*$ and $\mathrm{I}^*=\mathrm{I}\cap((\mathcal{P}^* \times \mathcal{L}^*) \cup (\mathcal{L}^* \times \mathcal{P}^*))$. As $\mathcal{S}$ satisfies the axiom of Pasch, $\mathcal{S}(L,M) = (\mathcal{P}^*,\mathcal{L}^*,\mathrm{I}^*)$ is a $\mathrm{pg}(s,\alpha-1,\alpha)$. Note that for $N_1, N_2 \in \mathcal{L}^*$, $N_1 \neq N_2$, the substructures $\mathcal{S}(N_1,N_2)$ and $\mathcal{S}(L,M)$ coincide. Moreover for each antiflag (x,N) of $\mathcal{S}$, there is exactly one substructure $\mathcal{S}(L,M)$ that contains both x and N. This substructure we will denote by $\mathcal{S}(x,N)$.

Let x and y be two non-collinear points of $\mathcal{S}$. There are $(t+1)/\alpha$ subgeometries $\mathcal{S}(L,M)$ of $\mathcal{S}$ that contain both x and y. We denote these subgeometries by $\mathcal{S}_i^* = (\mathcal{P}_i^*,\mathcal{L}_i^*,\mathrm{I}_i^*)$, $i = 1,\ldots,(t+1)/\alpha$. The *line of the second type* $\langle x,y\rangle$ is defined to be the set $\mathcal{P}_1^* \cap \ldots \cap \mathcal{P}_{(t+1)/\alpha}^*$. It follows immediately that no two distinct points of the line $\langle x,y\rangle$ are collinear in $\mathcal{S}$ (for more explanation see [5]), and that for z_1, $z_2 \in \langle x,y\rangle$, $z_1 \neq z_2$, the lines $\langle z_1,z_2\rangle$ and $\langle x,y\rangle$ coincide. As $\langle x,y\rangle$ is a set of two by two non-collinear points of the partial geometry $\mathcal{S}_i^*$, $i \in \{1,\ldots,(t+1)/\alpha\}$, it follows that $|\langle x,y\rangle| \leq s+1-s/\alpha$. If $|\langle x,y\rangle| = s+1-s/\alpha$ for all $x,y \in \mathcal{P}$, x not collinear with y, then the partial geometry $\mathcal{S}$ is called *regular.*

Theorem 2.1 ([5]) *The partial geometry* $\mathcal{S} = \mathrm{pg}(s,t,\alpha)$, $\alpha \notin \{1,s+1,t+1\}$, *is isomorphic to an* H_q^n *if and only if*

1. *$\mathcal{S}$ satisfies the axiom of Pasch;*
2. *$\mathcal{S}$ is regular;*
3. $2s > s^4 - \alpha s^3 + \alpha^2 s^2 + \alpha^3 s - 2\alpha^4$.

Remark. The third assumption of the theorem is derived from a necessary condition for a graph to be the point graph of a partial geometry. This necessary condition is commonly known as the *Bose condition for a graph.* Note that the third assumption of theorem 2.1 turns out to be a very strong condition, as for $\alpha \neq s$, this condition is almost never satisfied.

3 (α, β)-geometries that satisfy the axiom of Pasch

In this section we will generalize the concepts defined in section 2 for partial geometries to similar concepts for (α, β)-geometries. The terminology given in this section will be the one used in the rest of this paper.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, β)-geometry of order (s, t), satisfying the axiom of Pasch, with $1 < \alpha < \beta < t+1$. Let L and M be two distinct concurrent lines of $\mathcal{S}$, with $L \cap M = \{x\}$. A *substructure* $S(L, M)$ of $\mathcal{S}$ can be defined in the same way as is done for a partial geometry in section 2. However, as $\mathcal{S}$ is an (α, β)-geometry, for each antiflag (p, L_p) of $S(L, M)$, we have $i(p, L_p) = \alpha$ or $i(p, L_p) = \beta$. Hence it is possible that $S(L, M)$ contains a point z_1 through which there are α lines of $S(L, M)$ and a point z_2 through which there are β lines of $S(L, M)$. In this case $S(L, M)$ is clearly not an (α, β)-geometry, as by definition the number of lines through a point in an (α, β)-geometry has to be a constant. If the number of lines of $S(L, M)$ through a point of $S(L, M)$ is a constant, then $S(L, M)$ is a $\mathrm{pg}(s, \alpha - 1, \alpha)$ or a $\mathrm{pg}(s, \beta - 1, \beta)$. This is easy to prove, as $\mathcal{S}$ satisfies the axiom of Pasch. A substructure $S(L, M)$ that is a $\mathrm{pg}(s, \alpha - 1, \alpha)$ we call an *α-substructure*. A substructure $S(L, M)$ that is a $\mathrm{pg}(s, \beta - 1, \beta)$ we call a *β-substructure*. A substructure $S(L, M)$ that is not a partial geometry, we call a *mixed substructure*. In what follows, we will denote a substructure $S(L, M)$ sometimes as π, ρ or σ.

The number of substructures through two distinct non-collinear points of $\mathcal{S}$ is not necessarily a constant, as the number of lines through a point in a substructure is not a constant. This number is however a constant if $\beta = s+1$. Indeed, assume that $\beta = s + 1$ and let x and y be distinct non-collinear points of $\mathcal{S}$. Let L be any line of $\mathcal{S}$ through x. As $x \not\sim y$, exactly α points of L are collinear with y. Hence there are exactly $(t + 1)/\alpha$ substructures $S(L, M)$ that contain both x and y. As x and y were arbitrarily chosen distinct non-collinear points of $\mathcal{S}$, it follows that the number of substructures through any two distinct non-collinear points of $\mathcal{S}$ is a constant and equal to $(t + 1)/\alpha$.

We define a *line of the second type* $\langle x, y\rangle$ through two distinct non-collinear points x and y of $\mathcal{S}$ as the intersection of all substructures $S(L, M)$ containing both x and y. Note that there are at least two distinct substructures through x and y, as by assumption $t + 1 > \alpha$. From the definition, it follows that each two distinct points of the line $\langle x, y\rangle$ are non-collinear in $\mathcal{S}$. Note that the notation $\langle x, y\rangle$ is also used for a line of $\mathcal{S}$ through x and y, the context makes clear whether it is a line of $\mathcal{S}$ or a line of the second type.

An $(\alpha, s + 1)$-geometry $\mathcal{S}$ of order (s, t), with $1 < \alpha < s+1 < t+1$, that satisfies the axiom of Pasch, is called *regular with respect to non-collinear* *points* if and only if each line of the second type and each line of $\mathcal{S}$ that are both contained in a substructure $S(L, M)$, intersect in at least one point. Note that this implies that they intersect in exactly one point, as a line of the second type cannot contain two distinct collinear points of $\mathcal{S}$.

4 An important lemma

We will prove that for a proper $(\alpha, s+1)$-geometry $\mathcal{S}$ of order (s,t), $1 < \alpha < s+1 < t+1$, that satisfies the axiom of Pasch, that is regular with respect to non-collinear points, and such that there is at least one α-substructure, it follows that $\alpha = s$.

First we make the following observation. Let $\mathcal{S}$ be an $(\alpha, s+1)$-geometry that satisfies the axiom of Pasch and such that $\mathcal{S}$ is regular with respect to non-collinear points. Assume that $\mathcal{S}$ contains an α-substructure $S(L, M)$. The points and lines of $S(L, M)$ form a $\mathrm{pg}(s, \alpha-1, \alpha)$. As $\alpha < s+1$, $S(L, M)$ contains two non-collinear points x and y. As $\mathcal{S}$ is regular with respect to non-collinear points, each line of $S(L, M)$ contains exactly one point of $\langle x, y\rangle$. So $|\langle x, y\rangle| = s + 1 - s/\alpha$, and hence $\alpha | s$.

Lemma 4.1 *Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s,t), $1 < \alpha < s+1 < t+1$, that satisfies the axiom of Pasch, that is regular with respect to non-collinear points and such that there is at least one α-substructure. Then $\alpha = s$.*

Proof. Assume first that there is no mixed substructure $S(L, M)$. Then both an α-substructure and an $(s+1)$-substructure exist. Note that in an $(s+1)$-substructure, any two distinct points of $\mathcal{S}$ are collinear in $\mathcal{S}$.

Let x and y be two non-collinear points of $\mathcal{S}$. Each substructure through $\langle x, y\rangle$ is an α-substructure. As $\alpha < t+1$, there are at least two distinct substructures π_1 and π_2 through $\langle x, y\rangle$. Let p be a point of $\mathcal{S}$ in π_1, $p \notin \langle x, y\rangle$. Let N be a line of $\mathcal{S}$ contained in π_2. There are α or $s+1$ lines through p intersecting N in a point. As $\alpha > 1$, there exists a line L_1 of $\mathcal{S}$ through p intersecting N in a point, such that $L_1 \not\subset \pi_1$. Let L_2 be a line of $\mathcal{S}$ contained in π_1 and not incident with p. The line L_2 intersects $\langle x, y\rangle$ (and hence also π_2) in a point. Let $x_1 = p, x_2, \ldots, x_{s+1}$ be the points of L_1. Define $\mathcal{P}'$ as the set of points contained in the $s+1$ substructures $S(x_i, L_2)$, for $i = 1, \ldots, s+1$. As $\pi_1 = S(p, L_2)$, all points of π_1 belong to $\mathcal{P}'$. In particular the points of $\langle x, y\rangle$ belong to $\mathcal{P}'$.

We will prove that each line of $\mathcal{S}$ that contains at least two points of $\mathcal{P}'$, contains $s+1$ points of $\mathcal{P}'$. Let z and z' be points of $\mathcal{P}'$, $z \neq z'$. Suppose that $z \sim z'$. We denote the line of $\mathcal{S}$ containing z and z' by M. We need to show that all points of M are points of $\mathcal{P}'$. If $z \mathrm{I} L_2$, if $M = L_1$, or if $z' \in S(z, L_2)$, then the result follows immediately. So we may suppose that $z' \notin S(z, L_2)$, $M \neq L_1$ and that $z \not\mathrm{I} L_2$. We distinguish two cases.

Case 1: $L_1 \sim M$. Let w be the point of L_1 that is contained in the substructure $S(z', L_2)$. Let $z'' \in M$, $z \neq z'' \neq z'$. We have to prove that $z'' \in \mathcal{P}'$. If $z'' \in L_1$, then clearly $z'' \in \mathcal{P}'$. So suppose that $z'' \not\mathrm{I}\, L_1$. As w, $z' \in S(z', L_2)$, the line $\langle w, z' \rangle$ (which can be either a line of $\mathcal{S}$ or a line of the second type) has a point u in common with L_2. The line $\langle u, z'' \rangle$ (which can be either a line of $\mathcal{S}$ or a line of the second type) has a point w' in common with L_1, as both $\langle u, z'' \rangle$ and L_1 belong to the substructure $S(L_1, M)$. All the points of this line $\langle u, z'' \rangle = \langle u, w' \rangle$ are elements of $S(w', L_2)$. Hence z'' is a point of $S(w', L_2)$, $w' \in L_1$, and so $z'' \in \mathcal{P}'$.

Case 2: $L_1 \not\sim M$. Let M' be a line that does not belong to π_1, such that $z \mathrm{~I~} M'$, $L_1 \sim M'$ and M' skew to L_2. Note that M' exists, as there are at least $\alpha > 1$ lines through z intersecting L_1 and at most one of these lines contains a point of L_2. From case 1 it follows that the $s+1$ points of M' are contained in $\mathcal{P}'$. Moreover, the $s+1$ substructures $S(x_i, L_2)$, for $x_i \mathrm{~I~} L_1$ $(i = 1, \ldots, s+1)$, coincide with the $s+1$ substructures $S(x_i', L_2)$, for $x_i' \mathrm{~I~} M'$ $(i = 1, \ldots, s+1)$. By construction M and M' intersect. Applying again case 1 gives us that each point of the line M is contained in one of the substructures $S(x_i', L_2)$, for $x_i' \mathrm{~I~} M'$ $(i = 1, \ldots, s+1)$. Hence each point of M is contained in $\mathcal{P}'$.

So we have proved that if a line contains two points of $\mathcal{P}'$, then each point of this line belongs to $\mathcal{P}'$. Define $\mathcal{L}'$ as the set of lines of $\mathcal{S}$ containing at least two distinct points of $\mathcal{P}'$. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$, with I' the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Now we distinguish two cases.

Assume that there is an $(s+1)$-substructure ρ through L_1 in $\mathcal{S}'$. Let u be a point of ρ not incident with L_1. The point u is contained in $\mathcal{S}'$, hence $u \in S(x', L_2)$ for a point $x' \mathrm{~I~} L_1$. As u and x' belong to ρ, the line $\langle x', u \rangle$ is a line of $\mathcal{S}$ contained in $S(x', L_2)$. Every two lines of $\mathcal{S}$ in a substructure of $\mathcal{S}$ intersect, hence $\langle x', u \rangle$ intersects L_2 in a point v. So ρ contains the point v of L_2. As $L_2 \subset \pi_1$, $v \in \pi_1$. Moreover ρ contains the point p that is the intersection point of L_1 and π_1. It follows that ρ intersects π_1 in the line $\langle v, p \rangle$. As $\langle v, p \rangle$ belongs to ρ, it is a line of $\mathcal{S}$. In π_1 the line $\langle v, p \rangle$ intersects the line $\langle x, y \rangle$ of the second type in a point w (here we use the regularity of $\mathcal{S}$ with respect to non-collinear points). Let $L_1^w = \langle v, p \rangle, \ldots, L_{s+1}^w$ $(i = 1, \ldots, s+1)$, be the $s+1$ lines through w in ρ. As $x \neq y$, we have that $w \neq x$ or $w \neq y$. So we may assume that $w \neq x$. Each substructure $S(L_i^w, x)$ $(i = 1, \ldots, s+1)$ contains the line $\langle x, y \rangle$ of the second type, and hence is an α-substructure. These substructures contain all points of $\mathcal{S}'$. Through w there are $(s+1)\alpha$ lines of $\mathcal{S}$ in $\mathcal{S}'$, namely α lines in each substructure $S(x, L_i^w)$ $(i = 1, \ldots, s+1)$.

Now we count the lines of $\mathcal{S}$ through w in $\mathcal{S}'$ in another way. Let N_w be a line of $\mathcal{S}$ through w in π_1, $N_w \neq \langle p, v \rangle$. In each substructure through N_w in $\mathcal{S}'$ there are α or $s+1$ lines of $\mathcal{S}$ through w. From the previous paragraph, we know that ρ intersects π_1 in the line $\langle p, v \rangle$. As $w \in \langle p, v \rangle$, the line N_w intersects ρ in the point w. The $s+1$ lines L_i^w $(i = 1, \ldots, s+1)$ through w in ρ give $s+1$ substructures $S(N_w, L_i^v)$ through N_w in $\mathcal{S}'$. Counting the lines of $\mathcal{S}$ through w

in these substructures, we get that there are $cs + (s + 1 - c)(\alpha - 1) + 1$ lines of $\mathcal{S}$ through w in $\mathcal{S}'$, for $c \in \mathbb{N}$, $0 \leq c \leq s + 1$. It follows that $(s + 1)\alpha = cs + (s + 1 - c)(\alpha - 1) + 1$, or $c = s/(s + 1 - \alpha)$. So $(s + 1 - \alpha)|s$, and as $\alpha \neq 1$, it follows that $s/2 + 1 \leq \alpha$. We noted in the beginning of this section that $\alpha|s$. Hence $\alpha = s$.

Assume next that there is no $(s+1)$-substructure through L_1 in $\mathcal{S}'$. Then either $\mathcal{S}'$ contains an $(s + 1)$-substructure not through L_1, or $\mathcal{S}'$ contains no $(s + 1)$-substructure.

Assume that there is an $(s + 1)$-substructure contained in $\mathcal{S}'$. This $(s + 1)$-substructure in $\mathcal{S}'$ cannot contain the line $\langle x, y\rangle$. So it contains a line N of $\mathcal{S}$ that is skew to $\langle x, y\rangle$. The line N is not contained in π_1. If N is contained in one of the substructures $S(x_i, L_2)$, with x_i I L_1 ($i \in \{1, \ldots, s + 1\}$), then clearly N intersects L_2 and hence also π_1 in a point. If N is not contained in any of the substructures $S(x_i, L_2)$, x_i I L_1 ($i \in \{1, \ldots s + 1\}$), then it contains at most one point of each such substructure $S(x_i, L_2)$. Hence each point of N is contained in a distinct substructure $S(x_i, L_2)$, for x_i I L_1 ($i = 1, \ldots, s+1$). As $\pi_1 = S(p, L_2)$, with p a point of the line L_1, it follows that also in this case N intersects π_1 in a point. Let N' be a line of $\mathcal{S}$ in π_1, such that N' is skew to N. The $s + 1$ substructures $S(x_i, L_2)$, for x_i I L_1 ($i = 1, \ldots, s + 1$), coincide with the $s + 1$ substructures $S(x_i', N')$, for x_i' I N ($i = 1, \ldots, s + 1$). So, replacing L_1 by N and L_2 by N' in the previous part of the proof, we get that $\alpha = s$.

Assume now that there is no $(s + 1)$-substructure contained in $\mathcal{S}'$. As $\mathcal{S}$ is a proper $(\alpha, s + 1)$-geometry and there are no mixed substructures, $\mathcal{S}$ contains an $(s + 1)$-substructure ρ'. Let $L_{w'}$ be a line of $\mathcal{S}$ through x intersecting ρ' in a point w'. The substructure $S(L_{w'}, y)$ contains the line $\langle x, y\rangle$ of the second type, hence it is an α-substructure. In $S(L_{w'}, y)$ there are $s + 1 - \alpha$ lines of the second type through w'. Let $\langle w', u'\rangle$ be such a line of the second type through w'. Let M_1 and M_2 be two lines of $\mathcal{S}$ through w' in ρ'. Then $S(u', M_1)$ and $S(u', M_2)$ are both α-substructures intersecting ρ' in the lines M_1 and M_2 of $\mathcal{S}$. Let p' be a point of $S(u', M_2)$, $p' \notin \langle u', w'\rangle$. Let $L_{p'}$ be a line of $\mathcal{S}$ through p' intersecting M_1 in a point different from w'. Let M_3 be a line of $\mathcal{S}$ in $S(u', M_2)$, $p' \notin M_3$. Let $\mathcal{P}^*$ be the set of all the points of $\mathcal{S}$ contained in the substructures $S(x_i, M_3)$, for $x_i \in L_{p'}$ ($i = 1, \ldots, s+1$). Let $\mathcal{L}^*$ be the set of all lines intersecting $\mathcal{P}^*$ in at least two points. As before, it follows that all points of $\mathcal{S}$ on the lines that are element of $\mathcal{L}^*$, are elements of $\mathcal{P}^*$. Let I^* be the restriction of I to $(\mathcal{P}^* \times \mathcal{L}^*) \cup (\mathcal{L}^* \times \mathcal{P}^*)$. Then, replacing $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ by $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$, the result follows in the same way as in the above.

We conclude that if there is no mixed substructure, then $\alpha = s$. Now assume that there is a mixed substructure σ in $\mathcal{S}$. Let p be a point of σ through which there are α lines of $\mathcal{S}$ in σ. As $\alpha < s + 1$, there is a line $\langle p, p'\rangle$ of the second type through p in σ.

Let u be a point of σ through which there are $s + 1$ lines of $\mathcal{S}$. As $\mathcal{S}$ is regular with respect to non-collinear points, there cannot be a line of the second type through u in σ. In particular $u \notin \langle p, p' \rangle$, and hence $\langle p, p' \rangle$ contains exactly one point of each of the $s+1$ lines through u in σ. It follows that $|\langle p, p' \rangle| = s+1$.

Counting the lines of $\mathcal{S}$ in σ that intersect $\langle p, p' \rangle$, we get that there are $\alpha(s+1)$ such lines. These lines are all the lines of $\mathcal{S}$ in σ, because of regularity with respect to non-collinear points. Counting the lines of $\mathcal{S}$ in σ intersecting a line of $\mathcal{S}$ in σ in a point, it follows that there are $c(\alpha - 1) + (s + 1 - c)s + 1$ such lines, for a $c \in \mathbb{N}$, $0 \leq c \leq s+1$. As $\mathcal{S}$ satisfies the Pasch axiom, these are all the lines of $\mathcal{S}$ in σ. Hence $\alpha(s+1) = c(\alpha - 1) + (s+1-c)s + 1$, and hence $c = s + 1 - s/(s+1-\alpha)$. It follows that $(s+1-\alpha)|s$. So $s+1-\alpha = s$ or $s+1-\alpha \leq s/2$. If $s+1-\alpha = s$, then $\alpha = 1$, a contradiction with the assumption. So $s+1-\alpha \leq s/2$, and hence $s/2+1 \leq \alpha$.

We noted in the above that $\alpha|s$. From $s/2+1 \leq \alpha$, it now follows that $\alpha = s$.
□

5 The different types of substructures

Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s,t), $1 < \alpha < s+1 < t+1$, that satisfies the axiom of Pasch, that is regular with respect to non-collinear points and such that there is at least one α-substructure. From lemma 4.1 it follows that $\alpha = s$. We will now count the number of points and lines of $\mathcal{S}$ in an $(s+1)$-substructure, an s-substructure and a mixed substructure, assuming that such substructures exist.

By definition, the points and lines of $\mathcal{S}$ in an $(s+1)$-substructure form a partial geometry $\mathrm{pg}(s,s,s+1)$. Therefore we will use from now on the term *projective plane* instead of $(s+1)$-substructure. In a projective plane there are s^2+s+1 points of $\mathcal{S}$ and s^2+s+1 lines of $\mathcal{S}$, and every two points are collinear.

The points and lines of $\mathcal{S}$ in an s-substructure are the points and lines of a partial geometry $\mathrm{pg}(s, s-1, s)$. Hence we will use the term *dual affine plane* instead of s-substructure. In a dual affine plane π, there are s^2+s points of $\mathcal{S}$ and s^2 lines of $\mathcal{S}$. Through each point p of $\mathcal{S}$ in π there is exactly one line of the second type. From the regularity with respect to non-collinear points, it follows that a line of the second type in π contains s points of $\mathcal{S}$.

A mixed substructure contains exactly one line of the second type. Indeed, let σ be a mixed substructure and let y be a point of $\mathcal{S}$ in σ through which there are s lines of $\mathcal{S}$ in σ. Then σ contains a line $\langle y, z \rangle$ of the second type through y. Let p be a point of σ through which there are $s+1$ lines of $\mathcal{S}$. As $\mathcal{S}$ is regular with respect to non-collinear points, $\langle y, z \rangle$ has exactly one point in common with each of the $s+1$ lines of $\mathcal{S}$ through p in σ. Hence $|\langle y, z \rangle| = s+1$.

Let L be a line of $\mathcal{S}$ in σ. Counting the lines of $\mathcal{S}$ in σ that intersect L, we get that there are $1 + cs + (s+1-c)(s-1) = s^2 + c$ such lines, where c is the number of points of L through which there are $s+1$ lines of $\mathcal{S}$ in σ. Now we count the number of lines of $\mathcal{S}$ in σ in another way. Through each point of $\langle y, z \rangle$ there are s

lines of $\mathcal{S}$ in σ. Each line of $\mathcal{S}$ in σ intersects $\langle y, z\rangle$, so there are $(s+1)s$ lines of $\mathcal{S}$ in σ. It follows that $c = s$ or thus L contains exactly one point through which there are s lines of $\mathcal{S}$ in σ. As $\mathcal{S}$ is regular with respect to non-collinear points, this proves that σ contains exactly one line of the second type. Hence σ contains $s^2 + s + 1$ points of $\mathcal{S}$ and $s^2 + s$ lines of $\mathcal{S}$. From now on we will speak of a *punctured affine plane* instead of a mixed substructure.

6 The characterization of $\mathrm{H}_q^{n,m}$ by H. Cuypers

In [3] H. Cuypers characterized the $(q, q+1)$-geometry $\mathrm{H}_q^{n,m}$. His characterization theorem is in a certain sense more general than ours, but in another sense it is more restrictive. We will explain this more in detail, after a short description of his result.

A *delta space* $\mathcal{D}$ is a partial linear space (that has not necessarily an order), that satisfies the following axiom: for each antiflag (p, L) of $\mathcal{D}$, p is collinear with no, all but one or all points of L. To exclude some degenerate cases, it is assumed that each line has at least three points.

A subset X of the pointset of a delta space $\mathcal{D}$ is called a *subspace* if each line that contains two points of X, is completely contained in X. A subspace together with all the lines intersecting it in at least two points is again a partial linear space. Subspaces are usually identified with these partial linear spaces. The intersection of a set of subspaces is again a subspace. For a subset X of the point set of a delta space $\mathcal{D}$, we define $\langle X\rangle$ as the intersection of all the subspaces of $\mathcal{D}$ containing X, and call it the subspace of $\mathcal{D}$ *generated* by X. A *plane* is a subspace generated by two intersecting lines.

If a delta space is embeddable in a projective space, then it satisfies the axiom of Pasch and its planes can be embedded in a projective plane. It follows easily from the previous section that a delta space embedded in a projective plane is isomorphic to a projective plane, a dual affine plane or a projective plane from which a line is removed.

Theorem 6.1 ([3]) *Let $\mathcal{S}$ be a connected partial linear space (which has not necessarily an order). Suppose that all planes of $\mathcal{S}$ are projective or dual affine and that $\mathcal{S}$ contains at least two planes one of them isomorphic to a projective plane. Then $\mathcal{S}$ is isomorphic to $\mathrm{H}_q^{n,m}$.*

This theorem is stronger than our characterization theorem in the sense that the partial linear space $\mathcal{S}$ is not assumed to have an order. By connectedness it follows immediately that the number of points on a line is a constant and equal to $q+1$, where q is the order of the projective plane (which exists by assumption). The number of lines through a point is however not assumed to be constant. In our theorem, we start from an $(\alpha, s+1)$-geometry, which by definition has an order (s, t).

Moreover a priori it is possible in theorem 6.1 that for some antiflag (p, L), $i(p, L) = 0$. For an $(\alpha, s+1)$-geometry this cannot occur.

On the other hand, in theorem 6.1 the existence of mixed substructures is excluded, and for this reason the $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$ is not characterized by theorem 6.1. Also, our theorem is for $(\alpha, s+1)$-geometries, we do not assume from the start that $\alpha = s$.

Our characterization theorem can therefore be seen as a certain extension to theorem 6.1, although in some sense theorem 6.1 is more general than ours.

7 A characterization of $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$

In this section we prove our main characterization theorem.

Theorem 7.1 *Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s,t), $1 < \alpha < s+1 < t+1$, that satisfies the axiom of Pasch, that is regular with respect to non-collinear points, and such that there is at least one α-subgeometry. Then $\mathcal{S}$ is isomorphic to $\mathrm{H}_q^{n,m}$ or $\mathrm{SH}_q^{n,m}$.*

Proof. From lemma 4.1 it follows that $\alpha = s$.

Step 1. Let $\langle x, y\rangle$ and $\langle x, z\rangle$ be two distinct lines of the second type. We will construct a projective plane, an affine plane or a dual affine plane through the lines $\langle x, y\rangle$ and $\langle x, z\rangle$, such that the points in this plane are two by two non-collinear in $\mathcal{S}$.

Note that $y \not\sim z$ in $\mathcal{S}$. Indeed, if $y \sim z$ then on the line $\langle y, z\rangle$ of $\mathcal{S}$ there are at most $s-1$ points collinear with x. This is a contradiction as $\mathcal{S}$ is an $(s, s+1)$-geometry. In other words, non-collinearity is transitive.

Let L_x be a line of $\mathcal{S}$ through x. The plane $\langle L_x, y\rangle$ is a dual affine plane or a punctured affine plane. As $\langle x, y\rangle$ is contained in $\langle L_x, y\rangle$, it follows that $|\langle x, y\rangle| = s$ or $s+1$. In the same way it follows that $|\langle x, z\rangle| = s$ or $s+1$. We consider each possibility separately.

The case $|\langle x, y\rangle| = s$ and $|\langle x, z\rangle| = s$

Let L be a line of $\mathcal{S}$ through x. Then $\mathcal{S}(y, L)$ and $\mathcal{S}(z, L)$ are dual affine planes. As a dual affine plane contains exactly one line of the second type through x, $\mathcal{S}(y, L)$ and $\mathcal{S}(z, L)$ are distinct. Let M be a line of $\mathcal{S}$ through y in $\mathcal{S}(y, L)$. Let N be a line of $\mathcal{S}$ through z intersecting M in a point, such that $N \not\subset \mathcal{S}(z, L)$. Let $\mathcal{P}'$ be the set of points of the substructures $\mathcal{S}(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. As in lemma 4.1 it follows that each line that contains at least two elements of $\mathcal{P}'$, contains $s+1$ elements of $\mathcal{P}'$. Let $\mathcal{L}'$ be the set of lines of $\mathcal{S}$ intersecting $\mathcal{P}'$ in at least two points. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$, with I' the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$.

We will prove that all substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, are dual affine planes. To do so, we first prove that each contain a line of the second type.

By construction y and z belong to S', and $S(y, L) \neq S(z, L)$. This implies that no two points of the line $\langle y, z \rangle$ belong to the same substructure $S(z_j, L)$, for $z_j \in N$.

Let w be a point of the line $\langle y, z \rangle$ of the second type. If $w \sim x$, then $i(y, \langle w, x \rangle) \leq s - 1$, a contradiction as S is an $(s, s+1)$-geometry. Hence w is not collinear with x in S. As w was arbitrarily chosen, no point of $\langle y, z \rangle$ is collinear with x in S.

If $|\langle y, z \rangle| = s + 1$, then each of the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contains a line of the second type through x and a point of the line $\langle y, z \rangle$.

Assume now that $|\langle y, z \rangle| = s$. Each of the s substructures $S(L, z_i)$, $z_i \in N$, that contains a point of $\langle y, z \rangle$, contains a line of the second type through x. Let $S(L, z_k)$ be the remaining substructure through L and a point z_k of N. Denote the intersection point of M and L by u. Then $y, z, u \in S(M, N)$. Clearly $u \notin \langle y, z \rangle$, as $u \sim y$ and $\langle y, z \rangle$ is a line of the second type. Moreover $S(M, N)$ is a dual affine plane, as by assumption $|\langle y, z \rangle| = s$. As S is regular with respect to non-collinear points, each of the lines of S through u in $S(M, N)$ intersects the line $\langle y, z \rangle$. Hence $S(L, z_k)$ intersects $S(M, N)$ in the unique line of the second type through u in $S(M, N)$.

We conclude that each substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contains a line of the second type, where s of these lines are incident with x and one of them is incident with x or u.

Assume first that $s \neq 2$. We will prove that each substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contains at least two lines of the second type. Let $x' \in L$, $x \neq x' \neq u$. In the dual affine planes $S(y, L)$ and $S(z, L)$, there is a line of the second type through x'. We denote these lines by $\langle x', y' \rangle$ and $\langle x', z' \rangle$ respectively. As non-collinearity is transitive, the line $\langle y', z' \rangle$ is also a line of the second type. Let M' be a line of S through y' in $S(y, L)$ such that $u, x \notin M'$. Let N' be a line through z' intersecting M' in a point, such that N' is skew to L. Then in the same way as we did above (replace x, M, N, y, z by x', M', N', y', z'), it follows that either all $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contain a line of the second type through x', or s of them contain a line of the second type through x' and the remaining one contains a line of the second type through u', where $\{u'\} = L \cap M'$. From the construction it follows that $u' \in \{x, x', u\}$. In either case, all $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contain at least two distinct lines of the second type and hence it are all dual affine planes.

Assume next that $s = 2$. By assumption $S(y, L)$ and $S(z, L)$ are dual affine planes. Let $S(p, L)$, for $p \in N$, be the remaining substructure through L in S'.

Assume that there are three lines L, L' and L'' of S through x in $S(p, L)$. Then $S(y, L)$, $S(y, L')$ and $S(y, L'')$ contain the line $\langle x, y \rangle$ of the second type, and hence it are dual affine planes. A dual affine plane contains $s^2 + s = 6$ points, so we get that $|\mathcal{P}'| = 14$. However $S(y, L)$ and $S(z, L)$ contain together 9 different points of $\mathcal{P}'$. So $S(p, L)$ has to contain 8 points of $\mathcal{S}$. This is a contradiction, as a substructure $S(L, M)$ contains at most $s^2 + s + 1 = 7$ points of $\mathcal{S}$. So $S(p, L)$ contains a line of the second type through x.

Let x' be a point of L, $x' \neq x$. In the same way as above we prove that there is a line of the second type through x' in $S(p, L)$. Hence $S(p, L)$ contains at least two lines of the second type. This proves that $S(p, L)$ is a dual affine plane. So each substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, 2, 3)$, is a dual affine plane.

Now we will prove that all the substructures contained in $\mathcal{S}'$ are dual affine planes.

Assume first that there would be a projective plane ρ contained in $\mathcal{S}'$. From the above, it follows that ρ does not contain the line L. By definition of $\mathcal{S}'$, each point of ρ is contained in one of the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. Clearly ρ contains at least two points of a $S(L, z_j)$, for $z_j \in N$, as the $s^2 + s + 1$ points of $\mathcal{S}$ in ρ are contained in the $s + 1$ substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. The line through these two points intersects L in a point r. So $\rho \cap L = \{r\}$ and ρ intersects each $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$ in a line of $\mathcal{S}$.

Let w be a point of L, $w \neq r$. Each substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a dual affine plane, and hence it contains s lines of $\mathcal{S}'$ through w. Counting the lines of $\mathcal{S}'$ through w, we get that there are $(s+1)(s-1)+1 = s^2$ such lines.

A line through w and a point of ρ contains two points of $\mathcal{S}'$, hence it is a line of $\mathcal{S}'$. Conversely each line L_w of $\mathcal{S}'$ through w intersects ρ. Indeed, L_w is contained in a substructure $S(L, z_j)$, for $z_j \in N$. We proved above that ρ intersects $S(L, z_j)$ in a line of $\mathcal{S}$, and that in $S(L, z_j)$ each two lines of $\mathcal{S}$ intersect. It follows that the s^2 lines of $\mathcal{S}$ through w in $\mathcal{S}'$ intersect ρ in a point.

The projective plane ρ contains $s^2 + s + 1$ lines of $\mathcal{S}$. Assume that c of these lines contain s points collinear with w. Counting the flags (p, L_p), for $p \in \rho$ a point of $\mathcal{S}$ collinear with w and L_p a line of $\mathcal{S}$ in ρ, we get that $cs + (s^2 + s + 1 - c)(s+1) = (s+1)s^2$, or thus $c = s^2 + 2s + 1$. This is a contradiction, as by definition $c \leq s^2 + s + 1$. This proves that there is no projective plane contained in $\mathcal{S}'$.

Assume next that there is a punctured affine plane σ contained in $\mathcal{S}'$. Then σ contains exactly one line $\langle w_1, w_2 \rangle$ of the second type, with $|\langle w_1, w_2 \rangle| = s + 1$. From the above we know that L is not contained in σ. As in the case of a projective plane, one proves that σ contains a point r' of L. Let u be a point of L, $u \neq r'$. Through u there are s^2 lines of $\mathcal{S}$ contained in $\mathcal{S}'$, being the s lines through u in each dual affine plane $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. Clearly there are $s + 1$ lines through r' in σ, as $\langle w_1, w_2 \rangle$ cannot be contained in a dual affine plane. Each of these lines is contained in a different $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. Now all lines through u in $\mathcal{S}'$ are contained in the $S(L, z_i)$, for

$z_i \in N$ $(i = 1, \ldots, s+1)$, and so they intersect a line of σ through r' in a point. Hence all lines through u intersect σ.

There are $s^2 + s$ lines of $\mathcal{S}$ in σ. Let c be the number of lines of σ on which there are s points collinear with u. Let a be the number of points of σ, collinear with u, through which there are s lines of $\mathcal{S}$ in σ. Counting the flags (p, L_p), for $p \in \sigma$ a point of $\mathcal{S}$ collinear with u and L_p a line of $\mathcal{S}$ in σ, we get that $cs + (s^2 + s - c)(s+1) = as + (s^2 - a)(s+1)$, or thus $c = s^2 + s + a$. By definition $c \leq s^2 + s$, and hence $a = 0$. This implies that no point of the line $\langle w_1, w_2 \rangle$ of the second type is collinear with u.

Let u' be a point of L, $r' \neq u' \neq u$. In the same way as above we get that no point of $\langle w_1, w_2 \rangle$ is collinear with u'. As $\langle w_1, w_2 \rangle$ is contained in $\mathcal{S}'$, it follows that $w_1 \in \mathcal{P}'$. So $w_1 \in \mathcal{S}(L, z_k)$, for a $z_k \in N$. Hence the dual affine plane $\mathcal{S}(L, z_k)$ contains two lines of the second type through w_1, namely $\langle w_1, u \rangle$ and $\langle w_1, u' \rangle$. This is a contradiction. Hence there cannot be a punctured affine plane contained in $\mathcal{S}'$.

We conclude that each substructure contained in $\mathcal{S}'$ is a dual affine plane. There are $s+1$ dual affine planes through L in $\mathcal{S}'$. In each of them there is one line of the second type through x containing s points, so in total there are $(s+1)(s-1) = s^2 - 1$ points of $\mathcal{S}'$ that are not collinear with x. Now let $\mathcal{P}^*$ be the set of these $s^2 - 1$ points of $\mathcal{S}'$, together with x. As non-collinearity is transitive, every two elements of $\mathcal{P}^*$ are non-collinear in $\mathcal{S}$. As $\mathcal{S}'$ contains only dual affine planes, each line (of the second type) containing two elements of $\mathcal{P}^*$, contains s elements of $\mathcal{P}^*$. Let $\mathcal{L}^*$ be the set of lines of the second type containing at least two elements of $\mathcal{P}^*$. Let I^* be the natural incidence relation. Then $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is a $2 - (s^2, s, 1)$ design, i.e. an affine plane of order s. As an affine plane is generated by any of its triangles, $\mathcal{S}^*$ is independent of the choice of $\mathcal{S}'$.

The case $|\langle x, y \rangle| = s + 1$ and $|\langle x, z \rangle| = s + 1$

Let L be a line of $\mathcal{S}$ through x. Then $\mathcal{S}(y, L)$ and $\mathcal{S}(z, L)$ are punctured affine planes. Let M be a line of $\mathcal{S}$ through y in $\mathcal{S}(L, y)$, with $M \cap L = \{u\}$. Let N be a line of $\mathcal{S}$ through z intersecting M in y', $y' \neq u$. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be the incidence structure defined as follows: $\mathcal{P}'$ is the set of points of $\mathcal{S}$ in the substructures $\mathcal{S}(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, $\mathcal{L}'$ is the set of lines of $\mathcal{S}$ containing at least two elements of $\mathcal{P}'$, and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$.

Each point of the line $\langle y, z \rangle$ is contained in a distinct substructure $\mathcal{S}(L, z_i)$, for a $z_i \in N$ $(i \in \{1, \ldots, s+1\})$. Because of the transitivity of non-collinearity, each point of the line $\langle y, z \rangle$ is not collinear with x. Hence at least s substructures $\mathcal{S}(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contain a line of the second type through x.

If at least two substructures $\mathcal{S}(L, z_j)$ and $\mathcal{S}(L, z_k)$, for $z_j, z_k \in N$, $z_j \neq z_k$, are dual affine planes, then from the previous case it follows that each substructure in S' is a dual affine plane. This is a contradiction, as $S(L,z)$ is a punctured affine plane.

Hence at most one of the $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a dual affine plane, at least $s-1$ of these substructures are punctured affine planes, and at most one of these substructures is a projective plane. We deal with each of the possibilities separately.

Assume that exactly one of the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a dual affine plane and exactly one of them is a projective plane. In a dual affine plane there are $s^2 + s$ points of $\mathcal{S}$, in a projective plane and a punctured affine plane there are s^2+s+1 points of $\mathcal{S}$. So $|\mathcal{P}'| = s^3 + s^2 + s$.

The line $\langle x, y\rangle$ is a line of the second type contained in a punctured affine plane through L. Let $M_1, \ldots, M_{s+1} = L$ be the $s+1$ lines of $\mathcal{S}$ through x in the projective plane through L in $\mathcal{S}'$. Then each $S(y, M_i)$ $(i = 1, \ldots, s+1)$ is a punctured affine plane, as $|\langle x, y\rangle| = s+1$. Clearly each point of $S(y, M_i)$ $(i = 1, \ldots, s+1)$, is contained in $\mathcal{S}'$. Counting again the points of $\mathcal{S}'$, we get that $|\mathcal{P}'| \geq s^3 + s^2 + s + 1$, a contradiction.

Assume that exactly one of the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a projective plane and none of them is a dual affine plane. The existence of a projective plane $S(\tilde{z}, L)$, $\tilde{z} \in N$, shows that $|\langle y, z\rangle| = s$. Now let M_z be a line of $\mathcal{S}$ through z in $S(z, L)$, with $M_z \cap L = \{u\}$. Then $S(y, M_z)$ is a dual affine plane. So through u there are s lines of $\mathcal{S}$ in $S(y, M_z)$ and each of these lines contains a point of $\langle y, z\rangle$. The line of the second type through u in $S(y, M_z)$ therefore has to be contained in $S(\tilde{z}, L)$. This is a contradiction, as a projective plane contains no lines of the second type.

Assume that all the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, are punctured affine planes. Counting the points of $\mathcal{S}'$ we get that $|\mathcal{P}'| = s^3 + s^2 + s + 1$.

Let $L_1 = L, \ldots L_s$ be the s lines of $\mathcal{S}$ through x in $S(L, z)$. Then $S(y, L_i)$ are s punctured affine planes. Together they contain $s^3 + s + 1$ points of $\mathcal{S}'$. Hence exactly s^2 points of $\mathcal{S}'$ are not contained in a punctured affine plane $S(y, L_i)$ $(i = 1, \ldots, s)$. Let $\mathcal{P}^*$ be the set of these s^2 points and the $s+1$ points of the line $\langle x, y\rangle$.

We will prove that the points of $\mathcal{P}^*$ are two by two non-collinear. Let therefore p be a point of $\mathcal{P}^*$ and assume that $p \sim y$. Clearly $p \notin \langle x, y\rangle$. By definition of $\mathcal{P}^*$, the line $\langle y, p\rangle$ intersects $S(z, L)$ in a point of the line $\langle x, z\rangle$ of the second type. However, this implies that $i(x, \langle y, p\rangle) \leq s-1$, a contradiction as $\mathcal{S}$ is an $(s, s+1)$-geometry. Hence all points of $\mathcal{P}^*$ are non-collinear with y. By transitivity of the non-collinearity, it follows that every two points of $\mathcal{P}^*$ are non-collinear in $\mathcal{S}$. This implies that $\mathcal{P}^*$ is the set of all points of $\mathcal{S}'$ not collinear with x, union $\{x\}$. As $|\mathcal{P}^*| = s^2 + s + 1$, in each $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, exactly s points are not collinear with x.

We will now prove that $\mathcal{S}'$ contains no dual affine planes. Assume therefore that $\mathcal{S}'$ contains a dual affine plane π. By definition of $\mathcal{S}'$, the $s^2 + s$ points of $\mathcal{S}'$

in π are contained in the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. So there is a substructure $S(L, z_k)$, for $z_k \in N$, that contains two points of π. The line through these two points intersects L in a point u. Hence $\pi \cap L = \{u\}$. As π is a dual affine plane, we know that there is a line $\langle u, w\rangle$ of the second type through u, with $|\langle u, w\rangle| = s$. But $u \in L$, so $\langle u, w\rangle$ is contained in an $S(L, z_j)$, for a $z_j \in N$. This is in contradiction with the assumption. So $\mathcal{S}'$ cannot contain a dual affine plane. It follows that each line of the second type contained in $\mathcal{S}'$, contains $s+1$ points of $\mathcal{S}'$.

We define $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ as follows: $\mathcal{P}^*$ is as above, $\mathcal{L}^*$ is the set of lines (of the second type) containing at least two points of $\mathcal{P}^*$, and I^* is the natural incidence relation. Note that $\mathcal{L}^*$ is well defined because of the transtivity of non-collinearity. It is easy to see that $\mathcal{S}^*$ is a $2-(s^2+s+1, s+1, 1)$-design, i.e. a projective plane. As a projective plane is defined by any three of its points, $\mathcal{S}^*$ does not depend on the choice of $\mathcal{S}'$.

Assume that exactly one of the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$ is a dual affine plane and none of them is a projective plane. We will prove that through each point of $\mathcal{S}'$ there is a line of the second type on which there are s points of $\mathcal{S}'$. Let π be the dual affine plane through L. Through each point of π there is a line of the second type in π that contains s points of $\mathcal{S}$. Now let $p \in \mathcal{S}'$, $p \notin \pi$. By definition of $\mathcal{S}'$, p belongs to a punctured affine plane $S(z_j, L)$, for $z_j \in N$. So there is a line L_p through p intersecting L in a point u. Let $\langle u, u'\rangle$ be the line of the second type through u in π. Then $S(L_p, u')$ is a dual affine plane through p. So $S(L_p, u')$ contains a line of the second type through p on which there are s points of $\mathcal{S}'$. This shows that through each point of $\mathcal{S}'$ there is a line of the second type containing s points of $\mathcal{S}$.

Now assume that there would be two lines of the second type containing s points of $\mathcal{S}$ through a point $w \in \mathcal{S}'$. Let L_w be a line of $\mathcal{S}'$ through w. Then through L_w there are two dual affine planes contained in $\mathcal{S}'$. From the first part of the proof, it follows that all substructures contained in $\mathcal{S}'$ are dual affine planes, a contradiction. Hence through each point of $\mathcal{S}'$ there is exactly one line of the second type that contains s points of $\mathcal{S}'$.

Note that the line of the second type in each of the punctured affine planes through L in $\mathcal{S}'$ is incident with x. For the punctured affine planes containing a point of $\langle y, z\rangle$, this follows from transitivity of non-collinearity. Moreover if $|\langle y, z\rangle| = s$ and there would be a punctured affine plane $S(L, z_i)$, for $z_i \in N$, that contains no point of $\langle y, z\rangle$, then π would contain a point y' of $\langle y, z\rangle$. So the lines $\langle y', x\rangle$ and $\langle y', z\rangle$ are two lines of the second type through y' containing s points of $\mathcal{S}'$, a contradiction with what we proved above.

Let $\mathcal{P}^*$ be the set of points of $\mathcal{S}'$ that are not collinear with x, union the point x. As there are s punctured affine planes and one dual affine plane through L, it follows that there are s lines of the second type through x containing $s+1$ points of $\mathcal{S}$, while the remaining line of the second type through x contains s points of $\mathcal{S}$. Hence $|\mathcal{P}^*| = s^2 + s$. By transitivity of non-collinearity, we know that every two points of $\mathcal{P}^*$ are non-collinear.

Now we define an incidence structure $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ as follows: $\mathcal{P}^*$ is the set points of $\mathcal{S}'$ defined above, $\mathcal{L}^*$ is the set of all lines (of the second type) containing at least two points of $\mathcal{P}^*$, and I^* is the natural incidence relation. As through each point of $\mathcal{S}'$ there is at most one line of the second type containing s points of $\mathcal{S}$, $|\mathcal{P}^*| = s^2+s$. It follows that through each point of $\mathcal{P}^*$ there is exactly one line of $\mathcal{L}^*$ on which there are s points of $\mathcal{S}$. Now we add a new point $\tilde{w}$ to $\mathcal{P}^*$, and we define $\tilde{w}$ to be incident with each line of $\mathcal{L}^*$ that contains s points of $\mathcal{S}$. Then clearly this new incidence structure is a $2-(s^2+s+1, s+1, 1)$ design, i.e. a projective plane. As $\tilde{w}$ was not a point of $\mathcal{S}$, it follows that $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is a dual affine plane. A projective plane is uniquely defined by any three of its points, hence $\mathcal{S}^*$ is independent of the choice of $\mathcal{S}'$.

The case $|\langle x, y\rangle| = s$ and $|\langle x, z\rangle| = s+1$

Let L be a line of $\mathcal{S}$ through x. Then $S(L, y)$ is a dual affine plane, and $S(L, z)$ is a punctured affine plane. As is done in the previous two cases, we can construct a line N of $\mathcal{S}$ that intersects both $S(L, y)$ and $S(L, z)$ in a point not on L. Let $\mathcal{P}'$ be the set of all the points contained in the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$.

If there are two dual affine planes or two punctured affine planes $S(L, z_i)$ and $S(L, z_j)$, for $z_i, z_j \in N$, $z_i \neq z_j$, then we can apply one of the previous cases. So we may assume that all the substructures through L and a point of N, except for $S(L, y)$ and $S(L, z)$, are projective planes. However we proved above that at most one substructure $S(L, \tilde{z})$, for $\tilde{z} \in N$, is a projective plane. Hence $s = 2$.

Let $L_1 = L$, L_2 and L_3 be the three lines of $\mathcal{S}$ through x in $S(L, \tilde{z})$. The substructures $\langle y, L_i\rangle$, $(i = 1, 2, 3)$, contain the line $\langle x, y\rangle$ of the second type, with $|\langle x, y\rangle| = 2$, hence it are all dual affine planes. It follows that $|\mathcal{P}'| = 14$. The substructures $S(z, L_i)$ $(i = 1, 2, 3)$ contain the line $\langle x, z\rangle$ of the second type, with $|\langle x, z\rangle| = 3$, hence it are all punctured affine planes. So $|\mathcal{P}'| = 15$. This is a contradiction. Hence this case does not occur.

We now studied all the possibilities. The incidence structures $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$, with $\mathcal{P}^*$ as defined above, will be called planes of type IV, V and VI, when $\mathcal{S}^*$ is respectively a projective plane, an affine plane and a dual affine plane.

Step 2. We define parallelism among the lines of the second type containing s points of $\mathcal{S}$ as follows. Two lines of the second type containing s points of $\mathcal{S}$ are parallel if they coincide or if they are disjoint subsets of a dual affine plane or a plane of type V or VI.

Clearly the parallelism defined in this way is reflexive and symmetric. It remains to prove that it is also transitive. Let therefore $\langle x, y\rangle$, $\langle u, v\rangle$ and $\langle p, w\rangle$ be three lines of the second type containing s points of $\mathcal{S}$. Suppose that $\langle x, y\rangle$ is parallel to $\langle u, v\rangle$, that $\langle x, y\rangle$ is parallel to $\langle p, w\rangle$. If two of these lines coincide, then it follows immediately that $\langle u, v\rangle$ is parallel to $\langle p, w\rangle$. So we may assume that no two of them coincide.

From the definition of parallelism, it follows that the lines $\langle x, y \rangle$ and $\langle u, v \rangle$ are both contained in a dual affine plane, a plane of type V or a plane of type VI. Similarly, the lines $\langle x, y \rangle$ and $\langle p, w \rangle$ are both contained in a dual affine plane, a plane of type V or a plane of type VI. We have to consider three cases.

Assume that both the plane containing $\langle x, y \rangle$ and $\langle u, v \rangle$ and the one containing $\langle x, y \rangle$ and $\langle p, w \rangle$ are dual affine planes. Let π_1 (respectively π_2) be the dual affine plane containing $\langle x, y \rangle$ and $\langle u, v \rangle$ (respectively $\langle x, y \rangle$ and $\langle p, w \rangle$). If $\pi_1 = \pi_2$, then $\langle u, v \rangle$ and $\langle p, w \rangle$ are contained in the dual affine plane π_1. By definition of parallelism it follows that $\langle u, v \rangle$ and $\langle p, w \rangle$ are parallel. So we may assume that $\pi_1 \neq \pi_2$. In this case the lines $\langle u, v \rangle$ and $\langle p, w \rangle$ are clearly disjoint.

Let M be a line of $\mathcal{S}$ in π_1 and let N be a line of $\mathcal{S}$ in π_2 skew to M. We define an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows: $\mathcal{P}'$ is the set of points contained in the substructures $\mathcal{S}(M, z_i)$, with $z_i \in N$ $(i = 1, \ldots, s+1)$; $\mathcal{L}'$ is the set of lines containing at least two (and hence $s+1$) points of $\mathcal{P}'$, I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. By construction π_1 and π_2 are contained in $\mathcal{S}'$.

Let L_p be a line of $\mathcal{S}$ through p in π_2. Then L_p intersects $\langle x, y \rangle$ in a point w'. Let $L_{w'}$ be a line of $\mathcal{S}$ in π_1 through w'. The substructure $\mathcal{S}(L_p, L_{w'})$ is a dual affine plane, a projective plane or a punctured affine plane. So it contains s or $s+1$ lines of $\mathcal{S}$ through p and at most one line of the second type. We denote these lines by $N_1 = L_p, \ldots, N_{s+1}$, where N_{s+1} can be a line of $\mathcal{S}$ or a line of the second type. The substructures $\mathcal{S}(w, N_i)$ $(i = 1, \ldots, s)$ are s dual affine planes, as each of them contains the line $\langle p, w \rangle$. The substructure $\mathcal{S}(w, N_{s+1})$, is a dual affine plane, a plane of type V or VI.

Clearly $\mathcal{S}(w, N_i)$ $(i = 1, \ldots, s+1)$ intersect π_1 in a line. Assume that $\mathcal{S}(w, N_j)$, $j \in \{1, \ldots, s+1\}$ intersects π_1 in a line L of $\mathcal{S}$. Then L intersects $\langle x, y \rangle$ in a point u'. As $\langle p, w \rangle$ has to intersect each line of $\mathcal{S}$ in $\mathcal{S}(w, N_j)$, $\langle p, w \rangle$ intersects L and hence $\langle x, y \rangle$ in the point u'. This is a contradiction as $\langle p, w \rangle$ and $\langle x, y \rangle$ are parallel. The plane π_1 contains exactly $s+1$ lines of the second type. Hence $\langle u, v \rangle$ is one of the lines of the second type contained in either $\mathcal{S}(w, N_k)$, for a $k \in \{1, \ldots, s\}$, or in the plane $\mathcal{S}(w, N_{s+1})$, which can be a dual affine plane, or a plane of type V or VI. By definition of parallelism, it follows in each case that $\langle p, w \rangle$ is parallel to $\langle u, v \rangle$.

Assume that the plane containing $\langle x, y \rangle$ and $\langle u, v \rangle$ is a dual affine plane but the plane containing $\langle x, y \rangle$ and $\langle p, w \rangle$ is not a dual affine plane. Then $\langle x, y \rangle$ and $\langle p, w \rangle$ are contained in a plane of type V or type VI. We call this plane of type V or VI the plane ω, while we call the dual affine plane through $\langle x, y \rangle$ and $\langle u, v \rangle$ the plane π. Let N be a line of $\mathcal{S}$ in π. Let M be a line of $\mathcal{S}$ through the point $p \in \omega$ that intersects N in a point. Let M' be a line of $\mathcal{S}$ intersecting both $\mathcal{S}(M, x)$ and $\mathcal{S}(M, y)$ in a point not on M. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be the incidence structure defined as follows: $\mathcal{P}'$ is the set of points of $\mathcal{S}$ contained in the substructures $\mathcal{S}(M, x_i)$, for $x_i \in M'$ $(i = 1, \ldots, s+1)$, $\mathcal{L}'$ is the set of lines of $\mathcal{S}$ containing at least two points of $\mathcal{P}'$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Then

S' contains both π and ω, as it contains three points of each of them.

The line $\langle x,p\rangle$ is a line of the second type through x in ω. Let N_x be a line of $\mathcal{S}$ through x in π. Then $\mathcal{S}(p,N_x)$ is a dual affine plane or a punctured affine plane. So p is collinear with the s points of N_x different from x. Let $M_1,\dots M_s$ be the s lines of $\mathcal{S}$ through p and a point of N_x. Then each $\mathcal{S}(w,M_i)$ $(i=1,\dots,s)$ is a dual affine plane containing the line $\langle p,w\rangle$, and as in the previous case one proves that each such substructure intersects π in a line of the second type different from $\langle x,y\rangle$. Now in the dual affine plane π there are exactly $s+1$ lines of the second type. So $\langle u,v\rangle$ has to be one of the lines contained in a dual affine plane $\mathcal{S}(w,M_i)$, for a $i\in\{1,\dots,s\}$. So $\langle w,p\rangle$ is parallel to $\langle u,v\rangle$.

Assume that none of the two planes containing $\langle x,y\rangle$ and $\langle u,v\rangle$ respectively containing $\langle x,y\rangle$ and $\langle p,w\rangle$ is a dual affine plane. In this case the points of the lines $\langle x,y\rangle$, $\langle p,w\rangle$ and $\langle u,v\rangle$ belong to an equivalence class C of non-collinear points of $\mathcal{S}$. Suppose that $z\notin C$. Then z is collinear with each point of C. In particular z is collinear with x. The plane $\mathcal{S}(\langle z,x\rangle,y)$ is a dual affine plane. It contains a line of the second type through z that is parallel to $\langle x,y\rangle$. We denote this line by $\langle z,z'\rangle$. From the preceding case it follows that $\langle z,z'\rangle$ is parallel to both $\langle u,v\rangle$ and $\langle p,w\rangle$, and these lines are two by two disjoint.

As by assumption the plane containing $\langle x,y\rangle$ and $\langle p,w\rangle$ is not a dual affine plane, the line $\langle x,p\rangle$ is a line of the second type. So $\mathcal{S}(\langle z,x\rangle,p)$ is a dual affine plane or a punctured affine plane. Let L_p be a line of $\mathcal{S}$ through p in $\mathcal{S}(\langle z,x\rangle,p)$. Let L_y be a line of $\mathcal{S}$ through y in $\mathcal{S}(\langle z,x\rangle,y)$, such that L_y is skew to L_p. Then we can again define an incidence structure $\mathcal{S}'=(\mathcal{P}',\mathcal{L}',\mathrm{I}')$ as follows: $\mathcal{P}'$ is the set of points contained in the $\mathcal{S}(L_p,z_i)$, with $z_i\in L_y$ $(i=1,\dots,s+1)$; $\mathcal{L}'$ is the set of lines containing at least two (and hence $s+1$) points of $\mathcal{P}'$, I' is the restriction of I to $(\mathcal{P}'\times\mathcal{L}')\cup(\mathcal{L}'\times\mathcal{P}')$. As the plane containing $\langle x,y\rangle$ and $\langle p,w\rangle$ is a plane of type V or VI, and as from the above we know that there is exactly one plane of type V or VI in $\mathcal{S}'$ through the point p, it follows that the plane containing $\langle z,z'\rangle$ and $\langle p,w\rangle$ is a dual affine plane. In the same way one proves that $\langle z,z'\rangle$ and $\langle u,v\rangle$ are contained in a dual affine plane. From a preceding case it now follows that $\langle p,w\rangle$ and $\langle u,v\rangle$ are parallel.

So we proved that the parallelism defined above is an equivalence relation. Note that each parallel class is a partition of the point set of $\mathcal{S}$. The parallel classes, which we denote by $[\langle u,v\rangle]$, are called points of the second type, and the set of these classes is denoted by $\mathcal{P}^*$.

Step 3. We will define parallelism among the planes of type V.

Suppose that ω is a plane of type V. Let x be a point of $\mathcal{S}$, $x\notin\omega$. As the parallel classes of the lines of the second type containing s points of $\mathcal{S}$, partition the points of $\mathcal{S}$, there are $s+1$ lines $\langle x,y_1\rangle,\dots,\langle x,y_{s+1}\rangle$ that are parallel to lines of ω. We will prove that the lines $\langle x,y_1\rangle,\dots,\langle x,y_{s+1}\rangle$ are contained in a plane of type V.

Assume first that there is a line L of $\mathcal{S}$ containing x and a point u of ω. Let $\langle u,u'\rangle$ and $\langle u,u''\rangle$ be two lines through u in ω. Then $|\langle u,u'\rangle|=s$ and

$|\langle u, u''\rangle| = s$. Hence $S(L, u')$ and $S(L, u'')$ are dual affine planes. Let N be a line of $\mathcal{S}$ intersecting $S(L, u')$ and $S(L, u'')$ in a point not on L. We define an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows: $\mathcal{P}'$ is the set of points of $\mathcal{S}$ contained in $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, $\mathcal{L}'$ is the set of lines of $\mathcal{S}$ containing at least two (and hence $s+1$) points of $\mathcal{P}'$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$.

It is clear that x belongs to $\mathcal{S}'$. Also $S(L, u')$ and $S(L, u'')$ belong to $\mathcal{S}'$ and hence $\mathcal{S}'$ contains three distinct points u, u' and u'' of ω. This shows that ω is contained in $\mathcal{S}'$.

From the previous part of the proof, it follows that each substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a dual affine plane. As ω is contained in $\mathcal{S}'$, each $S(L, z_i)$ intersects ω in a line $\langle u, p_i\rangle$ of the second type through u. Hence each dual affine plane $S(L, z_i)$ contains a line of the second type through x parallel to $\langle u, p_i\rangle$. As $\langle x, y_i\rangle$ is the unique line through x parallel to $\langle u, p_i\rangle$, $\langle x, y_i\rangle$ is contained in $S(L, z_i)$. Hence the lines $\langle x, y_1\rangle, \ldots, \langle x, y_{s+1}\rangle$ are each contained in a distinct dual affine plane $S(L, z_i)$, for $z_i \in N$.

Let $\mathcal{P}''$ be the set of points of $\mathcal{S}'$ that are not collinear with x, union $\{x\}$. Then $\mathcal{P}''$ is the set of points of the lines $\langle x, y_1\rangle, \ldots \langle x, y_{s+1}\rangle$. Let $\mathcal{L}''$ be the set of lines of the second type containing at least two points of $\mathcal{P}''$. Let I'' be the restriction of I' to $(\mathcal{P}'' \times \mathcal{L}'') \cup (\mathcal{L}'' \times \mathcal{P}'')$. Then $\mathcal{S}''$ is a plane of type V. So we have proved that there is a plane of type V through x, with point set the points of the lines $\langle x, y_1\rangle, \ldots, \langle x, y_{s+1}\rangle$.

Next we assume that there is no line of $\mathcal{S}$ through x and a point of ω. In this case x and the points of ω belong to an equivalence class C of non-collinear points in $\mathcal{S}$. Suppose that $v \notin C$. Then v is collinear with x and with each point of ω. Let $\langle v, z_1\rangle, \ldots, \langle v, z_{s+1}\rangle$ be the lines containing v and parallel to $\langle x, y_1\rangle, \ldots, \langle x, y_{s+1}\rangle$. From the preceding case it follows that the points on these lines are the points of a plane of type V. Indeed, $\langle v, z_1\rangle, \ldots, \langle v, z_{s+1}\rangle$ are parallel to lines of ω and v is collinear in $\mathcal{S}$ with each point of ω. As $\langle v, x\rangle$ is a line of $\mathcal{S}$, the same argument shows that the points on $\langle x, y_1\rangle, \ldots, \langle x, y_{s+1}\rangle$ are the points of a plane of type V.

So we have proved that the points of the lines $\langle x, y_1\rangle, \ldots, \langle x, y_{s+1}\rangle$ are the points of a plane of type V. We define parallelism among planes of type V as follows: two planes ω and ω' of type V are parallel if some line of the second type in ω is parallel to a line of the second type in ω'. From the definition of parallelism for lines of the second type containing s points of $\mathcal{S}$, it follows that the new defined parallelism is an equivalence relation. Each parallel class of planes of type V partitions the point set of $\mathcal{S}$. The parallel classes, which we denote by $[\omega]$, are called lines of the third type, and the set of these classes is denoted by $\mathcal{L}^*$.

Step 4. We introduce a new incidence structure $\overline{\mathcal{S}} = (\overline{\mathcal{P}}, \overline{\mathcal{L}}, \overline{\mathrm{I}})$, with $\overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{P}^*$, with $\overline{\mathcal{L}}$ the set of all lines of $\mathcal{S}$, all lines of the second type and all lines of the third type and with incidence relation $\overline{\mathrm{I}}$ defined as follows.

1. for $x \in \mathcal{P}$ and $L \in \mathcal{L}$: $x \,\bar{\mathrm{I}}\, L \iff x \,\mathrm{I}\, L$;

2. for $x \in \mathcal{P}$ and $\langle y, z\rangle$ a line of the second type $x \,\bar{\mathrm{I}}\, \langle y, x\rangle \iff x \in \langle y, z\rangle$;

3. for $x \in \mathcal{P}$ and $[\omega] \in \mathcal{L}^*$: x is not incident with $[\omega]$;

4. for $[\langle y, z\rangle] \in \mathcal{P}^*$ and $L \in \mathcal{L}$: $[\langle y, z\rangle]$ is not incident with L;

5. for $[\langle y, z\rangle] \in \mathcal{P}^*$ and $\langle u, v\rangle$ a line of the second type:
   * If $|\langle u, v\rangle| = s$: $[\langle y, z\rangle] \,\bar{\mathrm{I}}\, \langle u, v\rangle \iff \langle u, v\rangle \in [\langle y, z\rangle]$;
   * If $|\langle u, v\rangle| = s + 1$: $[\langle y, z\rangle]$ is not incident with $\langle u, v\rangle$;

6. $[\langle y, z\rangle] \in \mathcal{P}^*$ and $[\omega] \in \mathcal{L}^*$: $[\langle y, z\rangle] \,\bar{\mathrm{I}}\, [\omega] \iff \langle y, z\rangle$ is parallel to a line in ω.

Step 5. We prove that $\overline{\mathcal{S}}$ is the design of points and lines of a projective space.

We first prove that each two distinct points of $\overline{\mathcal{S}}$ are incident with exactly one line of $\overline{\mathcal{S}}$.

Assume that $p_1, p_2 \in \mathcal{P}, p_1 \neq p_2$. Then either p_1 is collinear in $\mathcal{S}$ with p_2, in which case the line of $\mathcal{S}$ through p_1 and p_2 is the unique line of $\overline{\mathcal{S}}$ through p_1 and p_2; or p_1 and p_2 are not collinear in $\mathcal{S}$, in which case the line of the second type containing them is the unique line of $\overline{\mathcal{S}}$ through p_1 and p_2.

Assume that $p_1 \in \mathcal{P}$, $[\langle x, y\rangle] \in \mathcal{P}^*$. In this case the unique line of $\overline{\mathcal{S}}$ through p_1 and $[\langle x, y\rangle]$ is the line of the second type through p_1 that belongs to the parallel class of $\langle x, y\rangle$.

Assume that $[\langle x, y\rangle], [\langle u, v\rangle] \in \mathcal{P}^*$, $[\langle x, y\rangle] \neq [\langle u, v\rangle]$. If $\langle x, y\rangle$ and $\langle u, v\rangle$ have a point in common, then x, y, u, v are contained in a plane ω of type V. Note that ω cannot be a plane of type VI, because in a plane of type VI all lines of the second type containing s points of $\mathcal{S}$ belong to the same parallel class. So $[\langle x, y\rangle]$ and $[\langle u, v\rangle]$ are two points of the line $[\omega]$ and there is no other line in $\overline{\mathcal{S}}$ containing both these points. If $\langle x, y\rangle$ and $\langle u, v\rangle$ have no point in common, then we can choose a line in $[\langle x, y\rangle]$ that does have a point in common with $\langle u, v\rangle$ (namely the line through u and $[\langle x, y\rangle]$). So the same argument as before shows that $[\langle x, y\rangle]$ and $[\langle u, v\rangle]$ are on exactly one line of $\overline{\mathcal{S}}$.

Next we prove that every three distinct points of $\overline{\mathcal{S}}$, that are not incident with a common element of $\overline{\mathcal{L}}$, generate a projective plane. From the definition of $\overline{\mathcal{S}}$ it follows that a dual affine plane and a punctured affine plane induce projective planes. Also planes of type IV, V and VI are projective planes, containing no lines of $\mathcal{S}$ but lines of the second type and lines of $\mathcal{L}^*$. We have to consider the following cases.

Assume that $p_1, p_2, p_3 \in \mathcal{P}$. Then clearly there is either a dual affine plane, a projective plane, a punctured affine plane or a plane of type IV, V or VI containing p_1, p_2 and p_3. Hence in any case p_1, p_2 and p_3 are in a projective plane.

Assume that p_1, $p_2 \in \mathcal{P}$ and $[\langle x, y\rangle] \in \mathcal{P}^*$. The lines $\langle p_1, [\langle x, y\rangle]\rangle$ and $\langle p_2, [\langle x, y\rangle]\rangle$ are lines of the second type containing s points of $\mathcal{S}$.

If $\langle p_1, p_2\rangle$ is a line of $\mathcal{S}$, then the points of $\mathcal{S}$ in $S(\langle p_1, p_2\rangle, [\langle x, y\rangle])$ are the points of a dual affine plane, and hence in $\overline{\mathcal{S}}$ it is a projective plane.

If $\langle p_1, p_2\rangle$ is a line of the second type containing s points of $\mathcal{S}$, then let L_{p_1} be a line of $\mathcal{S}$ through p_1. The substructures $S(L_{p_1}, p_2)$ and $S(L_{p_1}, [\langle x, y\rangle])$ are both dual affine planes. Let M be a line of $\mathcal{S}$ intersecting $S(L_{p_1}, p_2)$ and $S(L_{p_1}, [\langle x, y\rangle])$ in a point not on L_{p_1}. We define an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows. Let $\mathcal{P}'$ be the set of points of $\mathcal{S}$ contained in $S(L_p, z_i)$, for z_i a point of M, let $\mathcal{L}'$ be the set of lines of $\mathcal{S}$ containing at least two points of $\mathcal{P}'$ and let I' be the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. As $\mathcal{S}'$ contains the dual affine planes $S(L_{p_1}, p_2)$ and $S(L_{p_1}, [\langle x, y\rangle])$, one proves as in the first part of the proof that there is a plane of type V through p_1, p_2 and $[\langle x, y\rangle]$.

If $\langle p_1, p_2\rangle$ is a line of the second type containing $s + 1$ points of $\mathcal{S}$, then as in the previous case we can define an incidence structure $\mathcal{S}'$. It then follows that there is a plane of type VI through p_1, p_2 and $[\langle x, y\rangle]$.

Assume that $p \in \mathcal{P}$ and $[\langle x, y\rangle]$, $[\langle u, v\rangle] \in \mathcal{P}^*$. The lines $\langle p, [\langle x, y\rangle]\rangle$ and $\langle p, [\langle u, v\rangle]\rangle$ are lines of the second type that contain s points of $\mathcal{S}$. We have proved above that there is a line $[\omega]$ of the third type, that contains both $[\langle x, y\rangle]$ and $[\langle u, v\rangle]$. We have also proved that each point of $\mathcal{S}$ belongs to a plane ω' of type V with parallel class $[\omega]$. Hence there is a plane ω_p of type V through p with parallel class $[\omega]$. The plane ω_p is a projective plane containing the points p, $[\langle x, y\rangle]$ and $[\langle u, v\rangle]$.

Assume that $[\langle x, y\rangle]$, $[\langle u, v\rangle]$, $[\langle p, q\rangle] \in \mathcal{P}^*$. Let w be a point of $\mathcal{P}$. The line through w and $[\langle x, y\rangle]$ (respectively $[\langle p, q\rangle]$ and $[\langle u, v\rangle]$) is a line of the second type that contains s points of $\mathcal{S}$. Let r_1 (respectively r_2 and r_3) be a point of $\mathcal{S}$ on this line, that is distinct from w. Let L be a line of $\mathcal{S}$ through w. The substructures $S(L, r_1)$ and $S(L, r_2)$ are both dual affine planes. As before, we can prove that the points w, r_1 and r_2 are contained in a plane ω of type V. Hence $\langle r_1, r_2\rangle$ is a line of the second type containing s points of $\mathcal{S}$, while $[\langle x, y\rangle]$ and $[\langle p, q\rangle]$ are both contained in the line $[\omega]$ that is an element of $\mathcal{L}^*$. Moreover for each point z of the line $\langle r_1, r_2\rangle$, $z \in \mathcal{P}$, the line $\langle w, z\rangle$ is a line of the second type containing s points of $\mathcal{S}$ and the point $[\langle w, z\rangle]$ belongs to the line $[\omega]$. Analougously, the substructures $S(L, r_1)$ and $S(L, r_3)$ are both dual affine planes, and hence it follows that the plane through w, r_1 and r_3 is a plane ω' of type V. The line $[\omega']$ is an element of $\mathcal{L}^*$, and it intersects $[\omega]$ in the point $[\langle w, r_1\rangle]$.

Let $\rho = (\mathcal{P}_w, \mathcal{L}_w, \mathrm{I}_w)$ be the incidence structure defined as follows. The point set $\mathcal{P}_w$ is the set of points of $\mathcal{P}^*$ that lie on a line of $\mathcal{L}^*$ that intersects both $[\omega]$ and $[\omega']$ in a point, together with the points of the lines of $\mathcal{L}^*$ through $[\langle x, y\rangle]$ that intersect a line $[\omega'']$ in a point, $[\omega'']$ being a line of $\mathcal{L}^*$ that intersects $[\omega]$ and $[\omega']$ in a distinct point. The line set $\mathcal{L}_w$ is the set of all lines of $\mathcal{L}^*$ containing two points of $\mathcal{P}^*$. Note that from the definition of $\mathcal{P}_w$ it follows that each point of such a line belongs to $\mathcal{P}_w$. Finally I_w is the restriction of I^* to $(\mathcal{P}_w \times \mathcal{L}_w) \cup (\mathcal{L}_w \times \mathcal{P}_w)$.

We will now prove that ρ is a projective plane. Let π be the projective plane through r_1, r_2 and r_3. Note that we proved in a previous case that π exists. Let z'

be an arbitrary point of $\mathcal{S}$ in the plane π.

If z' is a point of $\langle r_1, r_2\rangle$, then we know that the line $\langle w, z'\rangle$ contains a point of the line $[\omega]$, which is a point of $\mathcal{P}_w$. Assume now that $z' \notin \langle r_1, r_2\rangle$. The line $\langle z', r_3\rangle$ either intersects $\langle r_1, r_2\rangle$ in a point of $\mathcal{S}$, or it is parallel to $\langle r_1, r_2\rangle$, in which case it is a line of the second type containing s points of $\mathcal{S}$.

Assume first that $\langle z', r_3\rangle$ intersects $\langle r_1, r_2\rangle$ in a point z'' of $\mathcal{S}$. Then z'' is a point of the plane ω of type V, hence $\langle w, z''\rangle$ is a line of the second type containing s points of $\mathcal{S}$. Also $\langle w, r_3\rangle$ is a line of the second type containing s points of $\mathcal{S}$. Hence the substructures $\mathcal{S}(L, z'')$ and $\mathcal{S}(L, r_3)$ are both dual affine planes. As we did before, we can prove that the plane through w, z'' and r_3 is a plane ω^* of type V. It follows that $\langle w, z'\rangle$ is a line of the second type containing s points of $\mathcal{S}$. The line $[\omega^*]$ intersects $[\omega]$ in the point $[\langle w, z''\rangle]$, and it intersects $[\omega']$ in the point $[\langle w, r_3\rangle]$. Hence $[\omega^*]$ is a line of $\mathcal{L}_w$. The point $[\langle w, z'\rangle]$ lies on this line, hence it is a point of $\mathcal{P}_w$.

Assume next that $\langle z', r_3\rangle$ and $\langle r_1, r_2\rangle$ are parallel. Then these two lines both contain the point $[\langle z', r_3\rangle]$ of $\mathcal{P}^*$. The line through w and $[\langle z', r_3\rangle]$ is a line of the second type containing s points of $\mathcal{S}$. Let $\tilde{z}$ be a point of $\mathcal{S}$ on this line, $\tilde{z} \neq w$. The substructures $\mathcal{S}(L, \tilde{z})$ and $\mathcal{S}(L, r_3)$ are both dual affine planes. It follows that the plane containing w, r_3 and $\tilde{z}$ is a plane $\tilde{\omega}$ of type V. The line $[\tilde{\omega}]$ is an element of $\mathcal{L}^*$. This line intersects $[\omega]$ in the point $[\langle z', r_3\rangle]$, while it intersects $[\omega']$ in the point $[\langle w, r_3\rangle]$. Hence $[\tilde{\omega}]$ is an element of $\mathcal{L}_w$, and the point $[\langle w, z'\rangle]$ is an element of $\mathcal{P}_w$.

So with each point z of the plane $\langle r_1, r_2, r_3\rangle$ there corresponds a point $[\langle w, z\rangle]$ of ρ, and this point is unique. This proves that ρ is isomorphic to the projective plane $\langle r_1, r_2, r_3\rangle$. It follows also that ρ is a projective plane, and hence $[\langle x, y\rangle]$, $[\langle p, q\rangle]$ and $[\langle u, v\rangle]$ generate a projective plane.

We conclude that $\overline{\mathcal{S}}$ is the design of points and lines of a projective space $\mathrm{PG}(n, s)$. As each two distinct points of $\mathcal{S}^*$ generate a line of $\mathcal{S}^*$, it follows that $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is the design of points and lines of a projective subspace $\Psi[m]$ of $\mathrm{PG}(n, s)$. As not every line of $\overline{\mathcal{S}}$ contains a point of $\mathcal{S}^*$, it is clear that $m \leq n - 2$.

Assume that there is no punctured affine plane. The lines of $\overline{\mathcal{S}}$ are the lines of $\mathcal{S}$, the lines of the second type containing s points of $\mathcal{S}$ and a point of $\mathcal{S}^*$, and the lines of the form $[\omega]$, with ω a plane of type V. So $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n, s) \setminus \Psi[m]$, $\mathcal{L}$ is the set of all lines skew to $\Psi[m]$ and I is the incidence of $\mathrm{PG}(n, s)$. This proves that $\mathcal{S}$ is isomorphic to $\mathrm{H}_q^{n,m}$, $0 < m < n - 2$.

Assume next that there is a punctured affine plane. Then $\overline{\mathcal{S}}$ contains lines on which there are $s + 1$ points of $\mathcal{S}$, that are not lines of $\mathcal{S}$. Let $\mathcal{B}$ be the set of all these lines. As the points of $\mathcal{S}^*$ are the points of the subspace $\Psi[m]$, the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ that contain a point of $\mathcal{S}^*$, is a constant. As there are $t + 1$ lines of $\mathcal{S}$ through each point of $\mathcal{S}$, the number of lines of $\mathcal{B}$ through a point of $\mathcal{S}$ is also a constant.

A punctured affine plane contains exactly one line of $\mathcal{B}$. Neither a dual affine plane, nor a projective plane can contain a line of $\mathcal{B}$. Hence a plane that contains two lines of $\mathcal{B}$, cannot contain a line of $\mathcal{S}$. This proves that the lines of $\mathcal{B}$ through a point x of $\mathcal{S}$ are the lines through x in an r-dimensional subspace $\Pi_x[r]$ of $\mathrm{PG}(n, s)$, and this subspace contains no lines of $\mathcal{S}$ through x. It immediately follows that the subspace $\Pi_x[r]$ can not contain lines of $\mathcal{S}$, as otherwise on such a line L of $\mathcal{S}$ there would be no points that are collinear with x, a contradiction as $\mathcal{S}$ is an $(s, s+1)$-geometry.

Let y be a point of $\Pi_x[r]$, y different from x. Then the subspace $\Pi_y[r']$ coincides with $\Pi_x[r]$. Indeed, all lines through y in $\Pi_x[r]$ are lines that do not belong to $\mathcal{S}$, so surely $\Pi_x[r] \subset \Pi_y[r']$. Now assume that $\Pi_y[r']$ is not a subspace of $\Pi_x[r]$. Then $\Pi_y[r]$ would contain a line L of $\mathcal{S}$ through x. By definition of $\Pi_y[r']$, it follows that no point of L is collinear with y in $\mathcal{S}$, again a contradiction. This proves that $\Pi_y[r'] = \Pi_x[r]$. Hence for every point z of $\mathcal{S}$, the dimension of $\Pi_z[r'']$ is r. We will prove that $\Psi[m] \subset \Pi_z[r]$, for each point z of $\mathcal{S}$. Indeed, if $\Psi[m]$ would not be contained in $\Pi_z[r]$, then there would be a line N of $\mathcal{B}$ through z and a line $\langle z, u\rangle$, with u a point of $\Psi[m]$, such that the plane through u and N contains a line of $\mathcal{S}$ through z. This is a contradiction, as such a plane cannot exist. Hence $\Psi[m] \subset \Pi_z[r]$, for each point z of $\mathcal{S}$, and thus $r \geq m+2$. Two subspaces $\Pi_{p_1}[r]$ and $\Pi_{p_2}[r]$ either coincide, or they have no point of $\mathcal{S}$ in common. Hence the subspaces $\Pi_z[r]$, for $z \in \mathcal{S}$, partition the points of $\mathrm{PG}(n, s) \setminus \Psi[m]$. We conclude that $\mathcal{S}$ is isomorphic to $\mathrm{SH}_q^{n,m}$. □

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