ON THE SPECTRUM OF CRITICAL SETS IN BACK CIRCULANT LATIN SQUARES

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Abstract: In this paper we prove there exists a strong critical set of size m in the back circulant latin square of order n for all $\frac{n^2-1}{4} \le m \le \frac{n^2-n}{2}$, when n is odd. Moreover, when n is even we prove that there exists a strong critical set of size m in the back circulant latin square of order n for all $\frac{n^2-n}{2} - (n-2) \le m \le \frac{n^2-n}{2}$ and $m \in \{\frac{n^2}{4}, \frac{n^2}{4}+2, \frac{n^2}{4}+4, \dots, \frac{n^2-n}{2} - n\}$.

1. Background information

In any finite combinatorial configuration it is possible to identify defining subsets which uniquely determine the structure of the configuration and in some cases are minimal with respect to this property. For instance, consider the back circulant latin square B_n corresponding to the cyclic group of order n. That is, an $n \times n$ array in which cell (i,j), $0 \le i,j \le n-1$, contains the symbol $i+j \pmod n$. The papers [6] by Curran and van Rees, [5] by Cooper, Donovan and Seberry, and [7] by Donovan and Cooper all deal with identifying partially filled-in latin squares, called critical sets, which uniquely determine the back circulant latin square and which are minimal with respect to this property. Recently Cavenagh ([4]) has shown that the size of a critical set in B_n must be at least $O(n^{4/3})$. It is conjectured in [1] that the smallest possible size for a critical set in B_n (in fact for any latin square) is $\lfloor n^2/4 \rfloor$. More recently Bedford and Johnson, [3] and [9], have established the existence of weak critical sets (critical sets with a certain type of completion) in back circulant latin squares.

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In all of these papers the authors exploit the cyclic nature of the latin square. More specifically, for $0 \le r, i, j \le n-1$, let d_r denote the set of cells (i, j) which contain the symbol r where $i + j = r$ and d_{n+r} denote the set of cells (i, j) which contain the symbol r, where $i + j = n + r$. Then in the majority of results given above, the defining subset is given by $(\cup_{r=0}^{a} d_r) \cup (\cup_{r=a+1}^{n-2} d_{n+r})$ (where a is between 0 and $n-2$) or a closely related set. This leads to a general question: For which partitions $\mathcal{P} = \{I, J\}$ of the set $\{0, 1, 2, \ldots, n-2\}$ does $(\cup_{i\in I} d_i) \cup (\cup_{j\in J} d_{n+j})$ uniquely determine the structure of the back circulant latin square and which partitions are minimal with respect to this property? The current paper sheds light on the problem, while constructing new critical sets of order n and size $(n^2 - 1)/4, \ldots, (n^2 - n)/2$ when n is odd and $n^2/4, n^2/4 + 2, \ldots, (n^2 - n)/2 - n, (n^2 - n)/2 - n + 1, \ldots, (n^2 - n)/2$ when n is even. The existence of critical sets of these sizes in latin squares was first verified by Donovan and Howse in [8]. (See also [2].) The results in this paper, however, apply specifically to back circulant latin squares. In addition, new constructions of latin trades have rendered proofs more transparent.

2. Definitions

We start with basic definitions which allow us to state and prove our main results, Theorems 35, 38, 41 and 43.

Let $N = \{0, 1, \ldots, n-1\}$. A *partial latin square* P of order n is an $n \times n$ array with rows and columns indexed by N and entries chosen from N in such a way that each element of N occurs at most once in each row and at most once in each column of the array. For ease of exposition, a partial latin square P will be represented by a set of ordered triples $\{(i, j; P_{ij}) \mid$ element $P_{ij} \in N$ occurs in cell (i, j) of the array$\}$. The *transpose* of a a partial latin square P, denoted by P^T, is the partial latin square obtained by exchanging rows with columns:

$$P^T = \{(j, i; P_{ij}) \mid (i, j; P_{ij}) \in P\}.$$

If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $N = \{0, 1, \ldots, n-1\}$ in such a way that each element of N occurs precisely once in each row and precisely once in each column of the array. Let B_n denote the back circulant latin square of order n, based on the addition table for $\mathbb{Z}_n$. For all positive integers n, $B_n = \{(i, j; i + j \pmod{n}) \mid 0 \le i, j \le n-1\}$.

Later in this paper we will need precise definitions of certain subsets of a back circulant latin square, with this in mind we emphasise the following definitions.

Definition 1. For $0 \le r \le n-1$,

1. $d_r = \{(i,j;r) \mid i+j=r\}$ and
2. $d_{n+r} = \{(i,j;r) \mid i+j=n+r\}$.

Definition 2. Given a partition $\mathcal{P} = \{I, J\}$ of the set $\{0,1,2\ldots,n-2\}$, let $\mathcal{D}_{\mathcal{P}}$ be the following partial latin square in B_n:

$$\mathcal{D}_{\mathcal{P}} = (\bigcup_{i\in I} d_i) \bigcup (\bigcup_{j\in J} d_{n+j}).$$

Example 3. Let $\mathcal{P}' = \{I', J'\}$ and $\mathcal{P}'' = \{I'', J''\}$ be partitions of the set $\{0,1,2\ldots,5\}$, with $I' = \{0,1,4\}$, $J' = \{2,3,5\}$, $I'' = \{0,2,3,4\}$ and $J'' = \{1,5\}$. Figure 1 illustrates the partial latin squares $\mathcal{D}_{\mathcal{P}'}$ and $\mathcal{D}_{\mathcal{P}''}$, which are contained in the latin square B_7.

$\mathcal{D}_{\mathcal{P}'}$

0	1			4		
1			4			
		4				
	4					2
4					2	3
				2	3	
			2	3		5

$\mathcal{D}_{\mathcal{P}''}$

0		2	3	4		
	2	3	4			
2	3	4				1
3	4				1	
4				1		
			1			
		1				5

B_7

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

Figure 1

For ease of exposition we define:

Definition 4. For $0 \le r \le n-1$,

1. $\mathcal{D}_r = d_0 \cup d_1 \cup \ldots \cup d_{r-1}$, and
2. $\mathcal{D}_{n+r} = d_{n+r} \cup d_{n+r+1} \cup \ldots \cup d_{2n-1}$.

Now simple counting arguments verify:

Lemma 5. For $0 \le r \le n-1$,

1. $|d_r| = r+1$,
2. $|d_{n+r}| = n-(r+1)$,

3. $d_r \cup d_{n+r}$ consists of all the elements $(i, j; r)$ of B_n, where $i + j \equiv r \pmod{n}$,
4. $|\mathcal{D}_r| = r(r+1)/2$, and
5. $|\mathcal{D}_{n+r}| = (n-r)(n-r-1)/2$. □

Example 6. For instance

- $d_0 = \{(0,0;0)\}$, $d_n = \{(1, n-1; 0), (2, n-2; 0), \dots, (n-1, 1; 0)\}$, so $|d_0| = 1$, $|d_n| = n - 1$;
- $d_1 = \{(0,1;1),(1,0;1)\}$, $d_{n+1} = \{(2, n-1; 1), (3, n-2; 1), \dots, (n-1, 2; 1)\}$, so $|d_1| = 2$, $|d_{n+1}| = n - 2$;
- $d_{n-1} = \{(0, n-1; n-1), (1, n-2; n-1), \dots, (n-1, 0; n-1)\}$, $d_{2n-1} = \emptyset$, so $|d_{n-1}| = n$, $|d_{2n-1}| = 0$.

We can now define the partial latin squares that will be of chief consideration in this paper.

Definition 7. Let $\lfloor n/2 \rfloor \le r \le n-2$ and $\lfloor n/2 \rfloor - 1 \le s \le r - 1$. Then $\mathcal{D}_{r,s}$ is the following partial latin square in B_n:

$$\mathcal{D}_{r,s} \quad = \quad (\mathcal{D}_r \setminus d_s) \cup d_{n+s} \cup \mathcal{D}_{n+r}.$$

Now let $\lfloor (n+4)/2 \rfloor \le r \le n-2$ and $\lfloor (n+2)/2 \rfloor \le s \le r-1$. Then $\mathcal{E}_{r,s}$ is the following partial latin square in B_n:

$$\mathcal{E}_{r,s} \quad = \quad ((\mathcal{D}_r \setminus d_s) \setminus d_{\lfloor (n-2)/2 \rfloor}) \cup d_{\lfloor (3n-2)/2 \rfloor} \cup d_{n+s} \cup \mathcal{D}_{n+r}.$$

Example 8. The partial latin squares $\mathcal{D}_{7,4}$ and $\mathcal{E}_{7,5}$ in the back circulant latin square B_9 are given in Figure 2.

$\mathcal{D}_{7,4}$

0	1	2	3		5	6		
1	2	3		5	6			
2	3		5	6				
3		5	6					
	5	6						
5	6							4
6							4	
						4		
					4			7

$\mathcal{E}_{7,5}$

0	1	2		4		6		
1	2		4		6			
2		4		6				
	4		6					
4		6						3
	6						3	
6						3		5
					3		5	
				3		5		7

Figure 2

In addition we will study the partial latin squares C_r given below.

Definition 9. Let C_r, $1 \leq r \leq n-2$, be the following partial latin squares in B_n.

$$\begin{aligned} C_1 &= (\mathcal{D}_{n-1} \setminus \{(n-2,0;n-2),(0,1;1)\}) \cup \{(n-2,1;n-1)\}, \text{ and} \\ C_r &= (C_{r-1} \setminus \{(n-r-1,0;n-r-1),(0,r;r)\}) \\ &\quad \cup \{(n-r-1,r;n-1)\}, \quad r \geq 2. \end{aligned}$$

Example 10. The partial latin squares C_1, C_2, C_3 and C_4 in B_6 are shown in Figure 3.

C_1

0		2	3	4	
1	2	3	4		
2	3	4			
3	4				
	5				

C_2

0			3	4	
1	2	3	4		
2	3	4			
	4	5			
	5				

C_3

0				4	
1	2	3	4		
	3	4	5		
	4	5			
	5				

C_4

0					
	2	3	4	5	
	3	4	5		
	4	5			
	5				

Figure 3

For a given partial latin square P the set of cells $\mathcal{S}_P = \{(i,j) \mid (i,j;P_{ij}) \in P, \text{ for some } P_{ij} \in N\}$ is said to determine the *shape* of P and $|\mathcal{S}_P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells. For each r, $0 \leq r \leq n-1$, let $\mathcal{R}_P^r$ denote the set of entries occurring in row r of P. Formally, $\mathcal{R}_P^r = \{P_{rj} \mid (r,j;P_{rj}) \in P\}$. Similarly, for each c, $0 \leq c \leq n-1$, we define $\mathcal{C}_P^c = \{P_{ic} \mid (i,c;P_{ic}) \in P\}$.

A partial latin square Q of order n is said to be a *latin trade* (or *latin interchange*) if $Q \neq \emptyset$ and there exists a partial latin square Q' (called a *disjoint mate* of Q) of order n, such that

1. $\mathcal{S}_Q = \mathcal{S}_{Q'}$,
2. for each $(i,j) \in \mathcal{S}_Q$, $Q_{ij} \neq Q'_{ij}$,
3. for each r, $0 \leq r \leq n-1$, $\mathcal{R}_Q^r = \mathcal{R}_{Q'}^r$, and
4. for each c, $0 \leq c \leq n-1$, $\mathcal{C}_Q^c = \mathcal{C}_{Q'}^c$.

A *critical set* in a latin square L (of order n) is a partial latin square C in L, such that

(1) L is the only latin square of order n which has element k in cell (i,j) for each $(i,j;k) \in C$, and
(2) no proper subset of C satisfies (1).

A *uniquely completable set* (UC) in a latin square L of order n is a partial latin square in L which satisfies Condition (1) above. So a uniquely completable set P in a latin square L is a critical set if for each $(i,j;k) \in P$, there exists a latin trade Q in L such that $Q \cap P = \{(i,j;k)\}$. (This ensures that $P \setminus \{(i,j;k)\}$ has at least two completions: L and $(L \setminus Q) \cup Q'$.)

Let L be a latin square of order n and $U \subseteq L$ be a uniquely completable set. In addition let T be a partial latin square of order n such that $U \subseteq T \subset L$. We say that the addition of an ordered triple $t = (r,c;s)$ is *forced* (see [10]) in the process of completing T to L, if

(i) $\forall r' \neq r$, $\exists z \neq c$ such that $(r',z;s) \in T$ or $\exists z \neq s$ such that $(r',c;z) \in T$, or
(ii) $\forall c' \neq c$, $\exists z \neq r$ such that $(z,c';s) \in T$ or $\exists z \neq s$ such that $(r,c';z) \in T$, or
(iii) $\forall s' \neq s$, $\exists z \neq r$ such that $(z,c;s') \in T$ or $\exists z \neq c$ such that $(r,z;s') \in T$.

The uniquely completable set U is called *strong* if we can define a sequence of sets of ordered triples $U = F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_r = L$ such that each triple $t \in F_{i+1} \setminus F_i$ is forced in F_i for $1 \leq i \leq r-1$. A uniquely completable set is *super-strong* if each triple in this sequence is forced only by virtue of property (iii) in the above definition of forcing.

In Section 3 we show that $\mathcal{D}_\mathcal{P}$ (see Definition 2) is always a strong uniquely completable set. However the set $\mathcal{D}_\mathcal{P}$ is not necessarily a critical set in B_n. For instance consider the partial latin squares $\mathcal{D}_{\mathcal{P}'} \setminus \{(0,4;4)\}$ and $\mathcal{D}_{\mathcal{P}''} \setminus \{(2,2;4),(2,6;1)\}$ as set out in Example 3. These proper subsets of $\mathcal{D}_\mathcal{P}$ are also uniquely completable sets (in fact critical sets) in B_7. This example leads us to the following open problem.

Open Problem. Classify all the partitions $\mathcal{P} = \{I,J\}$ of the set $\{0,1,2\ldots,n-2\}$ such that $\mathcal{D}_\mathcal{P}$ is a critical set in B_n.

The above problem has been studied in the past and the next theorem provides some partial solutions to this question.

Theorem 11. Let $0 \leq r \leq n-1$. Then $\mathcal{D}_r \cup \mathcal{D}_{n+r}$ is a super-strong critical set in B_n.

Proof. This result is proved in [6], [5] and [7]. □

Applying Lemma 5 and Theorem 11 leads to the following corollary.

Corollary 12. Let $0 \le r \le n-1$. Then there exists a super-strong critical set of size $(r(r+1)+(n-r)(n-r-1))/2$ in the back circulant latin square of order n.

In this paper we generalise Theorem 11 and find some new partitions $\mathcal{P} = \{I, J\}$ of the set $\{0, 1, 2 \dots, n-2\}$ such that $\mathcal{D}_{\mathcal{P}}$ defines a critical set in B_n. Specifically, we will show that the partial latin squares given by $\mathcal{D}_{r,s}$ and $\mathcal{E}_{r,s}$ are strong critical sets in B_n. We also show that the partial latin square C_r is a strong critical set in B_n. In Section 3 we show that that these partial latin squares have unique completion to B_n. Then in Sections 4 and 5 we construct latin trades and verify that every element of the partial latin squares $\mathcal{D}_{r,s}$, $\mathcal{E}_{r,s}$ and C_r is necessary for unique completion to B_n. With these constructions we prove that there exists a strong critical set of size m in B_n for all $\frac{n^2-1}{4} \le m \le \frac{n^2-n}{2}$, when n is odd. Moreover, when n is even we prove that there exists a strong critical set of size m in B_n for all $\frac{n^2-n}{2} - (n-2) \le m \le \frac{n^2-n}{2}$ and $m \in \{\frac{n^2}{4}, \frac{n^2}{4}+2, \frac{n^2}{4}+4, \dots, \frac{n^2-n}{2} - n\}$.

3. Unique completeness

In this section we focus on the sets $\mathcal{D}_{\mathcal{P}}$ and C_r (see Definitions 2 and 9, respectively) and prove that both of these sets have unique completion.

Lemma 13. For any partition $\mathcal{P} = \{I, J\}$ of the set $\{0, 1, 2 \dots, n-2\}$, the partial latin square $\mathcal{D}_{\mathcal{P}}$ is a strong uniquely completable set in B_n.

Proof. The following is an algorithm which verifies that $\mathcal{D}_{\mathcal{P}}$ has a strong, unique completion to B_n.

Algorithm

- Let $X = \emptyset$, $F = \mathcal{D}_{\mathcal{P}}$ and $t = |J|$.
- For $i = 1$ to t
  - Let $x_i = \min\{j \mid j \in J \setminus X\}$, then for all $y \in N$ such that $y < x_i$, $d_y \subset F$ and for all $z \in N$ such that $z > x_i$, symbol x_i occurs in row and column z of F. Hence in the process of completing F for $k = 0, \dots, x_i$ cell $(k, x_i - k)$ must contain symbol x_i.
  - Set $X := X \cup \{x_i\}$ and $F := F \cup d_{x_i}$.
- End

At the conclusion of this process we obtain a superset of $\mathcal{D}_{n-1}$, which is a strong, uniquely completable set by Theorem 11. Therefore $\mathcal{D}_{\mathcal{P}}$ is a strong, uniquely completable set in B_n. □

Lemma 14. Let $1 \le r \le n-2$. The partial latin square C_r (given in Definition 9) is a strong completable set in B_n for all r.

Proof. (a) The empty cells in row zero of C_r are $(0, j)$, where $j \in \{1, 2, \dots, r\} \cup \{n-1\}$. We fill these empty cells from $j = n-1$ to $j = 1$. Since the cell

$(n-j-1, j)$ is filled with $n-1$ for $1 \le j \le r$ this forces the cell $(0, n-1)$ to contain $(n-1)$. On the other hand, $(d_j \setminus \{(0,j;j),(j,0;j)\}) \subseteq C_r$ for $j = r, r-1, r-2, \dots, 1$. This forces the cell $(0, j)$ to contain j for $j = r, r-1, r-2, \dots, 1$. We update C_r by filling the empty cells in row zero.
(b) The empty cells in column zero of C_r are $(i, 0)$, where $i \in \{n-1-r, n-r, n-r+1 \dots, n-1\}$. We fill these empty cells from $i = n-1$ to $i = n-1-r$. Since the cell $(i, n-1-i)$ is filled with $n-1$ for $n-1-r \le i \le n-2$ this forces the cell $(n-1, 0)$ to contain $n-1$. On the other hand, $(d_i \setminus \{(0,i;i),(i,0;i)\}) \subseteq C_r$ for $i = n-2, n-3, \dots, n-1-r$. This forces the cell $(i, 0)$ to contain i for $i = n-2, n-3, \dots, n-1-r$. We update C_r by filling the empty cells in column zero. The resultant partial latin square is a superset of $\mathcal{D}_{n-1}$, so it follows that C_r is a strong uniquely completable set in B_n. □

4. Latin trades

In this section we develop a theory of latin trades that will be used to show that the partial latin squares under consideration in this paper are indeed critical sets. Lemmas 18 and 21 derive latin trades that may exist in back circulant latin squares of any order. We also introduce an operation $\circ_r$ (see Lemma 23) to combine these latin trades to make new ones. All of the latin trades are designed to avoid particular sets of elements.

Example 15. Because of the complexity of the arguments that follow, in this section we focus on a specific example. Our approach is then generalised in Section 5. Consider the partial latin square $\mathcal{E}_{7,5}$ in B_9, given in Example 8. By the end of this section we will have shown that for each element $(i,j;k)$ in $\mathcal{E}_{7,5}$ there exists a latin trade I in B_n such that $I \cap \mathcal{E}_{7,5} = \{(i,j;k)\}$. An immediate consequence will be that $\mathcal{E}_{7,5}$ is a critical set in B_9.

First consider the element $(0,2;2)$ in $\mathcal{E}_{7,5}$. What sort of properties must a latin trade I have so that $I \cap \mathcal{E}_{7,5} = \{(0,2;2)\}$? Well, to begin with, I must intersect d_{n+2}, because I must have at least two cells containing the entry 2. Also I must avoid d_4 and d_6. So if we can construct a latin trade I which is contained in $\{(0,2;2)\} \cup (\bigcup d_p)$, (where $8 \le p \le 9+2$) we are done. In fact a latin trade with these properties is given in Figure 4.

		2						8
					8			2
		8			2			

Figure 4

The following construction will enable us to find latin trades for all the elements in d_0, d_1 and d_2 of $\mathcal{E}_{7,5}$. First we need the following definition, which provides a "box" in which to frame a latin trade.

Definition 16. Let $1 \le x, y \le n$. Define $L(x, y, n)$ to be the partial latin square in B_n formed by the intersection of the first $x+1$ rows with the first $y+1$ columns. More formally,

$$L(x, y, n) \quad = \quad \{(i, j; i+j \pmod{n}) \mid 0 \le i \le x, 0 \le j \le y\}.$$

The next lemma gives a result on the existence of latin trades in subsets of the back circulant square.

Lemma 17. Let $1 \le x, y \le n$. The partial latin square $L(x, y, n)$ contains no latin trades if $x + y < n$.

Proof. Suppose there exists a latin trade $I \in L(x, y; n)$, where $x + y < n$. Now, the cell $(0,0)$ is the only cell in $L(x, y, n)$ containing the symbol 0. Therefore $(0, 0; 0)$ cannot be contained in I. Since $x + y < n$, each cell (i, j) in $L(x, y; n)$ contains the entry $i + j$. Suppose we have shown that there exist no cells in I containing the entries 0 through to k, for some $k \le x + y$, and suppose that $(0, k+1; k+1) \in I$. Then there must be an element $(0, j; k+1) \in I'$ (where $j \ne k+1$) and thus an element $(k+1-j, j; k+1) \in I$, by the definition of a latin trade. Thus $0 \le j \le k$. But from our assumption, $(0, j; j) \notin I$, so $(0, j; k+1)$ cannot belong to I'. It follows that $(0, k+1; k+1)$ is not in I. Similarly, each element $(i, k+1-i; k+1) \notin I$, where $1 \le i \le k+1$. So the result follows. □

Lemma 18. Let $n \le 2x < 2n$. There exists a latin trade in the back circulant square B_n, denoted by $G_{x,n}$, with the following properties:

1. $G_{x,n} \subseteq L(x, x, n)$, and includes the elements $(0, 0; 0)$, $(0, x; x)$, $(x, 0; x)$, $(n-x, x; 0)$, $(x, n-x; 0)$.
2. $G_{x,n} \setminus \{(0, 0; 0)\}$ is a subset of $\bigcup d_p$, where $x \le p \le n$.
3. $G_{x,n}$ contains no entries in cells of the form (α, β) where $0 < \beta < n-x$.

Proof. A latin trade $G_{x,n}$ with the above properties is given in Theorem 4, Section 3 of [8]. We shall give an alternative and more straightforward construction.

We consider two cases: $n = 2x$ and $n < 2x$.

If $n = 2x$, our latin trade $G_{x,2x}$ is an *intercalate* (latin trade of size 4) containing the elements $(0,0;0), (x,0;x), (0,x;x)$ and $(x,x;0)$. Its disjoint mate, $G'_{x,2x}$ consists of the elements $(0,0;x)$, $(0,x;0)$, $(x,0;0)$ and $(x,x;x)$.

Otherwise $n < 2x$. Let $n - x = (2x - n)a + b$, where a is an integer greater than or equal to 0 and b is an integer such that $0 \leq b < 2x - n$. We consider two situations: $b = 0$ and $b > 0$.

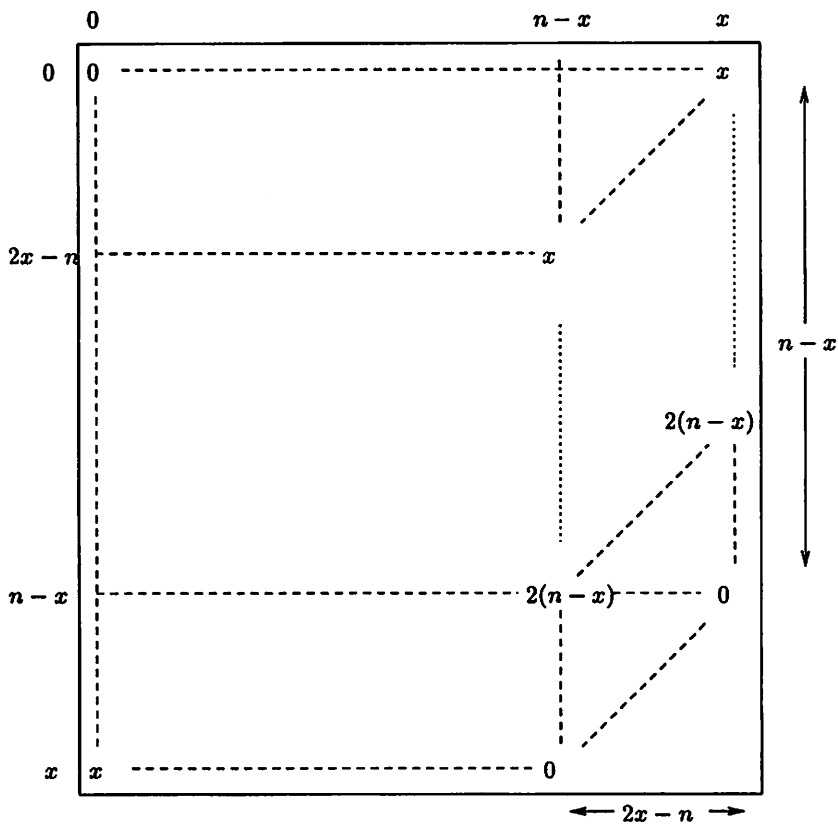


Figure 5: The case $b = 0$.

If $b = 0$, we call this an *initial case*. Consider the diagram in Figure 5. Since $2x - n$ divides $n - x$, we can "zig-zag" up columns $n - x$ and x until we reach $(0, x; x)$.

More formally, our latin trade is as follows (note that all entries are calculated (modulo n)):

$$\begin{aligned} G_{x,n} &= \{(0,0;0),(x,0;x)\} \cup \{(x-i(2x-n),n-x;n-i(2x-n)), \\ &\quad (x-(i+1)(2x-n),x;n-i(2x-n)) \mid 0 \le i \le a\}, \end{aligned}$$

with a disjoint mate obtained by swapping the two entries in each row:

$$\begin{aligned} G'_{x,n} &= \{(0,0;x),(x,0;0),(n-x,n-x;0),(x,n-x;x),(0,x;0), \\ &\quad (2x-n,x;x)\} \cup \{(x-i(2x-n),x;n-i(2x-n)), \\ &\quad (x-(i+1)(2x-n),n-x;n-i(2x-n)) \mid 1 \le i \le a-1\}. \end{aligned}$$

We verify that $G_{x,n}$ is indeed a latin trade with disjoint mate $G'_{x,n}$ by showing that the sets of entries from corresponding columns are the same. Firstly, the set of entries in column 0 is $\{0,x\}$ for both $G_{x,n}$ and $G'_{x,n}$. Secondly, the set of entries in column x of $G_{x,n}$ is equal to:

$$\{n-i(2x-n) \mid 0 \le i \le a\} = \{n-i(2x-n) \mid 1 \le i \le a-1\} \cup \{0, n-a(2x-n)\},$$

which is equal to the set of entries in column x of $G'_{x,n}$. Similarly, column $n-x$ contains the same entries in $G_{x,n}$ as in $G'_{x,n}$. The reader may confirm that $G_{x,n}$ lies in B_n by checking that $i+j$ is equivalent to k modulo n, for each triple $(i,j;k) \in G_{x,n}$.

Otherwise $b > 0$. Here we assume that the latin trade $G_{2x-n,2x-n+b}$ exists, and embed it in the top right-hand corner of $G_{x,n}$ (see Figure 6). Our assumption is justified at the end of the proof.

The latin trade $G_{x,n}$ is as follows:

$$\begin{aligned} G_{x,n} &= \{(0,0;0),(x,0;x)\} \cup \{(x-i(2x-n),n-x;n-i(2x-n)), \\ &\quad (x-(i+1)(2x-n),x;n-i(2x-n)) \mid 0 \le i \le a-1\} \\ &\quad \cup \{(2x-n+b,n-x;x+b)\} \cup \{(i,j+n-x;k+n-x) \mid \\ &\quad (i,j;k) \in G_{2x-n,2x-n+b} \text{ and } k \neq 0\} \\ &\quad \cup \{(i,j+n-x;x+b) \mid (i,j;0) \in G_{2x-n,2x-n+b} \setminus \{(0,0;0)\}\}. \end{aligned}$$

In each non-empty row of $G_{x,n}$ there are exactly two entries, so the disjoint mate $G'_{x,n}$ is obtained by swapping the entries in each row. More specifically:

$$\begin{aligned} G'_{x,n} &= \{(0,0;x),(x,0;0),(n-x,n-x;0),(x,n-x;x)\} \\ &\quad \cup \{(x-i(2x-n),x;n-i(2x-n)),(x-(i+1)(2x-n),n-x; \\ &\quad n-i(2x-n)) \mid 1 \le i \le a-1\} \cup \{(2x-n+b,x;x+b)\} \\ &\quad \cup \{(i,j+n-x;k+n-x) \mid (i,j;k) \in G'_{2x-n,2x-n+b} \text{ and } k \neq 0 \\ &\quad \text{and } (i,j) \neq (0,0)\} \cup \{(i,j+n-x;x+b) \mid \\ &\quad (i,j;0) \in G'_{2x-n,2x-n+b} \setminus \{(0,2x-n;0)\}\} \cup \{(0,x;0)\}, \end{aligned}$$

where $G'_{2x-n,2x-n+b}$ is the disjoint mate of $G_{2x-n,2x-n+b}$.

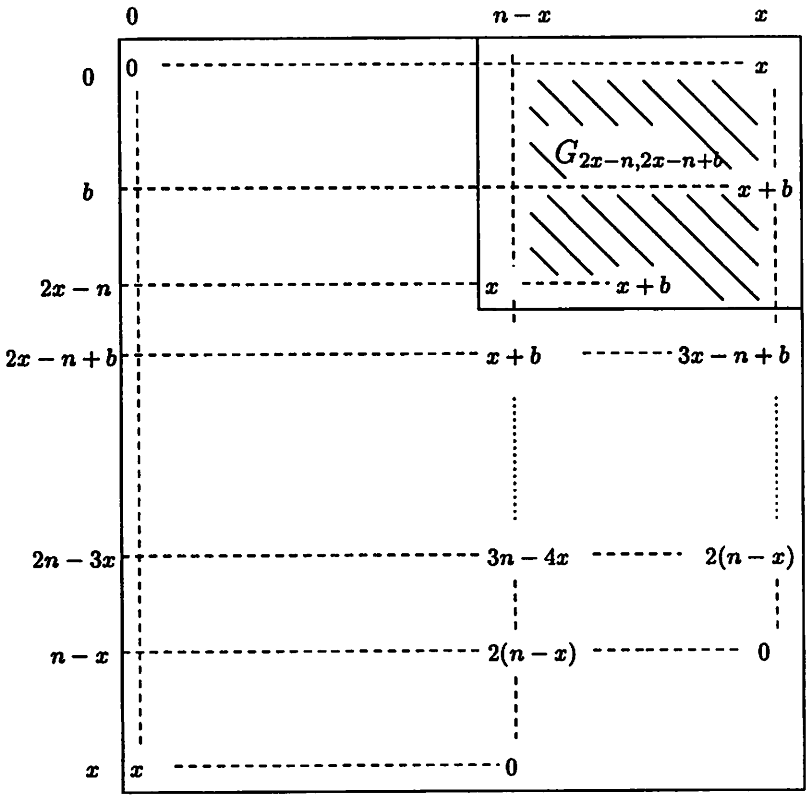


Figure 6: The case $b > 0$.

As before, we show that the set of entries from corresponding columns are the same. Firstly, the set of entries in column 0 is $\{0, x\}$ for both $G_{x,n}$ and $G'_{x,n}$. Secondly, by examining the definition of $G_{x,n}$ above, the set of entries in column $n - x$ of $G_{x,n}$ is equal to:

$$\{n - i(2x - n) \mid 0 \leq i \leq a - 1\} \cup \{x + b\}$$
$$\cup\{k + n - x \mid (i, 0; k) \in G_{2x-n,2x-n+b} \text{ and } k \neq 0\}.$$

But by Condition 3 of our lemma, this simplifies to:

$$\{n - i(2x - n) \mid 1 \leq i \leq a - 1\} \cup \{0, x, x + b\}.$$

Meanwhile, the set of entries in column $n - x$ of $G'_{x,n}$ is equal to

$$\{n - i(2x - n) \mid 1 \leq i \leq a - 1\} \cup \{0, x\} \cup \{k + n - x \mid$$
$$(i, 0; k) \in G'_{2x-n,2x-n+b} \text{ and } i, k \neq 0\} \cup \{x + b \mid (i, 0; 0) \in G'_{2x-n,2x-n+b}\}.$$

But from Condition 3 there are only two entries in column 0 and two entries in row 0 of $G_{2x-n,2x-n+b}$. So we must have $(0,0;2x-n)$, $(0,2x-n;0)$, $(2x-n,0;0) \in G'_{2x-n,2x-n+b}$. It follows that the sets of entries in column $n-x$ are equal.

Thirdly, the set of entries in column x of $G_{x,n}$ is equal to:

$$\{n-i(2x-n) \mid 0 \le i \le a-1\} \cup \{x+b\} \cup \{k+n-x \mid (i,2x-n;k) \in G_{2x-n,2x-n+b} \text{ and } k \neq 0\}.$$

The set of entries in column x of $G'_{x,n}$ is equal to:

$$\{n-i(2x-n) \mid 1 \le i \le a-1\} \cup \{0, x+b\} \cup \{k+n-x \mid (i,2x-n;k) \in G'_{2x-n,2x-n+b} \text{ and } k \neq 0\}.$$

Clearly these two sets are equal. For remaining columns, we use the fact that corresponding columns of $G_{2x-n,2x-n+b}$ and $G'_{2x-n,2x-n+b}$ contain the same set of entries.

Now we justify the assumption of the existence of $G_{2x-n,2x-n+b}$, where $b > 0$. Let $a_1 = a$ and $b_1 = b$. Next, let a_2 and b_2 be integers such that

$$b_1 = (2x-n-b_1)a_2 + b_2,$$

with $a_2 \ge 0$ and $0 \le b_2 < 2x-n-b_1$. If $b_2 = 0$ we can construct $G_{2x-n,2x-n+b_1}$ as it is an initial case, and therefore $G_{x,n}$ may be constructed. Otherwise $b_2 > 0$, and we require the existence of the latin trade $G_{2x-n-b_1,2x-n-b_1+b_2}$. The process continues recursively. We wish to show that an initial case is eventually reached.

In general, define a series of integers $b_1, b_2, \ldots$ as follows. Let $b_1 = b$. Assume $b_1, b_2, \ldots, b_{i-1}$ are defined for some integer $i > 1$. Then let a_i and b_i be integers such that

$$b_{i-1} = (2x-n-(b_1+b_2+\cdots+b_{i-1}))a_i + b_i,$$

with $a_i \ge 0$ and $0 \le b_i < 2x-n-(b_1+b_2+\cdots+b_{i-1})$. Noting that $2x-n-(b_1+b_2+\cdots+b_i) \ge 0$ and $b_i \ge 0$ for all i, it follows that there must be an integer k such that $b_k = 0$.

Thus a latin trade given by the initial case is eventually obtained, and we can construct $G_{x,n}$. □

Example 15 continued. We wish to find a latin trade that intersects $\mathcal{E}_{7,5}$ only in the element $(0,0;0)$. Here we begin by using Lemma 18 to construct the latin trade $G_{7,9}$. Here $n = 9$ and $x = 7$, so $n-x = 2$ and $2x-n = 5$. Thus $n-x = (2x-n) \times 0 + 2$, giving $b_1 = 2$ and $a_1 = 0$. So in constructing $G_{7,9}$ we need the smaller latin trade $G_{5,7}$. This in turn requires the latin trade $G_{3,5}$, with $a_2 = 0, b_2 = 2$. Finally $a_3 = 2$ and $b_3 = 0$, so $G_{3,5}$ is an initial case.

Note that $G_{7,9} \setminus \{(0,0;0)\}$ intersects only d_7, d_8 and d_9 in the latin square B_9. Thus the latin trade $G_{7,9}$ proves the necessity of the element $(0,0;0) \in \mathcal{E}_{r,s}$. However, we can do more by "shifting" $G_{7,9}$, as described in the next definition.

0			3
		3	4
		4	0
3		0	

$G_{3,5}$

$\hookrightarrow$

0					5
				5	6
				6	0
		5		0	
5		0			

$G_{5,7}$

$\hookrightarrow$

0							7
						7	8
						8	0
				7		0	
		7		0			
7		0					

$G_{7,9}$

Figure 7

Definition 19. Let I be a partial latin square in the back circulant latin square of order n. We define $I \oplus (i,j)$ to be the partial latin square in B_n given by:

$$I \oplus (i,j) = \{(\alpha + i, \beta + j; \gamma + i + j \pmod{n}) \mid (\alpha, \beta; \gamma) \in I\}.$$

Note that a latin trade is a partial latin square, so the previous definition can be used for latin trades. Also, if I is a latin trade, then $I \oplus (i,j)$ is also a latin trade. In fact, if we know that a latin trade I intersects only certain sets of the form d_p, then we can make a similar claim for the latin trade $I \oplus (i,j)$. However, to do so we must ensure that we have not "shifted" I past row or column $n-1$ of B_n.

Lemma 20. Let $I \subseteq L(x,y,n)$ be a latin trade in the latin square B_n that includes $(0,0;0)$. Let i and j be integers such that $i+j < n$, $i+x < n$ and $j+y < n$. Let M be a subset of $\{0,1,\ldots,n\}$ and suppose $I \setminus \{(0,0;0)\}$ is a subset of

$$\bigcup_{m \in M} d_m.$$

Then the latin trade $I \oplus (i,j) \setminus \{(i,j;i+j)\}$ is a subset of:

$$\bigcup_{m \in M} d_{m+i+j}.$$

Proof. The conditions $i+x < n$ and $j+y < n$ ensure that for all rows r and for all columns c of the latin trade $I \oplus (i,j)$, $r \leq n-1$ and $c \leq n-1$. Thus elements of d_m in I are mapped to elements of d_{m+i+j} in $I \oplus (i,j)$. □

Example 15 continued. Consider the latin trade $G_{7,9}$ constructed above. We noted that $G_{7,9} \setminus \{(0,0;0)\}$ intersects only d_7, d_8 and d_9. Now applying Lemma 20, the latin trades $G_{7,9} \oplus (0,1) \setminus \{(0,1;1)\}$ is a subset of $d_8 \cup d_9 \cup d_{10}$, and thus does not intersect $\mathcal{E}_{7,5}$. Thus the element $(0,1;1)$ is necessary in $\mathcal{E}_{7,5}$ for a unique completion. Similarly $(1,0;1)$ can be shown to be necessary in $\mathcal{E}_{7,5}$. Also the latin trades $G_{6,9} \oplus (0,2)$, $G_{6,9} \oplus (1,1)$ and $G_{6,9} \oplus (2,0)$ show the necessity of the entries $(0,2;2)$, $(1,2;2)$ and $(2,0;2)$. So using Lemma 18 and by translating latin trades, we have shown the necessity of six of the elements of $\mathcal{E}_{7,5}$.

Now consider the cells of $\mathcal{E}_{7,5}$ that contain the entry 4. Here we run into some difficulties if we try to use latin trades from Lemma 18. Consider $(0,4;4) \in \mathcal{E}_{7,5}$. It is infeasible to use a latin trade of the form $G_{x,9} \oplus (0,4)$, because x must be greater than $9/2$. We need a latin trade that is more rectangular in shape, and we must also be careful to avoid d_{9+3}. With these requirements in mind, we construct a new latin trade from the final columns of $G_{x,n}$.

Lemma 21. Let $0 < x, y$. There exists a latin trade, denoted by $H_{x,y}$ in the latin square B_{x+y} with the following properties:

1. $H_{x,y} \subseteq L(x, y, x+y)$, and includes the elements $(0,0;0)$, $(0,y;y)$, $(x,0;x)$ and $(x,y;0)$.
2. $H_{x,y} \setminus \{(0,0;0),(x,y;0)\}$ is a subset of $\bigcup d_p$, where $\min\{x,y\} \leq p \leq \max\{x,y\}$.

Proof. There are three cases to consider: $x = y$, $x > y$ and $x < y$. If $x = y$, use the intercalate $G_{x,2x}$ (see Lemma 18). If $x > y$, our required latin trade is constructed from the final columns of the latin trade $G_{x,2x-y}$ (given in Lemma 18). Condition 3 of Lemma 18 ensures that $G_{x,2x-y}$ contains no entries in columns β, where $0 < \beta < x - y$. We can create a new latin trade from $G_{x,2x-y}$, by removing the two elements $(0,0;0)$ and $(x,0;x)$ from column 0 and replacing them with an entry in cells $(0, x-y)$ and (x,x). Then we can relabel all entries so that our latin trade occurs in the latin square B_{x+y}.

More formally:

$$\begin{aligned} H_{x,y} = {} & \{(0,0;0),(x,y;0)\} \cup \{(i, j-(x-y); k-(x-y)) \mid \\ & (i,j;k) \in G_{x,2x-y}, j \geq x-y, k \neq 0\} \\ & \cup \{(i, j-(x-y); x)) \mid (i,j;k) \in G_{x,2x-y}, j \geq x-y, k = 0\}, \end{aligned}$$

with disjoint mate:

$$\begin{aligned} H'_{x,y} &= \{(0,0;y),(0,y;0),(x,0;0),(x,y;y)\} \\ &\cup\{(i,j-(x-y);k-(x-y)) \mid (i,j;k) \in G'_{x,2x-y}, \\ &j \geq x-y, k \neq 0, i \notin \{0,x\}\} \cup \{(i,j-(x-y);x) \mid \\ &(i,j;k) \in G'_{x,2x-y}, j \geq x-y, k=0, i \notin \{0,x\}\}, \end{aligned}$$

where $G'_{x,2x-y}$ is the disjoint mate of $G_{x,2x-y}$.

Finally if $x < y$, then let $H_{x,y} = (H_{y,x})^T$. □

Example 15 continued. In Figure 8 we give the latin trade $G_{5,8}$, constructed as in Lemma 18, together with $H_{5,2}$, which arises from the final three columns of $G_{5,8}$, as in Lemma 21. We also give the latin trade $H_{5,4}$ which indicates the kind of pattern that arises when there is a difference of one between x and y in $H_{x,y}$.

0					5
				5	6
			5	6	
			6		0
5			0		

$G_{5,8}$

0		2
	2	3
2	3	
3		5
5		0

$H_{5,2}$

0				4
			4	5
		4	5	
	4	5		
4	5			
5				0

$H_{5,4}$

Figure 8

Note that $H_{5,2} \setminus \{(0,0;0),(5,2;0)\}$ is a subset of $\cup d_p$, where $2 \leq p \leq 5$, as per Condition 2 of Lemma 21.

Now we can show that every element of d_4 is necessary in $\mathcal{E}_{7,5}$. For elements $(0,4;4)$, $(1,3;4)$ and $(2,2;4)$ use the latin trades $H_{6,3} \oplus (0,4)$, $H_{6,3} \oplus (1,3)$ and $H_{6,3} \oplus (2,2)$ respectively. For elements $(3,1;4)$ and $(4,0;4)$ use the latin trades $H_{3,6} \oplus (3,1)$ and $H_{3,6} \oplus (4,0)$.

In fact we can also use the latin trades from the previous lemma to show the necessity of the elements of d_{9+3} and d_{9+5}.

The next lemma is similar to Lemma 20. However, here our aim is to intersect the bottom right hand corner of I with the element $(i,j;i+j-n)$.

Lemma 22. Let $I \subseteq L(x,y,x+y)$ be a latin trade in the latin square B_{x+y} that includes the ordered triples $(0,0;0)$ and $(x,y;0)$. Let i and j be integers such that $i > x$ and $j > y$. Let M be a subset of $\{0,1,\ldots,x+y\}$ and suppose $I \setminus \{(0,0;0),(x,y;0)\}$ is a subset of

$$\bigcup_{m \in M} d_m.$$

Then the latin trade $I \oplus (i-x, j-y) \setminus \{(i,j;i+j-(x+y)), (i-x, j-y; i+j-(x+y))\}$ is a subset of

$$\bigcup_{m \in M} d_{m+i+j-(x+y)}.$$

Proof. The conditions $i > x$ and $j > y$ ensure that $I \oplus (i-x, j-y)$ is bounded to the left and above by row 0 and column 0 respectively. □

Example 15 continued. Now we use the latin trades $H_{4,5}$ and $H_{5,4}$ constructed by Lemma 21 and apply Lemma 22, translating the latin trades so that the bottom right-hand element intersects the appropriate element of $\mathcal{E}_{7,5}$. So for elements $(4,8;3)$, $(5,7;3)$, $(6,6;3)$ and $(7,5;3)$ use the latin trades $H_{4,5} \oplus (0,3)$, $H_{4,5} \oplus (1,2)$, $H_{4,5} \oplus (2,1)$ and $H_{4,5} \oplus (3,0)$. For element $(8,4;3)$, use the latin trade $H_{5,4} \oplus (3,0)$. For elements $(6,8;5)$, $(7,7;5)$ and $(8,6;5)$ use the latin trades $H_{4,5} \oplus (2,3)$, $H_{4,5} \oplus (3,2)$ and $H_{4,5} \oplus (4,1)$.

The element $(8,8;7) \in d_{9+7}$ is necessary because without it, we can swap the entries 7 and 8 throughout the latin square. This leaves us with the elements of d_6. Latin trades of rectangular shape will again be useful here, but how do we guarantee that they will avoid *both* d_{9+3} and d_{9+5}? We can do so by combining two latin trades from Lemma 21.

In the following lemma, $I_{x_1,y}$ could be equal to $H_{x_1,y}$, whereas $I_{x_2,y}$ could be equal to either $H_{x_2,y}$ or $G_{x_2,z}$ (with $y = x_2$).

Lemma 23. Let $x_2, y \le z \le x_2 + y$ and let $I_{x_1,y}$ and $I_{x_2,y}$ be latin trades contained in $L(x_1, y, x_1 + y)$ and $L(x_2, y, z)$ respectively such that:

1. $I_{x_1,y}$ includes the elements $(0,0;0)$, $(0,y;y)$, $(x_1,0;x_1)$ and $(x_1,y;0)$.
2. $I_{x_2,y}$ includes the elements $(0,0;0)$, $(0,y;y)$, $(x_2,0;x_2)$, $(x_2, z-x_2;0)$ and $(z-y, y; 0)$.
3. $I_{x_2,y}$ contains no entries in cells of the form (α, β), where $\alpha + \beta > z$.
4. Either row x_1 of $I_{x_1,y}$ or row 0 of $I_{x_2,y}$ contains no other entries.

Then, in the latin square $B_{x_1+x_2+y}$, there exists a latin trade (denoted by $I_{x_1,y} \circ_r I_{x_2,y}$) obtained by adding x_1 to the non-zero entries of $I_{x_2,y}$ and overlapping non-zero entries in the last row of $I_{x_1,y}$ with those in the first row of $I_{x_2,y}$. More formally,

$$\begin{aligned} I_{x_1,y} \circ_r I_{x_2,y} &= (I_{x_1,y} \setminus \{(x_1,y;0)\}) \\ &\cup \{(i+x_1, j; k+x_1) \mid (i,j;k) \in I_{x_2,y} \text{ and } k \neq 0\} \\ &\cup \{(i+x_1, j; 0) \mid (i,j;0) \in I_{x_2,y} \text{ and } (i,j) \neq (0,0)\}. \end{aligned}$$

Proof. Let $I'_{x_1,y}$ and $I'_{x_2,y}$ be the disjoint mates of $I_{x_1,y}$ and $I_{x_2,y}$ respectively. Since 0 occurs only twice as an entry in cells of $I_{x_1,y}$, we must have

$(0,y;0),(x_1,0;0) \in I'_{x_1,y}$. Since $(z-y,y;0) \in I_{x_2,y}$ and there are no cells of the form $(z-y,\alpha)$ with $\alpha > y$, we must have $(0,y;0) \in I'_{x_2,y}$. By similar reasoning, $(x_2,0;0) \in I'_{x_2,y}$. Let

$$\begin{aligned}\mathcal{I}' &= (I'_{x_1,y} \setminus \{(x_1,0;0)\}) \cup \{(i+x_1,j;k+x_1) \mid (i,j;k) \in I'_{x_2,y} \text{ and } \\ &\quad k \neq 0\} \cup \{(i+x_1,j;0) \mid (i,j;0) \in I'_{x_2,y} \text{ and } (i,j) \neq (0,y)\}.\end{aligned}$$

For ease of proof let $\mathcal{I}$ represent $I_{x_1,y} \circ_r I_{x_2,y}$. It is clear that $\mathcal{I}$ and $\mathcal{I}'$ have the same size and shape.

Consider an arbitrary ordered triple $(\alpha,\beta;\gamma)$ from $\mathcal{I}$. We will show that γ occurs somewhere in both row α and column β of $\mathcal{I}'$. First suppose that $(\alpha,\beta;\gamma) \in I_{x_1,y} \setminus \{(x_1,y;0)\}$. Then there exists an element $(\alpha',\beta;\gamma) \in I'_{x_1,y} \setminus \{(0,y;0)\}$. If $(\alpha',\beta;\gamma) = (x_1,0;0)$, then observe that $(x_1+x_2,0;0) \in \mathcal{I}'$. Otherwise $(\alpha',\beta;\gamma) \in \mathcal{I}'$. Thus γ occurs in column β of $\mathcal{I}'$. There also exists $(\alpha,\beta';\gamma) \in I'_{x_1,y} \setminus \{(x_1,0;0)\} \subseteq \mathcal{I}'$ and so γ occurs in row α of $\mathcal{I}'$.

Next suppose that $(\alpha,\beta;\gamma) = (i+x_1,j;k+x_1)$, where $(i,j;k) \in I_{x_2,y}$ and $k \neq 0$. Then, there exists $(i',j;k) \in I'_{x_2,y}$ and $(i'+x_1,j;k+x_1) \in \mathcal{I}'$. Thus γ occurs in column β of $\mathcal{I}'$. There also exists $(i,j';k) \in I'_{x_2,y}$. Then we have $(i+x_1,j';k+x_1) \in \mathcal{I}'$ and so γ occurs in row α of $\mathcal{I}'$.

Finally suppose that $(\alpha,\beta;\gamma) = (i+x_1,j;0)$ and $(i,j) \neq (0,0)$. Then $(i,j;0) \in I_{x_2,y}$. If $j=y$, then $(0,y;0) \in I'_{x_1,y}$, so $(0,y;0) \in \mathcal{I}'$, otherwise $j \neq y$ and there exists $(i',j;0) \in I'_{x_2,y}$. Since $j \neq 0$, $(i'+x_1,j;0) \in \mathcal{I}'$. Thus γ occurs in column β of $\mathcal{I}'$. There also exists $(i,j';0) \in I'_{x_2,y}$. Since $(i,j) \neq (0,0)$, $j' \neq y$. So we have $(i+x_1,j';0) \in \mathcal{I}'$, and so γ occurs in row α.

Thus $\mathcal{I}$ and $\mathcal{I}'$ are row and column balanced. □

Corollary 24. Let $I_{x_1,y}$ and $I_{x_2,y}$ be latin trades contained in $L(x_1,y,x_1+y)$ and $L(x_2,y,x_2+y)$ respectively such that

1. $I_{x_1,y}$ includes the elements $(0,0;0)$, $(0,y;y)$, $(x_1,0;x_1)$ and $(x_1,y;0)$,
2. $I_{x_2,y}$ includes the elements $(0,0;0)$, $(0,y;y)$, $(x_2,0;x_2)$ and $(x_2,y;0)$, and
3. either row x_1 of $I_{x_1,y}$ or row 0 of $I_{x_2,y}$ contains no other entries.

In addition let z be an integer such that $0 < z < x_1+x_2+y$ and $z \notin \{x_1,y,x_1+y,x_1+x_2\}$. If $I_{x_1,y}$ does not intersect d_z *and* $I_{x_2,y}$ does not intersect d_{z-x_1}, then the latin trade $I_{x_1,y} \circ_r I_{x_2,y}$, as defined in Lemma 23, does not intersect d_z.

Corollary 25. If either $x_1 \geq y$ or $x_2 \geq y$, then the latin trade $H_{x_1,y} \circ_r H_{x_2,y}$ is well defined as per Lemma 23. Moreover, $H_{x_1,y} \circ_r H_{x_2,y} \setminus \{(0,0;0), (x_1+x_2,y;0)\}$ intersects only subsets of the form d_p, for p in the following ranges:

1. $\min\{x_1,y\} \leq p \leq \max\{x_1,y\}$, and

2. $\min\{x_1 + x_2, x_1 + y\} \leq p \leq \max\{x_1 + x_2, x_1 + y\}$.

Proof. The condition ($x_1 \geq y$ or $x_2 \geq y$) ensures that either the last row of $H_{x_1,y}$ or the first row of $H_{x_2,y}$ contains only two entries, so that Condition 3 of Lemma 23 is satisfied. Otherwise the result follows from Corollary 24. □

Example 15 continued. In Figure 9 we show how the latin trades $H_{5,3}$ and $H_{2,3}$ may be adjoined by the operation $\circ_r$ to create a new trade in B_{10}.

0			3
		3	4
		4	5
3		5	
5			0

$H_{5,3}$

0		2	3
	2	3	
2	3		0

$H_{2,3}$

0			3
		3	4
		4	5
3		5	
5		7	8
	7	8	
7	8		0

$H_{5,3} \circ_r H_{2,3}$

Figure 9

Note that, $x_1 = 5$, $y = 3$ and $x_2 = 2$, $\min\{5,3\} = 3$, $\max\{5,3\} = 5$, $\min\{5+2, 5+3\} = 7$, and $\max\{5+2, 5+3\} = 8$. Thus $H_{5,3} \circ_r H_{2,3} \setminus \{(0,0;0),(7,3;0)\}$ intersects only subsets of the form d_p, where $3 \leq p \leq 5$ or $7 \leq p \leq 8$, as per Corollary 25.

Consider the element $(0,6;6)$ of $\mathcal{E}_{7,5}$. Here we use the latin trade $(H_{5,2} \circ_r H_{2,2}) \oplus (0,6)$. Corollary 25 ensures that $(H_{5,2} \circ_r H_{2,2})$ avoids the sets d_6 and d_8. So when we translate this latin trade across by six rows, it will avoid the sets d_{12} and d_{14} as required. The remaining elements of $\mathcal{E}_{7,5}$ (except for $(8,8;7)$) are shown to be necessary using latin trades $H_{4,5} \oplus (0,3)$, $H_{4,5} \oplus (1,2)$, $H_{4,5} \oplus (2,1)$, $H_{4,5} \oplus (3,0)$, $H_{5,4} \oplus (3,0)$, $H_{6,3} \oplus (0,5)$, $H_{6,3} \oplus (1,4)$ and $H_{6,3} \oplus (2,3)$. Finally there is a latin trade containing every ordered triple with either entry 7 or entry 8 (we exchange entries to obtain the disjoint mate); this shows the necessity of $(8,8;7)$. Thus $\mathcal{E}_{7,5}$ is indeed a critical set!

5. Necessity of elements

The latin trades developed in the previous section will be used to show the necessity of elements in the partial latin squares $\mathcal{D}_{r,s}$, $\mathcal{E}_{r,s}$ (see Definition 7) and C_r (see Definition 9) for unique completion.

For each partial latin square P listed above it will be shown that for each $(i,j;k) \in P$ there exists a latin trade $I \in B_n$ which intersects P in

$(i, j; k)$ alone. Lemmas 26 and 28 deal with the partial latin square $\mathcal{D}_{r,s}$, Lemmas 29 and 30 deal with the partial latin square $\mathcal{E}_{r,s}$ and Lemma 33 deals with the partial latin square C_r.

In Lemmas 26 and 28 care must be taken to show that the latin trade does not intersect d_{n+s}. Hence in Lemma 26 we verify the existence of latin trades which do not contain the symbol z, where z is in an appropriate range. In Lemmas 29 and 30 care must be taken to show that the latin trades intersect neither d_{n+s} nor $d_{n+\lfloor (n-2)/2 \rfloor}$. Hence in Lemma 27 we carefully state a set of constraints for which there exists two solutions and then use this lemma in the proof of Lemmas 28 and 30. In Lemma 29 we verify the existence of latin trades which do not contain the symbols z_1 and z_2, where z_1 and z_2 are in an appropriate range.

Lemma 26. If $2 \le y \le x$ and $z \notin \{0, x, y, x+y\}$, there exists a latin trade $J_{x,y}(z)$ in the latin square B_{x+y} with the following properties:

1. $J_{x,y}(z) \subseteq L(x, y, x+y)$, and includes the elements $(0,0;0)$,$(0,y;y)$, $(x,0;x)$ and $(x,y;0)$.
2. $J_{x,y}(z) \setminus \{(0,0;0),(x,y;0)\}$ is a subset of $\bigcup d_p$, where $y \le p \le x$.
3. $J_{x,y}(z)$ does not intersect d_z.

Proof. If either $z > x$ or $z < y$, the latin trade $H_{x,y}$ given in Lemma 21 suffices. Otherwise $z < x$ and $z > y$, and we use the latin trade $H_{z-y+1,y}$ or $H_{x+y-z-1,y}$. Observe that $z > \max\{z-y+1, y\}$ and $z < \min\{x+(z-y+1), y+(z-y+1)\}$. Thus from Corollary 25, our latin trade does not intersect d_z. □

In the following lemma we verify the solution of a set of constraints. We make use of this result in Lemmas 28 and Lemma 30.

Lemma 27. Let i, j, y and n be positive integers ($i \le j$, $i+j \ge \lfloor n/2 \rfloor$) under the following set of constraints:

1. $2 \le y$;
2. $n-2-(i+j) \le y$;
3. $i+1 \le y$;
4. $y \le n/2$;
5. $y \le n-1-j$; and
6. $n \ge 5$.

If, in turn, $i+j \le n-3$, then there exists at least one integer y that satisfies the above inequalities. If, as well, $i+j \le n-4$, then there exist at least two solutions for y. Finally if $i+j \le n-3$, $n > 5$ and $(i,j) \notin \{(0,n-3),((n-3)/2,(n-3)/2)\}$, there exist at least two distinct solutions for y.

Proof. We combine each lower bound for y with each upper bound for y.

Firstly, $2 \leq y \leq n/2$ has two solutions if $n > 5$ and one solution if $n = 5$. Consider $2 \leq y \leq n-1-j$. If $i+j \leq n-3$, then $j \leq n-3$. If $j = n-3$, then $i = 0$, and there is exactly one solution for y. Otherwise $j \leq n-4$, and there are two solutions for y in the inequality $2 \leq y \leq n-1-j$. Next consider $i+1 \leq y \leq n/2$. If $i+j \leq n-3$ and $i \leq j$, we have that $i \leq (n-3)/2$. If $i = (n-3)/2$, we must have $j = (n-3)/2$, and there is one solution for y. Otherwise $i \leq (n-4)/2$, and there are at least two solutions for y in the equation $i+1 \leq y \leq n/2$. The inequality $i+1 \leq y \leq n-1-j$ has at least two solutions for y because $i+j \leq n-3$. The inequality $n-2-(i+j) \leq y \leq n/2$ has two solutions because $(i+j) \geq \lfloor n/2 \rfloor$. Finally, the inequality $n-2-(i+j) \leq y \leq n-1-j$ has at least two solutions for y, because $i \geq 0$. □

Lemma 28. Let $\lfloor n/2 \rfloor \leq r \leq n-2$, $\lfloor n/2 \rfloor - 1 \leq s \leq r-1$. Then for each entry $(i,j;k) \in \mathcal{D}_{r,s}$ (see Definition 7), there exists a latin trade I in B_n such that

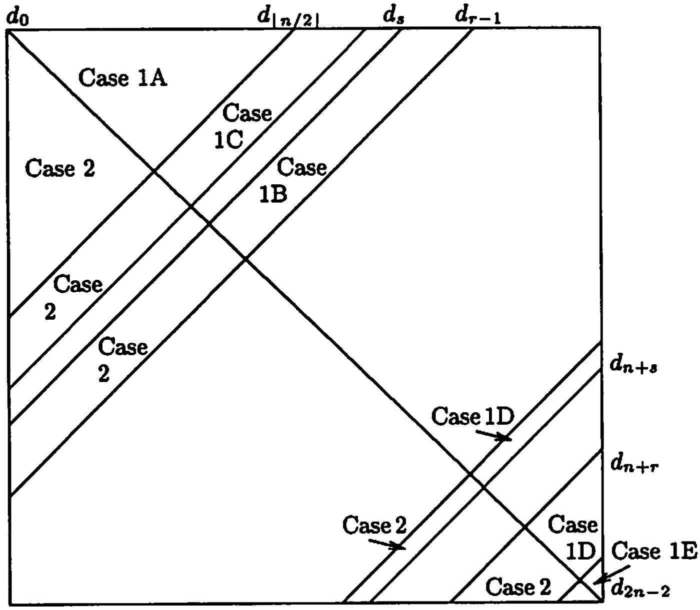
$$I \cap \mathcal{D}_{r,s} = \{(i,j;k)\}.$$

Proof. We split our proof into two cases. In Case 1, $i \leq j$ and in Case 2, $i > j$. Case 1 is then to be split into the subcases 1A to 1E. (See Figure 10 for a pictorial representation of the cases.)

Case 1A: $i+j \leq \lfloor n/2 \rfloor - 1$, $i+j < s$. Note that $i+j \leq n/2 - 1$ is equivalent to $n \leq 2(n-1-i-j)$. So using Lemma 18, we can construct the latin trade $G_{n-1-i-j,n}$ which contains the element $(0,0;0)$, with all other entries in subsets of the form d_p, where $n-1-i-j \leq p \leq n$. Observing that $i+(n-1-i-j) \leq n-1$ and $j+(n-1-i-j) \leq n-1$ and applying Lemma 20, $G_{n-1-i-j,n} \oplus (i,j)$ contains only $(i,j;i+j)$ and elements of the set: $\bigcup d_p$, where $n-1 \leq p \leq n+i+j$. Since $i+j < s \leq r-1$, $G_{n-1-i-j,n} \oplus (i,j) \setminus \{(i,j;i+j)\}$ does not intersect $((\mathcal{D}_r \setminus d_s) \cup (\mathcal{D}_{n+r} \cup d_{n+s}))$. Hence our latin trade intersects $\mathcal{D}_{r,s}$ only in the required element.

Case 1B: $\lfloor n/2 \rfloor \leq i+j \leq r-1$, and $s < i+j$. The cases $(i,j) = (0,n-3)$ and $(i,j) = ((n-3)/2,(n-3)/2)$ are dealt with at the end of this case. Let $t = s-(i+j)+n$.

We will use Lemma 26 to construct a latin trade $J_{n-y,y}(t)$ for an appropriately chosen integer y. Then we wish to place the latin trade $J_{n-y,y}(t) \oplus (i,j)$ so that it contains $(i,j;i+j)$, and $(i+n-y,j+y;i+j)$, and otherwise intersects only $\bigcup d_p$ where $r \leq p \leq n+r-1$ and $p \neq n+s$. In order to do this we need all of the conditions listed below. Conditions 1 and 2 ensure that y is in the appropriate range for Lemma 26 and Conditions 3 and 4 ensure that Lemma 20 can be applied. Then by Lemma 26 and Lemma 20, Conditions 5 and 6 ensure that $J_{n-y,y}(t) \oplus (i,j) \setminus \{(i,j;i+j),(i+n-y,j+y;i+j)\}$ is a subset of $\bigcup d_p$, where $r \leq p \leq n+r-1$.



Case Analysis of Lemma 28

Figure 10

1. $2 \leq y \leq n/2$ and
2. $t \notin \{y, n-y, n\}$.
3. $y + j \leq n - 1$,
4. $n - y + i \leq n - 1$ (or, equivalently, $y \geq i + 1$)
5. $y + (i + j) \geq n - 2 (\geq r)$, and
6. $(i + j) + (n - y) \leq n + r - 1$.

Condition 6 is always true because $(i+j) \leq r-1$. We cannot have $t = n$ in Condition 2 because $s \neq (i+j)$. Assuming Condition 1 is true, we cannot have $t = y$ in Condition 2. To show this, recall that $i + j \leq r - 1 \leq n - 3$. Also $s \geq \lfloor n/2 \rfloor - 1$, implying that $t = s-(i+j)+n \geq n/2-2-(n-3)+n > n/2 \geq y$.

So we wish to show that there exist two integer solutions for y under the following constraints: $y \geq 2$, $y \geq n - 2 - (i + j)$, $y \geq i + 1$, $y \leq n/2$ and $y \leq n - 1 - j$. We can then choose *one* solution y for which $t \neq n - y$, and thus satisfy *all* Conditions 1 through to 6.

This follows from Lemma 27, except in the following cases: $n \leq 5$, $(i, j) = (0, n - 3)$ and $(i, j) = ((n - 3)/2, (n - 3)/2)$.

If $n \le 4$, Case 1B does not arise. If $n = 5$, Case 1B only occurs if $r = 3$, $s = 1$ and $i + j = 2$. Here the trade $H_{3,2} \oplus (i, j)$ suffices.

If $(i, j) = (0, n-3)$ and $s \ne n-5$, our only choice for y is 2, but here $t = s - (n-3) + n = s + 3 \ne n - 2$ and so $t \ne n - y$. Thus we can use the latin trade $J_{n-2,2}(t) \oplus (0, n-3)$. In the special case where $(i, j) = (0, n-3)$ and $s = n - 5$ we use the following latin trade:

$$(H_{n-3,2} \circ_r G_{2,3}) \oplus (0, n-3).$$

The subset d_{n+s} is avoided.

If $(i, j) = ((n-3)/2, (n-3)/2)$, n is odd, and we may choose $y = (n-1)/2$. Suppose that $t = n - y$. Then $t = s - (n-3) + n = s + 3$ and $t = (n+1)/2$. But this implies that $s = (n-5)/2$, contradicting $s \ge \lfloor n/2 \rfloor - 1$. Thus $t \ne n - y$, and our choice $y = (n-1)/2$ satisfies Conditions 1 through to 6.

Case 1C: $\lfloor n/2 \rfloor \le i + j < s \le r - 1$.

Here we use Lemma 21 to construct $H_{n-y,y}$ for a carefully chosen integer y. We will then use Lemma 20 to place the latin trade $H_{n-y,y} \oplus (i, j)$ so that it contains $(i, j; i+j)$ and $(i+n-y, j+y; i+j)$ and otherwise intersects only $\bigcup d_p$ where $r \le p \le n + s - 1$. In order to do this we need Conditions 3, 4 and 5 from Case 1B, as well as $i + j + (n - y) \le n + s - 1$. But this last inequality follows from the fact that $i + j < s$.

This is a subset of the inequalities from Lemma 27, and $i+j < s \le n-3$, so we certainly have at least one value for y.

Case 1D: $n + r \le i + j \le 2n - 3$, or $i + j = n + s$. We first construct the latin trade $H_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ from Lemma 21. This latin trade contains the ordered triples $(0, 0; 0)$, $(\lfloor n/2 \rfloor, \lceil n/2 \rceil; 0)$, and all other entries occurring in $d_{\lfloor n/2 \rfloor} \cup d_{\lceil n/2 \rceil}$.

Now, $i, j \le n - 1$ and $i + j \ge n + s \ge n + \lfloor n/2 \rfloor - 1$ together imply that $j \ge \lceil n/2 \rceil$. Furthermore $j \le n - 1$ and $i + j \ge n + \lfloor n/2 \rfloor - 1$ imply that $i \ge \lfloor n/2 \rfloor$. So from Lemma 22 the latin trade

$$H_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \oplus (i - \lfloor n/2 \rfloor, j - \lceil n/2 \rceil)$$

contains the elements $(i - \lfloor n/2 \rfloor, j - \lceil n/2 \rceil; i + j - n)$, $(i, j; i + j - n)$ with all other elements chosen from $d_{i+j+\lfloor n/2 \rfloor - n}$ or $d_{i+j+\lceil n/2 \rceil - n}$.

But

$$i + j + \lfloor n/2 \rfloor - n \ge n + s + \lfloor n/2 \rfloor - n \ge 2\lfloor n/2 \rfloor - 1 \ge n - 2$$

and

$$i + j + \lceil n/2 \rceil - n \le n + \lceil n/2 \rceil - 3.$$

Therefore all ordered triples in

$$H_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \oplus (i - \lfloor n/2 \rfloor, j - \lceil n/2 \rceil)$$

except for $(i - \lfloor n/2 \rfloor, j - \lceil n/2 \rceil; i + j)$ (which lies in $d_{i+j} \notin \mathcal{D}_{r,s}$) and $(i, j; i + j - n)$ are in the following set: $\bigcup d_p$, where $r \leq n - 2 \leq p \leq \lceil n/2 \rceil + n - 3 \leq n + s - 1$. So our latin trade intersects $\mathcal{D}_{r,s}$ only in the required element.

Case 1E: $i + j = 2n - 2$. For the final case we use the latin trade consisting of all cells containing entries $n - 2$ or $n - 1$:

$$\{(\alpha, n - 2 - \alpha; n - 2), (\alpha, n - 1 - \alpha; n - 1) \mid 0 \leq \alpha \leq n - 1\}.$$

Case 2: Finally if $i > j$, we may use the transpose of the above latin trades, as both $\mathcal{D}_{r,s}$ and B_n are symmetric. □

In proving the next result (specifically Cases 2 and 4 of the proof) we utilize the construction given in Lemma 26. To aid the reader's understanding of this proof a diagram outlining this construction is given in Figure 11.

Lemma 29. Let $2 \leq y \leq x$, $y < z_1 < z_1 + 1 < z_2 < x + y$, $z_1, z_2 \neq x$ and $(z_1, z_2) \neq (y + 1, 2y)$. Then there exists a latin trade $K_{x,y}(z_1, z_2)$ in the latin square B_{x+y} with the following properties:

1. $K_{x,y}(z_1, z_2) \subseteq L(x, y, x + y)$, and includes the elements $(0, 0; 0)$, $(0, y; y)$, $(x, 0; x)$ and $(x, y; 0)$.
2. $K_{x,y}(z_1, z_2)$ intersects neither d_{z_1} nor d_{z_2}.
3. $K_{x,y}(z_1, z_2) \setminus \{(0, 0; 0), (x, y; 0)\}$ is a subset of $\bigcup d_p$, where $\min\{y, z_1 - y + 1\} \leq p \leq x$.

Proof. Case 1: $z_1 > x$. For this case $z_2 > z_1 > \max\{x, y\}$, so we may use the latin trade $H_{x,y}$ from Lemma 21.

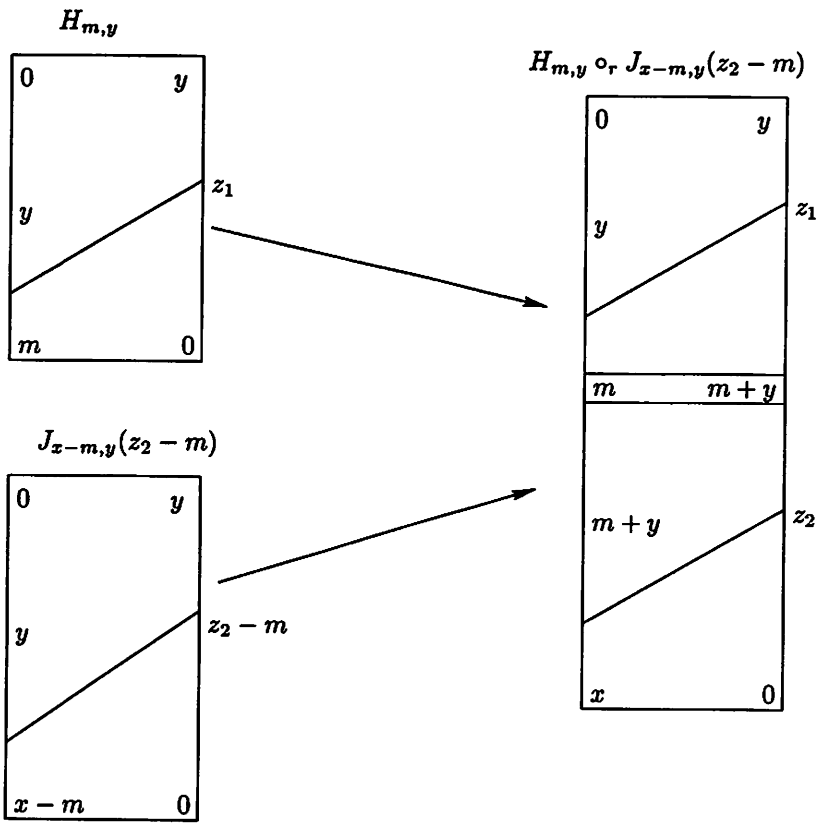
Case 2: $z_1 < x$, and either $x < 2y$ or $z_1 \geq 2y$. We wish to use Lemma 26 to construct the latin trade $J_{x+y-z_1-1,y}(z_2 + y - z_1 - 1)$. To ensure this is possible, we require the following conditions to hold:

1. $y \leq x + y - z_1 - 1$.
2. $y \neq z_2 + y - z_1 - 1$.
3. $x + y - z_1 - 1 \neq z_2 + y - z_1 - 1$.
4. $(x + y - z_1 - 1) + y \neq z_2 + y - z_1 - 1$.

But Conditions 1, 2, 3 and 4 are implied by $z_1 < x$, $z_2 \neq z_1 + 1$, $z_2 \neq x$ and $z_2 < y + x$ respectively. The latin trade $H_{z_1-y+1,y}$ exists from Lemma 21, because $z_1 > y$. Combining these latin trades gives our required latin trade:

$$K_{x,y}(z_1, z_2) = H_{z_1-y+1,y} \circ_r J_{x+y-z_1-1,y}(z_2 + y - z_1 - 1).$$

Observe that $H_{z_1-y+1,y}$ intersects neither d_{z_1} nor d_{z_2}. This follows because $z_1 > y$, $z_1 > z_1 - y + 1$ and $z_1 \neq (z_1 - y + 1) + y$. Also $z_2 > z_1 + 1 = (z_1 - y + 1) + y$. We next show that $J_{x+y-z_1-1,y}(z_2 + y - z_1 - 1)$



Overlapping of trades in Cases 2 and 4 of Lemma 29

Figure 11

avoids $d_{z_1-(z_1-y+1)} = d_{y-1}$ and $d_{z_2-(z_1-y+1)}$. The latter is true because by definition, $J_{x+y-z_1-1,y}(z_2+y-z_1-1)$ avoids $d_{z_2+y-z_1-1}$. The former is true because $y-1<y$.

So from Corollary 24, our latin trade $K_{x,y}(z_1,z_2)$ will intersect neither d_{z_1} nor d_{z_2} as required.

Case 3: $z_1 < x$, $x > 2y$, $y+1 < z_1 < 2y$ and $z_2 = 2y$. The required latin trade is:

$$K_{x,y}(z_1,z_2) = H_{y+1,y} \circ_r H_{x-y-1,y}.$$

Observe that $\max\{y, y+1\} < z_1 < (y+1)+y$. Therefore from Lemma 21, $H_{y+1,y}$ avoids d_{z_1}. Also $z_2 = 2y$ lies strictly between $y+1$ and $(y+1)+y$, so $H_{y+1,y}$ also avoids d_{z_2}.

Next we show that $H_{x-y-1,y}$ intersects neither $d_{z_1-(y+1)}$ nor $d_{z_2-(y+1)}$. This follows from the fact that $z_1-(y+1) < z_2-(y+1) < \min\{y, x-y-1\}$. Now apply Corollary 24 and we know that the combined latin trade misses the required elements.

Case 4: $z_1 < x$, $x \geq 2y$, $z_1 < 2y$ and $z_2 \neq 2y$. The required latin trade is:

$$K_{x,y}(z_1, z_2) = H_{y,y} \circ_r J_{x-y,y}(z_2 - y).$$

For $J_{x-y,y}(z_2 - y)$ to exist from Lemma 26 we require $y \leq x - y$ (which follows from $x \geq 2y$) and $z_2 - y \notin \{0, y, x-y, x\}$ (which follows from $z_2 > y$, $z_2 \neq 2y$, $z_2 \neq x$ and $z_2 < x+y$). To prove $H_{y,y}$ avoids d_{z_1} and d_{z_2}, observe that $y < z_1 < 2y$, $y < z_2$ and $z_2 \neq 2y$. Next we prove $J_{x-y,y}(z_2 - y)$ intersects neither d_{z_1-y} nor d_{z_2-y}. For the former observe that $z_1 - y < y$ and the latter is clear from the definition of $J_{x-y,y}(z_2 - y)$. So by Corollary 24, the latin trade $K_{x,y}(z_1, z_2)$ avoids both d_{z_1} and d_{z_2}.

□

Lemma 30. Let $\lfloor (n+4)/2 \rfloor \leq r \leq n-2$ and $\lfloor (n+2)/2 \rfloor \leq s \leq r-1$. Let $\mathcal{E}_{r,s}$ be as in Definition 7. Then for each entry $(i, j; k) \in \mathcal{E}_{r,s}$ there exists a latin trade I in B_n such that $I \cap \mathcal{E}_{r,s} = \{(i, j; k)\}$.

Proof. We split our proof into two cases. In Case 1, $i \leq j$ and in Case 2, $i > j$. Case 1 is then to be split into the subcases 1A to 1E. As the partial latin square $\mathcal{E}_{r,s}$ is similar to $\mathcal{D}_{r,s}$, most cases can be verified using Lemma 28. However care must be taken to ensure that $d_{\lfloor (3n-2)/2 \rfloor}$ is also avoided.

Case 1A: $i+j < \lfloor (n-2)/2 \rfloor$. Apply Case 1A in Lemma 28. Since $n+i+j < \lfloor (3n-2)/2 \rfloor$, $d_{\lfloor (3n-2)/2 \rfloor}$ is avoided.

Case 1B: $\lfloor n/2 \rfloor \leq i+j < s$. Here we need the latin trade $J_{n-y,y}(t) \oplus (i, j)$ constructed as in Lemma 26, with $t = \lfloor (3n-2)/2 \rfloor - (i+j)$. For this to exist, and not to intersect the required entries we need:

1. $2 \leq y \leq n/2$.
2. $t \notin \{y, n-y, n\}$.
3. $y + j \leq n - 1$.
4. $n - y + i \leq n - 1$.
5. $y + (i + j) \geq (n - 2)$.
6. $(i + j) + (n - y) \leq n + s - 1$.

Note that Conditions 1 and 2 above will be required for the implementation of Lemma 26, Conditions 3 and 4 for the implementation of Lemma 20 and Condition 5 is required since we require $y + i + j \geq r$.

Condition 6 above follows from the fact that $i+j < s$. We cannot have $t = n$ in Condition 2 because $i+j \geq \lfloor n/2 \rfloor$. Next, $i+j < s$ and $s \leq n-3$. Since $t = \lfloor (3n-2)/2 \rfloor - (i+j)$ it follows that $t > \lfloor (3n-2)/2 \rfloor - (n-3) \geq$

$\lfloor n/2 \rfloor + 2$. But $y \le n/2$ by Condition 1. So $t > y$. But from Lemma 27, as $i + j < s \le n - 3$ there are at least two solutions to Conditions 1, 3, 4 and 5, and thus we may choose one of these solutions not equal to t. Then apply Lemma 22 to ensure the latin trade is correctly placed.

Case 1C: $s < i + j \le r - 1$. We deal with this case at the end of proof.

Case 1D: $\lfloor (3n-2)/2 \rfloor \le i + j \le 2n - 3$. Apply Case 1D in Lemma 28. Since $n + \lceil n/2 \rceil - 3 \le \lfloor (3n-2)/2 \rfloor$, $d_{\lfloor (3n-2)/2 \rfloor}$ is avoided.

Case 1E: $i + j = 2n - 2$. Apply Case 1E in Lemma 28.

Now we settle the Case 1C ($s < i + j \le r - 1$). Note that this case exists only for $n \ge 9$. Here we will construct the latin trade

$$K_{n-y,y}(\lfloor (3n-2)/2 \rfloor - (i+j), n + s - (i+j))$$

using Lemma 29. We must choose y carefully to ensure that the following conditions of Lemma 29 are satisfied:

1. $2 \le y \le n/2$.
2. $y < \lfloor (3n-2)/2 \rfloor - (i+j)$.
3. $\lfloor (3n-2)/2 \rfloor - (i+j) + 1 < n + s - (i+j)$.
4. $n + s - (i+j) < n$.
5. $n - y \ne \lfloor (3n-2)/2 \rfloor - (i+j)$.
6. $n - y \ne n + s - (i+j)$.
7. $(\lfloor (3n-2)/2 \rfloor - (i+j), n + s - (i+j)) \ne (y+1, 2y)$.

Some of these inequalities may be quickly taken care of. Conditions 3 and 4 follow from the fact that $\lfloor (n+2)/2 \rfloor \le s < i + j$. If we assume Condition 1 we can remove Condition 7. To see this, suppose that:

$$\begin{aligned} \lfloor (n-2)/2 \rfloor - (i+j-n) &= y+1 \text{ and} \\ s - (i+j-n) &= 2y. \end{aligned}$$

Manipulating to cancel y gives

$$s + (i+j) = 2\lfloor n/2 \rfloor + n - 4. \qquad (1)$$

But $i+j \le r-1 \le n-3$ and $s < i+j$, so $s + (i+j) \le 2n-7$, contradicting (1).

Next we want to ensure that $K_{n-y,y}(\lfloor (3n-2)/2 \rfloor - (i+j), n+s-(i+j)) \oplus (i,j)$ does not intersect $\mathcal{E}_{r,s}$. To prove this we need:

8. $n - y + i \le n - 1$.
9. $y + j \le n - 1$.
10. $y + (i+j) \ge n - 2$.
11. $\lfloor (3n-2)/2 \rfloor - (i+j) - y + 1 + (i+j) \ge n - 2$.
12. $n - y + (i+j) \le n + r - 1$.

By Lemma 20 and Condition 3 of Lemma 29 these Conditions ensure that $K_{n-y,y}(\lfloor(3n-2)/2\rfloor-(i+j), n+s-(i+j))\oplus(i,j)$ intersects only $(i,j;i+j)$, and the following: $\bigcup d_p$, where $n-2 \leq p \leq r-1$ and $p \notin \{n+i+j, n+s, \lfloor(3n-2)/2\rfloor\}$. Condition 11 is equivalent to $y \leq \lfloor n/2\rfloor+2$ so it is implied by Condition 1. Condition 12 is met because $i+j \leq r-1$.

We wish to show that there exists two (consecutive) integer solutions for y under the following constraints: $y \geq 2$, $y \geq n-2-(i+j)$, $y \geq i+1$ and $y \leq n/2$, $y \leq n-1-j$. We can then choose *one* solution for y so that *both* Conditions 5 and 6 are satisfied, because Condition 3 states that $\lfloor(3n-2)/2\rfloor-(i+j)$ and $n+s-(i+j)$ are *not* consecutive.

From Lemma 27, we can find such a y except possibly when $n \leq 5$, $(i,j) \in \{(0,n-3), ((n-3)/2,(n-3)/2)\}$. We can rule the first inequality out because $n \geq 9$, as stated at the beginning of this case.

If $(i,j) = ((n-3)/2,(n-3)/2)$, n is odd, and we choose $y=(n-1)/2$. For these values of i, j and y, $n-y \neq \lfloor(3n-2)/2\rfloor-(i+j)$. Moreover $s \geq \lfloor(n+2)/2\rfloor$ implies $n-y \neq n+s-(i+j)$.

If $(i,j)=(0,n-3)$ and $s \neq n-5$, our only choice for y is 2, but here $n-y \neq n+s-(i+j)$ and $n-y \neq \lfloor(3n-2)/2\rfloor-(i+j)$ (since $n \geq 9$). Thus we can use the latin trade $J_{n-2,2}(n+s-(i+j)) \oplus (0,n-3)$.

Finally consider the special case where $(i,j)=(0,n-3)$ and $s=n-5$. If $n=9$, we must have $r=7$ and $s=5$, contradicting $s=n-5$. Similarly if $n=10$, $s \neq 5$. Thus $n \geq 11$, $n-3 \geq \lfloor(n+4)/2\rfloor$ and we use the following latin trade:

$$(J_{n-3,2}(\lfloor(n+4)/2\rfloor) \circ_r G_{2,3}) \oplus (0,n-3).$$

<u>Case 2</u>: Finally if $i > j$, we may use the transpose of the above latin trades, as both $\mathcal{E}_{r,s}$ and B_n are symmetric. □

Next we turn our attention to the partial latin square C_r. It is worth noting that C_r is similar to $\mathcal{D}_{n-1}$ except that C_r contains some occurrences of the symbol $n-1$ and C_r has additional empty cells in row 0 and column 0. To prove that all the entries of C_r are necessary for unique completion one constructs latin trades which intersect each entry $(i,j;k) \in C_r$ in that entry alone. Care must be taken to ensure these latin trades avoid the required entries.

Definition 31. Let $I \subseteq L(x,x,n)$ be a latin trade. Then the *reversal of I about the entry x*, denoted by $I^R(x)$, is the latin trade in B_n formed by taking the mirror image of I along the set of cells d_x of $L(x,x,n)$. More formally:

$$I^R(x) = \{(x-j, x-i; 2x-k(\text{mod } n)) \mid (i,j;k) \in I\}.$$

Example 32. In Figure 12 we give the latin trade $G_{4,5} \subseteq B_5$ (from Lemma 18) on the left, with $G^R_{4,5}(4)$ on the right.

0				4
			4	0
		4	0	
	4	0		
4	0			

			3	4
		3	4	
	3	4		
3	4			
4				3

Figure 12

Lemma 33. Let C_r, $1 \leq r \leq n-2$, be the partial latin squares given in Definition 9. Then for each element $(i,j;i+j) \in C_r$ there exists a latin trade I in B_n such that $I \cap C_r = \{(i,j;i+j)\}$.

Proof. Let $(i,j;i+j)$ be an element of C_r. We split our proof into nine cases.

Case 1: $0 < j \leq (n-2)/2$, $i \geq (n-2)/2$ and $i+j \leq n-2$. Here we use the latin trade $H_{n-1-i,i+1} \oplus (i,j)$. From Lemma 21, and since $i+1 \geq n-1-i$ (equivalently $i \geq (n-2)/2$), we know that $H_{n-1-i,i+1} \setminus \{(0,0;0)\}$ is a subset of $\cup d_p$, where $n-1-i \leq p \leq n$. Since $i+n-1-i = n-1 < n$ and $j+i+1 \leq n-1 < n$, from Lemma 20 we know that $(H_{n-1-i,i+1} \oplus (i,j)) \setminus \{(i,j;i+j)\}$ is a subset of $\bigcup d_p$, where $n-1+j \leq p \leq n+(i+j)$. Since $j > 0$, this partial latin square does not intersect C_r. It follows that $H_{n-1-i,i+1} \oplus (i,j) \cap C_r = \{(i,j;i+j)\}$.

Case 2: $i = 0$, $0 < j \leq (n-2)/2$. Note that from the definition of C_r, we must have $j > r$. Here we use the latin trade $G_{n-1-j,n} \oplus (0,j)$ from Lemma 18, which exists because $0 < j \leq (n-2)/2$. From Lemma 18, we know that $G_{n-1-j,n} \setminus \{(0,0;0)\}$ is a subset of $\cup d_p$, where $n-1-j \leq p \leq n$. Since $n-1-j < n$ and $n-1-j+j = n-1 < n$, from Lemma 20 we know that $(G_{n-1-j,n} \oplus (0,j)) \setminus \{(0,j;j)\}$ is a subset of $\bigcup d_p$, where $n-1 \leq p \leq n+j$.
The only place this partial latin square might intersect C_r, then, is in the subset d_{n-1} when $i = 0$. But the entry $n-1$ occurs only in columns 0 through to r of C_r, and we have $j > r$. Thus we can safely say that this partial latin square avoids $d_{n-1} \cap C_r$. It follows that $G_{n-1-j,n} \oplus (0,j) \cap C_r = \{(0,j;j)\}$.

Case 3: $j > (n-2)/2$ and $i+j \leq n-2$. Here we use the latin trade $H_{j+1,n-1-j} \oplus (i,j)$ from Lemma 21. Since $j > (n-2)/2$, $j+1 > n-1-j$, and we have that $H_{j+1,n-1-j} \setminus \{(0,0;0)\}$ is a subset of $\cup d_p$, where $n-1-j \leq p \leq n$. Since $i+j+1 < n$ and $n-1-j+j = n-1 < n$, from Lemma 20 we know that $(H_{j+1,n-1-j} \oplus (i,j)) \setminus \{(i,j;i+j)\}$ is a subset of $\bigcup d_p$, where $n-1+i \leq p \leq n+i+j$.

Now, $n-1+i \geq n-1$, with equality only possible if $i=0$. But if $i=0$, from the definition of C_r, we must have $j > r$. And the entry $n-1$ occurs only in columns 0 through to r of C_r. Thus we can safely say that this partial latin square avoids $d_{n-1} \cap C_r$. It follows that $H_{j+1,n-1-j} \oplus (i,j) \cap C_r = \{(i,j;i+j)\}$.

Case 4: $i,j \geq 1$ and $i+j \leq n/2$. We use the latin trade $G_{n-(i+j),n} \oplus (i,j)$, constructed in Lemma 18, which exists because $n-(i+j) \geq n/2$. Since $n-(i+j)+i < n$ and $n-(i+j)+j < n$, from Lemmas 18 and 20, we know that $G_{n-(i+j),n} \oplus (i,j) \setminus \{(i,j;i+j)\}$ is contained in $\cup d_p$, where $n \leq p \leq n+(i+j)$, which does not intersect C_r.

Case 5: $i,j \geq 1$, $i+j > n/2$, $i,j \leq (n-2)/2$. We use the latin trade $G_{\lceil n/2 \rceil,n} \oplus (i,j)$, constructed in Lemma 18. Since $\lceil n/2 \rceil + i < n$ and $\lceil n/2 \rceil + j < n$ then from Lemmas 18 and 20, we know that $G_{\lceil n/2 \rceil,n} \oplus (i,j) \setminus \{(i,j;i+j)\}$ is contained in $\cup d_p$, where $n \leq n/2 + \lceil n/2 \rceil < p \leq n+(i+j)$, which does not intersect C_r.

Case 6: $i+j = n-1$. Here we use the latin trade $H_{n-1-i,i+1} \oplus (i,j)$. This will overlap column 0. However if $(i,j;n-1) \in C_r$, then $i \geq n-r-1$. Also C_r has elements of the form $(0,x;x)$ only if $x < n-r-1$.

Case 7: $j=0$, $i < (n-r-2)/2$. Here we use the reversal of a latin trade (see Definition 31). Our latin trade is as follows:

$$(G^R_{n-r-i-1,n-r}(n-r-i-1) \circ_r H_{r,n-i-1-r}) \oplus (i, r+i+1).$$

The intersection of this trade with column 0 is a subset of $\{(0,i,i)\} \cup \{(0,x;x) \mid n-r-1 \leq x \leq n-1\}$. Also it is easy to see that this latin trade avoids $d_{n-1} \cap C_r$.

Case 8: $j=0$, $(n-r-2)/2 \leq i \leq (n-2)/2$. Our latin trade is:

$$\begin{aligned}&\{(i,n-i-1;n-1),(i,0;i),(n-1,0;n-1)\}\\ &\cup(H_{n-2i-2,i+1} \setminus \{(n-2i-2,i+1;0)\} \oplus (2i+1,n-i-1)).\end{aligned}$$

This latin trade will intersect column 0 only in the element $(i,0;i)$ and elements of the form $(x,0;x)$, where $2i+1 \leq x \leq n-1$. But $2i+1 \geq n-r-1$, so the latin trade only intersects C_r in the required element.

Case 9: $j=0$, $i > (n-2)/2$. Here we use the latin trade $H_{n-1-i,i+1} \oplus (i,j)$. This will overlap column 0. However, since $n-1-i < i+1$, this latin trade intersects column 0 only in $(0,i;i)$ and $(0,n-1;n-1)$. Also $(i,0;i) \in C_r$ implies that $(i,n-1+i;n-1) \notin C_r$. Thus the latin trade meets C_r in the required element only. □

Example 34. In Figure 13 we give a latin trade I in B_6 such that $I \cap C_1 = \{(0,0;0)\}$. This is Case 7 of Lemma 33, and $I = (G^R_{4,5}(4) \circ_r H_{1,4}) \oplus (0,2)$.

The partial latin square C_1 is given on the left, while I is given on the right.

0		2	3	4	
1	2	3	4		
2	3	4			
3	4				
	5				

0					5
				5	0
			5	0	
		5	0		
4		0	1	2	3
5		1	2	3	4

Figure 13

6. Main results

We are now in a position to state and prove our main results.

Theorem 35. Let $\lfloor n/2 \rfloor \le r \le n-2$ and $\lfloor (n-2)/2 \rfloor \le s \le r-1$. Then the partial latin square $\mathcal{D}_{r,s}$ (see Definition 7) is a strong critical set in B_n.

Proof. Let $I = \{0, 1, \dots, r-1\} \setminus \{s\}$ and $J = \{r, r+1, \dots, n-2\} \cup \{s\}$. Obviously, $\mathcal{P} = \{I, J\}$ is a partition of the set $\{0, 1, 2 \dots, n-2\}$. Moreover,

$$\mathcal{D}_{\mathcal{P}} = (\bigcup_{i \in I} d_i) \bigcup (\bigcup_{j \in J} d_{n+j}) = (\mathcal{D}_r \setminus d_s) \bigcup (\mathcal{D}_{n+r} \cup d_{n+s}).$$

So, by Lemma 13, $(\mathcal{D}_r \setminus d_s) \bigcup (\mathcal{D}_{n+r} \cup d_{n+s})$ is a strong uniquely completable set in B_n. Also, Lemma 28 shows that each element of $(\mathcal{D}_r \setminus d_s) \bigcup (\mathcal{D}_{n+r} \cup d_{n+s})$ is necessary for the unique completion. □

Corollary 36. Let n be odd, $(n-1)/2 \le r \le n-2$ and $(n-3)/2 \le s \le r-1$. Then, in B_n, there exists a critical set of size

$$\frac{r(r+1)}{2} + \frac{(n-r)(n-r-1)}{2} - (s+1) + (n-s-1).$$

Corollary 37. Let n be even, $n/2 \le r \le n-2$ and $(n-2)/2 \le s \le r-1$. Then, in B_n, there exists a critical set of size

$$\frac{r(r+1)}{2} + \frac{(n-r)(n-r-1)}{2} - (s+1) + (n-s-1).$$

Theorem 38. Let $\lfloor (n+4)/2 \rfloor \le r \le n-2$ and $\lfloor (n+2)/2 \rfloor \le s \le r-1$. Then the partial latin square $\mathcal{E}_{r,s}$ (see Definition 7) is a strong critical set in B_n.

Proof. Let $I = \{0, 1, \dots, r-1\} \setminus \{s, \lfloor (n-2)/2 \rfloor\}$ and $J = \{r, r+1, \dots, n-2\} \cup \{s, \lfloor (n-2)/2 \rfloor\}$. Obviously, $\mathcal{P} = \{I, J\}$ is a partition of the set $\{0, 1, 2 \dots, n-2\}$. Moreover,

$$\mathcal{D}_{\mathcal{P}} = (\bigcup_{i \in I} d_i) \bigcup (\bigcup_{j \in J} d_{n+j}) = \mathcal{E}_{r,s}.$$

So, by Lemma 13, $\mathcal{E}_{r,s}$ is a strong critical set in B_n. On the other hand, Lemma 30 shows that each element of $\mathcal{E}_{r,s}$ is necessary for the unique completion. This completes the proof. □

When n is odd by Lemma 5 we have $|d_{n+(n-3)/2}| = |d_{(n-3)/2}| + 1$. This leads to the following corollary.

Corollary 39. Let n be odd, $(n+5)/2 \leq r \leq n-2$ and $(n+1)/2 \leq s \leq r-1$. Then, in B_n, there exists a critical set of size

$$\frac{r(r+1)}{2} + \frac{(n-r)(n-r-1)}{2} - (s+1) + (n-s-1) + 1.$$

Remark 40. By Lemma 5 we have $|d_{n+(n-2)/2}| = |d_{(n-2)/2}| = n/2$ for n even. So Theorems 35 and 38 generate strong critical sets in B_n of the same sizes.

Theorem 41. The partial Latin square C_r (see Definition 9) is a strong critical set in B_n.

Proof. By Lemma 14, each C_r is a strong completable set in B_n. On the other hand, Lemma 33 shows that each element of C_r is necessary for the unique completion. This completes the proof. □

Corollary 42. Let $((n^2-n)/2)-(n-2) \leq t \leq (n^2-n)/2$. Then there exists a strong critical set of size t in B_n.

Theorem 43. There exists a strong critical set of size m in the back circulant square of order n for all $\frac{n^2-1}{4} \leq m \leq \frac{n^2-n}{2}$, when n is odd, and for all $\frac{n^2-n}{2}-(n-2) \leq m \leq \frac{n^2-n}{2}$ and $m \in \{n^2/4, (n^2/4)+2, (n^2/4)+4, \dots, \frac{n^2-n}{2}-n\}$ when n is even.

Finally we make a comment about what the complete spectrum for the size of a critical set in back circulant square of order n might be. We give two conjectures made by Bate and van Rees [1]:

Conjecture 1 For all $n \geq 1$, the size of the smallest critical set in a back-circulant latin square is $\lfloor n^2/4 \rfloor$.

Conjecture 2 For all even $n \geq 6$, there exists no critical set of order $\lfloor n^2/4 \rfloor + 1$.

Bate and van Rees [1] showed these conjectures to be true for all $n \leq 12$ ($n \neq 11$). This indicates that some of the "holes" in the spectrum may not be filled when n is even.

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