ON OPTIMAL ORIENTATIONS OF TENSOR PRODUCT OF COMPLETE GRAPHS

R. Lakshmi and P. Paulraja
Department of Mathematics
Annamalai University
Annamalainagar-608 002
Tamilnadu, India.

Abstract. For a graph G, let $\mathcal{D}(G)$ be the set of strong orientations of G. Define $\vec{d}(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\}$ and $\rho(G) = \vec{d}(G) - d(G)$, where $d(D)$ (resp. $d(G)$) denotes the diameter of the digraph D (resp. graph G). In this paper, we determine the exact value of $\rho(K_r \times K_s)$, for $r \leq s$ and $(r,s) \notin \{(3,5),(3,6),(4,4)\}$, where $K_r \times K_s$ denotes the tensor product of K_r and K_s. Using the results obtained here, a known result on $\rho(G)$, where G is a regular complete multipartite graph is deduced as corollary.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity*, denoted by $e_G(v)$, of v is defined as $e_G(v) = \max\{d_G(v,x) \mid x \in V(G)\}$, where $d_G(v,x)$ denotes the distance from v to x in G. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{e_G(v) \mid v \in V(G)\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has neither loops nor multiple arcs (that is, a pair of arcs with same tail and same head). For $v \in V(D)$, the notations $e_D(v)$ and $d(D)$ are defined as in the undirected graph. For $x, y \in V(D)$, we write $x \rightarrow y$ or $y \leftarrow x$ if $(x,y) \in A(D)$. For $V' \subset V(D)$ and $x \notin V'$, by $x \rightarrow V'$ (or $V' \leftarrow x$), we mean that x is adjacent to all the vertices of V' i.e., $(x,v) \in A(D)$ for all $v \in V'$. For vertices $x_1, x_2, \ldots, x_k$ of V and a subset V' of $V \setminus \{x_1, x_2, \ldots, x_k\}$, we write $x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_k \rightarrow V'$ for the set $\{x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_k \rightarrow v' \mid v' \in V'\}$ of directed paths, where $x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_k \rightarrow v'$ represents the directed path with arcs

$x_1 \to x_2, x_2 \to x_3, \ldots, x_{k-1} \to x_k$ and $x_k \to v'$. For $v \in V(D)$, $N_D^+(v)$ denotes the set of out-neighbours of v in D. For $x \in V(D)$ and $V' \subseteq V(D)$, by $d_D(x, V') \leq k$, we mean $d_D(x, a) \leq k$, for all $a \in V'$. We call a digraph D to be *k-regular* if $d_D^+(v) = d_D^-(v) = k$ for every $v \in V(D)$. A digraph D is *vertex-transitive* if for every pair of vertices u, $v \in V(D)$, there is an automorphism that maps u to v.

For graphs G and H, the *tensor product*, $G \times H$, of G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{(u,v)(x,y) : ux \in E(G)$ and $vy \in E(H)\}$. Let $V(G) = \{x_1, x_2, \ldots, x_n\}$ and $V(H) = \{y_1, y_2, \ldots, y_m\}$; for our convenience we write $V(G \times H)$ as follows: $V(G \times H) = V(G) \times V(H) = \bigcup_{i=1}^{n} \{\{x_i\} \times V(H)\} = \bigcup_{j=1}^{m} \{V(G) \times \{y_j\}\}$. We call $X_i = \{x_i\} \times V(H)$ as a *G-layer* and $Y_j = V(G) \times \{y_j\}$ as a *H-layer* in $G \times H$. If G and H are connected and nontrivial, then $G \times H$ is connected if and only if at least one of G and H is nonbipartite. Clearly, the tensor product is commutative. Hence while considering $K_r \times K_s$ we always assume that $r \leq s$.

For graphs G and H, the *cartesian product*, $G \Box H$, of G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \Box H) = \{(u,v)(x,y) : v = y$ and $ux \in E(G)$ or $u = x$ and $vy \in E(H)\}$.

Let K_n denote the complete graph of order n. For our discussion, we assume that $V(K_n) = \{0, 1, \ldots, n-1\}$.

An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' celebrated one-way street theorem states that a connected graph G has a strong orientation if and only if G is 2-edge-connected [16]. Given a 2-edge-connected graph G, let $\mathcal{D}(G)$ be the set of all strong orientations of G. The *orientation number* of G is defined to be $\vec{d}(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\}$. In [11], $\vec{d}(G) - d(G)$ is defined as $\rho(G)$.

Any orientation D in $\mathcal{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G. The problem of evaluating the orientation number of an arbitrary connected graph is very difficult. Chvátal and Thomassen [3], among other results, obtained $\vec{d}(G) \leq d(2d+1)$ if $d \geq 3$ and $\vec{d}(G) \leq 6$ if $d = 2$, where d is the diameter of the 2-edge-connected graph G. Further, they have shown that the problem of deciding whether a graph admits an orientation of diameter 2 is NP-hard. Goldberg [4] evaluated the extreme value of the diameter of a strong digraph with n vertices and $n+m$ arcs; it states that if G is a 2-edge-connected graph with n vertices and $n+m$ edges, where $n \geq 4$ and $m \geq 1$, then $\vec{d}(G) \geq \left\lceil \frac{2(n-1)}{m+1} \right\rceil$. The parameter $\vec{d}(G)$ has also been studied in various particular classes of graphs including the complete n-partite graphs, see ([2], [5], [6], [7], [9], [10], [12], [13], [14], [15] and [17]) and the cartesian product of graphs (see the references in [11]). Optimal orientations have variety of applications, see [11]. For further results on orientations of graphs see [11], a recent survey by Koh and Tay.

Notations and terminology not defined here can be seen in [1].

In this paper, we focus on the orientation number of $K_r \times K_s$. $K_r \times K_s$ is connected if and only if $r \geq 2$, $s \geq 2$ and $(r,s) \neq (2,2)$ or $(r,s) = (1,1)$. In fact, $K_r \times K_s$ is isomorphic to both $K_r(s) - E(sK_r)$ and $K_s(r) - E(rK_s)$, where $K_r(s)$ denotes the complete r-partite graph in which each partite set has s vertices and sK_r denotes s disjoint copies of K_r. As any two distinct G-layers X_i and X_j of $K_r \times K_s$ induce a subgraph isomorphic to $K_{s,s} - F_0$, where $F_0 = \{(i,k)(j,k) : k \in V(K_s)\}$ is the 1-factor of $K_{s,s}$, $d(K_r \times K_s) = 2$ or 3 according as $\min\{r,s\} \geq 3$ or $\min\{r,s\} = 2$ and $r \neq s$.

In this paper, we obtain the exact value of $\rho(K_r \times K_s)$ for almost all the values of r and s. It is shown that for $r \leq s$ and $(r,s) \notin \{(3,5),(3,6),(4,4)\}$,

$$\rho(K_r \times K_s) = \begin{cases} 2 & if\ (r,s) \in \{(2,3),(2,4)\}, \\ 1 & if\ (r,s) \in \{(3,3),(3,4)\}, \\ 0 & otherwise. \end{cases}$$

For the exceptional values, $(r,s) \in \{(3,5), (3,6), (4,4)\}$, it has been proved that $\rho(K_r \times K_s) \leq 1$. Further, as $K_r \times K_s$ is a spanning subgraph of $K_r(s)$, we deduce that $\rho(K_r(s)) = 0$ for $r \geq 3$, $s \geq 3$ and $(r,s) \notin \{(3,3),(3,4),(3,5),(3,6),(4,3),(4,4),(5,3),(6,3)\}$. For $r \geq 2$ and $s \geq 2$, $\rho(K_r(s)) = 0$ was proved in [6].

2 Optimal orientations of $K_r \times K_s$

In the sequel, we use the following notations. Let $V(K_r) = \{0,1,\ldots,r-1\}$ and $V(K_s) = \{0,1,\ldots,s-1\}$. The vertex $(i+k, j+l)$ of $K_r \times K_s$ represents the vertex $((i+k)(mod\, r), (j+l)(mod\, s))$. Recall that for $i \in V(K_r)$ and $j \in V(K_s)$, $X_i = \{(i,k) : k \in V(K_s)\}$, $Y_j = \{(k,j) : k \in V(K_r)\}$ and $V(K_r \times K_s) = \bigcup_{i=0}^{r-1} X_i = \bigcup_{j=0}^{s-1} Y_i$. Throughout this paper, addition in the subscripts of the G-layer X_i and the H-layer Y_j are taken, respectively, modulo r and modulo s. The main theorem of this paper follows from a sequence of lemmas that we shall prove below.

Lemma 2.1. *If both r and s are even and $r,s \geq 6$, then $\rho(K_r \times K_s) = 0$.*

Proof. Delete the set of edges

$E_1 = \{(i,j)(i+\frac{r}{2}, j+\frac{s}{2}) : i \in V(K_r),\ j \in V(K_s)\}$

of $K_r \times K_s$ and denote the resulting subgraph by H. We orient H so that for any $i \in V(K_r)$ and $j \in V(K_s)$,

$(i,j) \rightarrow \{\{(i+1,j+1),(i+1,j+2),\ldots,(i+1,j+\frac{s}{2})\} \cup$

$$\left\{\bigcup_{k=i+2}^{i+\frac{s}{2}-1}\{(k,j+2),(k,j+3),\ldots,(k,j+\tfrac{s}{2}),(k,j-1)\}\right\}\cup$$
$$\left\{\bigcup_{k=i+\frac{s}{2}}^{i-2}\{(k,j+2),(k,j+3),\ldots,(k,j+\tfrac{s}{2}-1),(k,j-1)\}\right\}\cup$$
$$\{(i-1,j+1),(i-1,j+2),\ldots,(i-1,j+\tfrac{s}{2}-1)\}\}.$$

Let D be the resulting digraph. D is vertex-transitive (see (1) of Appendix) and $\frac{rs-r-s}{2}$-regular. We claim that $d(D) = 2$; it follows that $\vec{d}(K_r \times K_s) = 2$ and, this in turn implies that $\rho(K_r \times K_s) = 0$. $d(D) = 2$ is shown by proving $e_D((i,j)) = 2$ for all $(i,j) \in V(D)$. To prove this, we show that $d_D((i,j),X_i) \leq 2$, $d_D((i,j),X_{i+1}) \leq 2$, $d_D((i,j),X_{i+l}) \leq 2$ for every $l \in \{2,3,\ldots,r-3\}$, $d_D((i,j),X_{i-2}) \leq 2$ and $d_D((i,j),X_{i-1}) \leq 2$ in order.

The existence of the paths $(i,j) \to (i+2,j-1) \to (i,j+1)$, $(i,j) \to (i+1,j+1) \to \{(i,j+2),(i,j+3),\ldots,(i,j+\frac{s}{2})\}$ and $(i,j) \to (i+1,j+\frac{s}{2}) \to \{(i,j+\frac{s}{2}+1),(i,j+\frac{s}{2}+2),\ldots,(i,j-1)\}$ in D shows that $d_D((i,j),X_i\backslash\{(i,j)\}) \leq 2$, and hence $d_D((i,j),X_i) \leq 2$.

The existence of the paths $(i,j) \to (i+2,j+\frac{s}{2}) \to \{(i+1,j+\frac{s}{2}+1),(i+1,j+\frac{s}{2}+2),\ldots,(i+1,j-1)\}$ and $(i,j) \to (i+2,j-1) \to (i+1,j)$ in D shows that $d_D((i,j),X_{i+1}\backslash N_D^+((i,j))) \leq 2$, and hence $d_D((i,j),X_{i+1}) \leq 2$.

For every $l \in \{2,3,\ldots,r-3\}$, the existence of the paths $(i,j) \to (i+l+1,j+\frac{s}{2}-1) \to \{(i+l,j+\frac{s}{2}),(i+l,j+\frac{s}{2}+1),\ldots,(i+l,j-2)\}$ and $(i,j) \to (i+l+1,j-1) \to \{(i+l,j),(i+l,j+1)\}$ in D shows that $d_D((i,j),X_{i+l}\backslash N_D^+((i,j))) \leq 2$, and hence $d_D((i,j),X_{i+l}) \leq 2$.

The existence of the paths $(i,j) \to (i-3,j+\frac{s}{2}-1) \to \{(i-2,j+\frac{s}{2}),(i-2,j+\frac{s}{2}+1),\ \ldots,(i-2,j-2)\}$ and $(i,j) \to (i-3,j-1) \to \{(i-2,j),(i-2,j+1)\}$ in D shows that $d_D((i,j),X_{i-2}\backslash N_D^+((i,j))) \leq 2$, and hence $d_D((i,j),X_{i-2}) \leq 2$.

The existence of the paths $(i,j) \to (i-2,j+\frac{s}{2}-1) \to \{(i-1,j+\frac{s}{2}),(i-1,j+\frac{s}{2}+1),\ldots,(i-1,j-1)\}$ and $(i,j) \to (i-2,j-1) \to (i-1,j)$ in D shows that $d_D((i,j),X_{i-1}\backslash N_D^+((i,j))) \leq 2$, and hence $d_D((i,j),X_{i-1}) \leq 2$. ■

Lemma 2.2. *If $r \geq 5$ is an integer and $s \geq 7$ is odd, then $\rho(K_r \times K_s) = 0$. Furthermore, $\rho(K_5 \times K_5) = 0$.*

Proof. We orient $K_r \times K_s$ so that for any $i \in V(K_r)$ and $j \in V(K_s)$,

$(i,j) \to \{(k,j+2),(k,j+3),\ldots,(k,j+\frac{s-1}{2}),(k,j-1)\}$

whenever $k \notin \{i-1,i+1\}$ and

$(i,j) \to \{(k,j+1),(k,j+2),\ldots,(k,j+\frac{s-1}{2})\}$ for $k \in \{i-1,i+1\}$.

We denote the resulting digraph by D. Clearly, D is $(r-1)(\frac{s-1}{2})$-regular and vertex-transitive (see (1) of Appendix). We claim that $d(D) = 2$.

For $r = s = 5$, the verification of $e_D((0,0)) = 2$ is easy and hence it is omitted; consequently $d(D) = 2$, as D is vertex-transitive.

We complete the proof by showing $d_D((i,j),X_i) \le 2$, $d_D((i,j),X_{i+1}) \le 2$, $d_D((i,j),X_{i+2}) \le 2$ and for $t \in V(K_r)\backslash\{0,1,2\}$, $d_D((i,j),X_{i+t}) \le 2$, in order, when $r \ge 5$ and $s \ge 7$ is odd.

The existence of the paths $(i,j) \to (i+1,j+1) \to \{(i,j+2),(i,j+3),\ldots,(i,j+\frac{s+1}{2})\}$, $(i,j) \to (i+1,j+\frac{s-1}{2}) \to \{(i,j+\frac{s+3}{2}),(i,j+\frac{s+5}{2}),\ldots,(i,j-1)\}$ and $(i,j) \to (i+2,j-1) \to (i,j+1)$ in D shows that $d_D((i,j),X_i\backslash\{(i,j)\}) \le 2$, and hence $d_D((i,j),X_i) \le 2$.

The existence of the paths $(i,j) \to (i+2,j-1) \to (i+1,j)$, $(i,j) \to (i+2,j+\frac{s-3}{2}) \to \{(i+1,j+\frac{s+1}{2}),(i+1,j+\frac{s+3}{2}),\ldots,(i+1,j-2)\}$ and $(i,j) \to (i+2,j+\frac{s-1}{2}) \to (i+1,j-1)$ in D shows that $d_D((i,j),X_{i+1}\backslash N_D^+((i,j))) \le 2$, and hence $d_D((i,j),X_{i+1}) \le 2$.

The existence of the paths $(i,j) \to (i+1,j+\frac{s-1}{2}) \to \{(i+2,j+\frac{s+1}{2}),(i+2,j+\frac{s+3}{2}),\ldots,(i+2,j-2)\}$, $(i,j) \to (i-1,j+1) \to (i+2,j)$ and $(i,j) \to (i-1,j+2) \to (i+2,j+1)$ in D shows that $d_D((i,j),X_{i+2}\backslash N_D^+((i,j))) \le 2$, and hence $d_D((i,j),X_{i+2}) \le 2$.

For any $t \in V(K_r)\backslash\{0,1,2\}$, the existence of the paths $(i,j) \to (i+2,j+2) \to \{(i+t,j+\frac{s+1}{2}),(i+t,j+\frac{s+3}{2})\}$, $(i,j) \to (i+2,j+\frac{s-1}{2}) \to \{(i+t,j+\frac{s+5}{2}),(i+t,j+\frac{s+7}{2}),\ldots,(i+t,j-1)\}$, $(i,j) \to (i+2,j-1) \to (i+t,j+1)$ and $(i,j) \to (i+1,j+1) \to (i+t,j)$ in D shows that $d_D((i,j),X_{i+t}\backslash N_D^+((i,j))) \le 2$, and hence $d_D((i,j),X_{i+t}) \le 2$. ■

Lemma 2.3. *For $s \ge 5$, $\rho(K_2\times K_s) = 0$ and $\rho(K_2\times K_3) = 2 = \rho(K_2\times K_4)$.*

Proof. Clearly, $K_2 \times K_3 \cong C_6$, the cycle of length 6, and hence $\rho(K_2 \times K_3) = \rho(C_6) = 2$. Moreover, as $K_2 \times K_4 \cong Q_3$, the 3-cube, and $\rho(Q_3) = 2$ [11], $\rho(K_2 \times K_4) = 2$. Next we prove that for $s \ge 5$, $\rho(K_2 \times K_s) = 0$. First we consider the case $s = 5$. We orient $K_2 \times K_5$ so that

$(0,0) \to \{(1,1), (1,2)\}$, $(1,0) \to \{(0,3), (0,4)\}$,
$(0,1) \to \{(1,0), (1,4)\}$, $(1,1) \to \{(0,2), (0,3)\}$,
$(0,2) \to \{(1,0), (1,3)\}$, $(1,2) \to \{(0,1), (0,4)\}$,
$(0,3) \to \{(1,2), (1,4)\}$, $(1,3) \to \{(0,0), (0,1)\}$,
$(0,4) \to \{(1,1), (1,3)\}$, $(1,4) \to \{(0,0), (0,2)\}$.

From this orientation of $K_2 \times K_5$ it can be seen that $\rho(K_2 \times K_5) = 0$ as $d(K_2 \times K_5) = \vec{d}(K_2 \times K_5) = 3$. Hence we suppose that $s \ge 6$.

To show that $\vec{d}(K_2 \times K_s) = 3$, $s \ge 6$, it suffices to provide an orientation of $K_2 \times K_s$ so that the resulting digraph D has diameter 3. We define the orientation as follows:

If s is even, orient the edges of $K_2 \times K_s$ so that $(0,j) \to \{(1,j-1),(1,j+1),(1,j+2),\ldots,(1,j+\frac{s}{2}-1)\}$ and $(0,j) \leftarrow \{(1,j+\frac{s}{2}),(1,j+\frac{s}{2}+1),\ldots,(1,j-2)\}$, $j \in V(K_s)$. Consequently, in the resulting digraph, say, D_1, $(1,j) \to \{(0,j+2),(0,j+3),\ldots,(0,j+\frac{s}{2})\}$ and $(1,j) \leftarrow \{(0,j+\frac{s}{2}+1),(0,j+\frac{s}{2}+2),\ldots,(0,j-1),(0,j+1)\}$.

If $s \geq 7$ is odd, then orient the edges of $K_2 \times K_s$ so that for any $i \in \{0,1\}$ and $j \in V(K_s)$, $(i,j) \to \{(i+1,j+1),(i+1,j+2),\ldots,(i+1,j+\frac{s-3}{2}),(i+1,j+\frac{s+1}{2})\}$. The resulting digraph, say, D_2 is vertex-transitive (see (1) of Appendix). From the orientations described above, it is easy to check that the resulting digraphs D_1 and D_2 have diameter 3. Hence $\rho(K_2 \times K_s) = 0$. ■

Next we consider the case $r = 3$ and s is odd. As $K_3 \times K_3$ is isomorphic to the cartesian product $K_3 \Box K_3$ (see p.183 of [8]), $\rho(K_3 \times K_3) = 1$ [11]. Hence we consider $s \geq 5$.

Lemma 2.4. $\rho(K_3 \times K_5) \leq 1$.

Proof. Orient $K_3 \times K_5$ so that for any $i \in V(K_3)$ and $j \in V(K_5)$,
$(i,j) \to \{(i+1,j+1),(i+1,j+2),(i+2,j+1),(i+2,j+2)\}$.
Let D be the resulting digraph. To verify that $d(D) \leq 3$, it is enough to verify that $e_D((0,0)) \leq 3$. The verification is easy and hence it is omitted. ■

Lemma 2.5. *If $s \geq 7$ is odd, then $\rho(K_3 \times K_s) = 0$.*

Proof. We orient $K_3 \times K_s$ so that for any $j \in V(K_s)$,

$$\begin{aligned}
(0,j) &\to \{(1,j+1),(1,j+2),\ldots,(1,j+\tfrac{s-1}{2}),\\
&\qquad (2,j+3),(2,j+4),\ldots,(2,j+\tfrac{s-1}{2}),(2,j-2),(2,j-1)\},\\
(1,j) &\to \{(0,j+1),(0,j+2),\ldots,(0,j+\tfrac{s-1}{2}),\\
&\qquad (2,j+2),(2,j+3),\ldots,(2,j+\tfrac{s-1}{2}),(2,j-1)\},\\
(2,j) &\to \{(0,j+3),(0,j+4),\ldots,(0,j+\tfrac{s-1}{2}),(0,j-2),(0,j-1),\\
&\qquad (1,j+2),(1,j+3),\ldots,(1,j+\tfrac{s-1}{2}),(1,j-1)\}.
\end{aligned}$$

Let D be the resulting digraph. Clearly, D is $(s-1)$-regular. To show that $d(D) = 2$, it is enough to verify that for each $i \in V(K_3)$, $e_D((i,0)) = 2$. We achieve this by showing $d_D((i,0),X_k) \leq 2$ for i, $k \in V(K_3)$.

First we prove that $d_D((0,0),X_0) \leq 2$, $d_D((0,0),X_1) \leq 2$ and $d_D((0,0),X_2) \leq 2$, in order. The existence of the paths $(0,0) \to (1,1) \to \{(0,2),(0,3),\ldots,(0,\frac{s+1}{2})\}$, $(0,0) \to (1,\frac{s-1}{2}) \to \{(0,\frac{s+3}{2}),(0,\frac{s+5}{2}),\ldots,(0,s-1)\}$ and $(0,0) \to (2,s-2) \to (0,1)$ in D shows that $d_D((0,0),X_0\backslash\{(0,0)\}) \leq 2$, and hence $d_D((0,0),X_0) \leq 2$. To show that $d_D((0,0),X_1) \leq 2$, we consider two cases, namely, $s = 7$ and $s \geq 9$. If $s = 7$, the existence of the paths $(0,0) \to \{(1,1),(1,2),(1,3)\}$, $(0,0) \to (2,3) \to \{(1,5),(1,6)\}$ and $(0,0) \to (2,5) \to \{(1,0),(1,4)\}$ in D shows that $d_D((0,0),X_1) \leq 2$. If $s \geq 9$, the existence of the paths $(0,0) \to \{(1,1),(1,2),\ldots,(1,\frac{s-1}{2})\}$, $(0,0) \to (2,\frac{s-3}{2}) \to \{(1,\frac{s+1}{2}),(1,\frac{s+3}{2}),\ldots,(1,s-2)\}$, $(0,0) \to (2,s-2) \to (1,0)$ and $(0,0) \to (2,\frac{s-1}{2}) \to (1,s-1)$ in D shows that $d_D((0,0),X_1) \leq 2$. Next, we prove that $d_D((0,0),X_2) \leq 2$. The existence of the paths $(0,0) \to \{(2,3),(2,4),\ldots,(2,\frac{s-1}{2}),(2,s-2),(2,s-1)\}$, $(0,0) \to (1,\frac{s-5}{2}) \to$

$\{(2,\frac{s+1}{2}),(2,\frac{s+3}{2}),\ldots,(2,s-3)\}$, $(0,0)\to(1,1)\to(2,0)$, $(0,0)\to(1,2)\to(2,1)$ and $(0,0)\to(1,3)\to(2,2)$ in D proves that $d_D((0,0),X_2)\le 2$.

Next we verify that $d_D((1,0),X_0)\le 2$, $d_D((1,0),X_1)\le 2$ and $d_D((1,0),X_2)\le 2$, in order. To show that $d_D((1,0),X_0)\le 2$, we consider two cases, namely, $s=7$ and $s\ge 9$. If $s=7$, the existence of the paths $(1,0)\to\{(0,1),(0,2),(0,3)\}$, $(1,0)\to(2,2)\to(0,0)$, $(1,0)\to(2,6)\to\{(0,4),(0,5)\}$ and $(1,0)\to(2,3)\to(0,6)$ in D shows that $d_D((1,0),X_0)\le 2$. If $s\ge 9$, the existence of the paths $(1,0)\to\{(0,1),(0,2),\ldots,(0,\frac{s-1}{2})\}$, $(1,0)\to(2,2)\to(0,0)$, $(1,0)\to(2,\frac{s-5}{2})\to\{(0,\frac{s+1}{2}),(0,\frac{s+3}{2}),\ldots,(0,s-3)\}$ and $(1,0)\to(2,\frac{s-1}{2})\to\{(0,s-2),(0,s-1)\}$ in D shows that $d_D((1,0),X_0)\le 2$. We next show that $d_D((1,0),X_1)\le 2$. The existence of the paths $(1,0)\to(0,1)\to\{(1,2),(1,3),\ldots,(1,\frac{s+1}{2})\}$, $(1,0)\to(0,\frac{s-1}{2})\to\{(1,\frac{s+3}{2}),(1,\frac{s+5}{2}),\ldots,(1,s-1)\}$ and $(1,0)\to(2,2)\to(1,1)$ in D shows that $d_D((1,0),X_1\backslash\{(1,0)\})\le 2$, and hence $d_D((1,0),X_1)\le 2$. The verification of $d_D((1,0),X_2)\le 2$ follows by the existence of the paths $(1,0)\to\{(2,2),(2,3),\ldots,(2,\frac{s-1}{2}),(2,s-1)\}$, $(1,0)\to(0,2)\to\{(2,0),(2,1)\}$, $(1,0)\to(0,\frac{s-5}{2})\to\{(2,\frac{s+1}{2}),(2,\frac{s+3}{2}),\ldots,(2,s-3)\}$ and $(1,0)\to(0,\frac{s-3}{2})\to(2,s-2)$ in D.

Finally, we prove that $d_D((2,0),X_0)\le 2$, $d_D((2,0),X_1)\le 2$ and $d_D((2,0),X_2)\le 2$, in order. The existence of the paths $(2,0)\to\{(0,3),(0,4),\ldots,(0,\frac{s-1}{2}),(0,s-2),(0,s-1)\}$, $(2,0)\to(1,s-1)\to\{(0,0),(0,1),(0,2)\}$ and $(2,0)\to(1,\frac{s-1}{2})\to\{(0,\frac{s+1}{2}),(0,\frac{s+3}{2}),\ldots,(0,s-3)\}$ in D shows that $d_D((2,0),X_0)\le 2$. We next verify that $d_D((2,0),X_1)\le 2$. The existence of the paths $(2,0)\to\{(1,2),(1,3),\ldots,(1,\frac{s-1}{2}),(1,s-1)\}$, $(2,0)\to(0,s-1)\to\{(1,0),(1,1)\}$ and $(2,0)\to(0,\frac{s-1}{2})\to\{(1,\frac{s+1}{2}),(1,\frac{s+3}{2}),\ldots,(1,s-2)\}$ in D guarantees that $d_D((2,0),X_1)\le 2$. To show that $d_D((2,0),X_2)\le 2$, we consider two cases, namely, $s=7$ and $s\ge 9$. If $s=7$, the existence of the paths $(2,0)\to(1,2)\to\{(2,4),(2,5)\}$, $(2,0)\to(1,3)\to(2,6)$, $(2,0)\to(0,3)\to\{(2,1),(2,2)\}$ and $(2,0)\to(0,5)\to(2,3)$ in D proves that $d_D((2,0),X_2\backslash\{(2,0)\})\le 2$, and hence $d_D((2,0),X_2)\le 2$. If $s\ge 9$, the existence of the paths $(2,0)\to(1,2)\to\{(2,4),(2,5),\ldots,(2,\frac{s+3}{2})\}$, $(2,0)\to(1,\frac{s-1}{2})\to\{(2,\frac{s+5}{2}),(2,\frac{s+7}{2}),\ldots,(2,s-1)\}$, $(2,0)\to(0,s-2)\to\{(2,1),(2,2)\}$ and $(2,0)\to(1,4)\to(2,3)$ in D shows that $d_D((2,0),X_2\backslash\{(2,0)\})\le 2$, and hence $d_D((2,0),X_2)\le 2$. ■

Lemma 2.6. $\rho(K_3\times K_4)=1$.

Proof. If possible assume that there is an orientation of $K_3\times K_4$ so that the resulting digraph D has diameter 2. Consider the vertex $(0,0)$. If $(0,0)$ has only one out-neighbour, say, (i,j) in D, then the vertices $(i,j+1),(i,j+2),(i,j+3),(i+1,j),(i+2,j)$ cannot be of distance at most 2 from $(0,0)$, as these vertices are nonadjacent to (i,j). Similarly, if $(0,0)$ has exactly two out-neighbours, then it can be easily verified that they cannot be in a single layer. If they are (i,j) and (r,s) with $i\ne r$ and $j\ne s$, then $d_D((0,0),(i,s))$

and $d_D((0,0),(r,j))$ are both greater than 2. Therefore, we conclude that $d_D^+((0,0)) > 2$. Similarly, $d_D^-((0,0)) > 2$. Thus $d_D^+((0,0)) = 3 = d_D^-((0,0))$. In general we can conclude that for any vertex (i,j), $d_D^+((i,j)) = 3 = d_D^-((i,j))$.

If all the out-neighbours of $(0,0)$ are in the i^{th} K_3-layer, then $(i,0)$ cannot be reachable by a directed path of length at most 2 from $(0,0)$. Therefore, the out-neighbours of $(0,0)$ must be in two different K_3-layers. Out of the three out-neighbours of $(0,0)$, assume without loss of generality that two of them be $(1,1)$ and $(1,2)$. If the third out-neighbour of $(0,0)$ is $(2,3)$, then $d_D((0,0),(1,3)) > 2$, a contradiction. Therefore $(2,3)$ is an in-neighbour of $(0,0)$. Again, without loss of generality assume that $(2,1)$ is the remaining out-neighbour of $(0,0)$. Consequently, $(1,3)$, $(2,2)$ and $(2,3)$ are the in-neighbours of $(0,0)$. As $d_D((0,0),(0,1)) = d_D((0,0),(1,0)) = d_D((0,0),(1,3)) = d_D((0,0),(2,2)) = 2$, $((1,2),(0,1))$, $((2,1),(1,0))$, $((2,1),(1,3))$ and $((1,1),(2,2))$, respectively, are arcs of D.

Next we consider the vertex $(2,1)$. Already we have obtained two of its out-neighbours, namely, $(1,0)$ and $(1,3)$. Using the above argument, it is clear that $(0,3)$ is the remaining out-neighbour of $(2,1)$, otherwise we may not be able to reach the vertex $(1,2)$ by a directed path of length at most 2 from $(2,1)$. Consequently, $(2,1) \leftarrow \{(0,0),(0,2),(1,2)\}$. As $d_D((2,1),(1,1)) = d_D((2,1),(1,2)) = d_D((2,1),(2,3)) = 2$, $((0,3),(1,1))$, $((0,3),(1,2))$ and $((1,0),(2,3))$, respectively, are arcs of D.

Next we consider the vertex $(0,3)$. As we have already obtained two of its out-neighbours, namely, $(1,1)$ and $(1,2)$, the third out-neighbour should be $(2,2)$; otherwise, the vertex $(1,0)$ cannot be reachable by a directed path of length at most 2 from $(0,3)$. Hence $(0,3) \leftarrow \{(1,0),(2,0),(2,1)\}$. As $d_D((0,3),(0,2)) = d_D((0,3),(1,0)) = d_D((0,3),(1,3)) = 2$, $((1,1),(0,2)),((2,2),(1,0))$ and $((2,2),(1,3))$, respectively, are arcs of D. Finally, we consider the vertex $(2,2)$. So far we have $(2,2) \rightarrow \{(0,0),(1,0),(1,3)\}$. Hence $(2,2) \leftarrow \{(0,1),(0,3),(1,1)\}$. Again as $d_D((2,2),(2,0)) = 2$, $((1,3),(2,0))$ is an arc of D. Now $d_D((2,0),(0,0)) > 2$ as $(1,1)$ and $(1,2)$ are in $N_D^+((0,0))$ and $(2,0)$ is an in-neighbour of $(1,3)$. This contradiction shows that $\vec{d}(K_3 \times K_4) \geq 3$.

We orient $K_3 \times K_4$ so that
$(0,0) \rightarrow \{(1,1),(1,2),(2,1)\}$, $(1,0) \rightarrow \{(0,1),(2,1),(2,2)\}$,
$(2,0) \rightarrow \{(0,1),(0,2),(1,1)\}$, $(0,1) \rightarrow \{(1,2),(2,2),(2,3)\}$,
$(1,1) \rightarrow \{(0,2),(0,3),(2,2)\}$, $(2,1) \rightarrow \{(0,2),(1,2),(1,3)\}$,
$(0,2) \rightarrow \{(1,0),(1,3),(2,3)\}$, $(1,2) \rightarrow \{(0,3),(2,0),(2,3)\}$,
$(2,2) \rightarrow \{(0,0),(0,3),(1,3)\}$, $(0,3) \rightarrow \{(1,0),(2,0),(2,1)\}$,
$(1,3) \rightarrow \{(0,0),(0,1),(2,0)\}$, $(2,3) \rightarrow \{(0,0),(1,0),(1,1)\}$.
This orientation is of diameter 3 and hence $\vec{d}(K_3 \times K_4) = 3$. ■

Lemma 2.7. $\rho(K_3 \times K_6) \leq 1$.

Proof. Orient $K_3 \times K_6$ so that for any $i \in V(K_3)$ and $j \in V(K_6)$,

$(i,j) \to \{(i+1,j+1),(i+1,j+2),(i+1,j+3),(i+2,j+1),(i+2,j+2)\}$.
Let D be the resulting digraph. To verify that $d(D) \le 3$, it is enough to verify that $e_D((0,0)) \le 3$. The verification is easy and hence it is omitted. ■

Lemma 2.8. *If $s \ge 8$ is even, then $\rho(K_3 \times K_s) = 0$.*

Proof. We orient $K_3 \times K_s$ so that for any $j \in V(K_s)$,

$$\begin{array}{rcl} (0,j) & \to & \{(1,j+1),(1,j+2),\ldots,(1,j+\frac{s}{2}), \\ & & (2,j+4),(2,j+5),\ldots,(2,j+\frac{s}{2}),(2,j-2),(2,j-1)\}, \\ (1,j) & \to & \{(0,j+1),(0,j+2),\ldots,(0,j+\frac{s}{2}-1), \\ & & (2,j+2),(2,j+3),\ldots,(2,j+\frac{s}{2}),(2,j-1)\}, \\ (2,j) & \to & \{(0,j+3),(0,j+4),\ldots, \\ & & (0,j+\frac{s}{2}-1),(0,j-3),(0,j-2),(0,j-1), \\ & & (1,j+2),(1,j+3),\ldots,(1,j+\frac{s}{2}-1),(1,j-1)\}. \end{array}$$

Let D be the resulting digraph. Clearly, D is $(s-1)$-regular. To show that $d(D) = 2$, it is enough, because of the symmetry of the graph $K_3 \times K_s$ and the orientation, to show that for each $i \in V(K_3)$, $e_D((i,0)) = 2$.

First we prove that $d_D((0,0),X_0) \le 2$, $d_D((0,0),X_1) \le 2$ and $d_D((0,0),X_2) \le 2$, in order. The existence of the paths $(0,0) \to (1,1) \to \{(0,2),(0,3),\ldots,(0,\frac{s}{2})\}$, $(0,0) \to (1,\frac{s}{2}) \to \{(0,\frac{s+2}{2}),(0,\frac{s+4}{2}),\ldots,(0,s-1)\}$ and $(0,0) \to (2,s-2) \to (0,1)$ in D verifies that $d_D((0,0),X_0\backslash\{(0,0)\}) \le 2$, and hence $d_D((0,0),X_0) \le 2$. To show that $d_D((0,0),X_1) \le 2$, we consider two cases, namely, $s = 8$ and $s \ge 10$. If $s = 8$, the existence of the paths $(0,0) \to \{(1,1),(1,2),(1,3),(1,4)\}$, $(0,0) \to (2,4) \to \{(1,6),(1,7)\}$ and $(0,0) \to (2,6) \to \{(1,0),(1,5)\}$ in D proves that $d_D((0,0),X_1) \le 2$. If $s \ge 10$, the existence of the paths $(0,0) \to \{(1,1),(1,2),\ldots,(1,\frac{s}{2})\}$, $(0,0) \to (2,\frac{s-2}{2}) \to \{(1,\frac{s+2}{2}),(1,\frac{s+4}{2}),\ldots,(1,s-2)\}$, $(0,0) \to (2,s-2) \to (1,0)$ and $(0,0) \to (2,\frac{s}{2}) \to (1,s-1)$ in D shows that $d_D((0,0),X_1) \le 2$. Next we prove that $d_D((0,0),X_2) \le 2$. The existence of the paths $(0,0) \to \{(2,4),(2,5),\ldots,(2,\frac{s}{2}),(2,s-2),(2,s-1)\}$, $(0,0) \to (1,\frac{s-2}{2}) \to \{(2,\frac{s+2}{2}),(2,\frac{s+4}{2}),\ldots,(2,s-3)\}$, $(0,0) \to (1,1) \to (2,0)$, $(0,0) \to (1,2) \to (2,1)$, $(0,0) \to (1,3) \to (2,2)$ and $(0,0) \to (1,4) \to (2,3)$ in D shows that $d_D((0,0),X_2) \le 2$.

Next we show that $d_D((1,0),X_0) \le 2$, $d_D((1,0),X_1) \le 2$ and $d_D((1,0),X_2) \le 2$, in order. To show that $d_D((1,0),X_0) \le 2$, we consider three cases, namely, $s = 8$, $s = 10$ and $s \ge 12$. If $s = 8$, the existence of the paths $(1,0) \to \{(0,1),(0,2),(0,3)\}$, $(1,0) \to (2,2) \to \{(0,0),(0,5),(0,7)\}$ and $(1,0) \to (2,7) \to \{(0,4),(0,6)\}$ in D guarantees that $d_D((1,0),X_0) \le 2$. If $s = 10$, the existence of the paths $(1,0) \to \{(0,1),(0,2),(0,3),(0,4)\}$, $(1,0) \to (2,2) \to \{(0,0),(0,5),(0,6)\}$, $(1,0) \to (2,4) \to \{(0,7),(0,8)\}$ and $(1,0) \to (2,5) \to (0,9)$ in D proves that $d_D((1,0),X_0) \le 2$. If $s \ge 12$, the existence of the paths $(1,0) \to \{(0,1),(0,2),\ldots,(0,\frac{s-2}{2})\}$, $(1,0) \to (2,2) \to$

$(0,0)$, $(1,0) \to (2, \frac{s-6}{2}) \to \{(0, \frac{s}{2}), (0, \frac{s+2}{2}), \dots, (0, s-4)\}$ and $(1,0) \to (2, \frac{s}{2}) \to \{(0, s-3), (0, s-2), (0, s-1)\}$ in D shows that $d_D((1,0), X_0) \leq 2$. Next we show that $d_D((1,0), X_1) \leq 2$. The existence of the paths $(1,0) \to (0,1) \to \{(1,2), (1,3), \dots, (1, \frac{s+2}{2})\}$, $(1,0) \to (0, \frac{s-2}{2}) \to \{(1, \frac{s+4}{2}), (1, \frac{s+6}{2}), \dots, (1, s-1)\}$ and $(1,0) \to (2, s-1) \to (1,1)$ in D proves that $d_D((1,0), X_1 \backslash \{(1,0)\}) \leq 2$, and hence $d_D((1,0), X_1) \leq 2$. To prove $d_D((1,0), X_2) \leq 2$, we consider two cases, namely, $s = 8$ and $s \geq 10$. If $s = 8$, the existence of the paths $(1,0) \to \{(2,2), (2,3), (2,4), (2,7)\}$, $(1,0) \to (0,1) \to \{(2,0), (2,5)\}$ and $(1,0) \to (0,2) \to \{(2,1), (2,6)\}$ in D shows that $d_D((1,0), X_2) \leq 2$. If $s \geq 10$, the existence of the paths $(1,0) \to \{(2,2), (2,3), \dots, (2, \frac{s}{2}), (2, s-1)\}$, $(1,0) \to (0,2) \to \{(2,0), (2,1), (2, \frac{s+2}{2}), (2, \frac{s+4}{2})\}$ and $(1,0) \to (0, \frac{s-4}{2}) \to \{(2, \frac{s+6}{2}), (2, \frac{s+8}{2}), \dots, (2, s-2)\}$ in D proves that $d_D((1,0), X_2) \leq 2$.

Finally, we show that $d_D((2,0), X_0) \leq 2$, $d_D((2,0), X_1) \leq 2$ and $d_D((2,0), X_2) \leq 2$, in order. The existence of the paths $(2,0) \to \{(0,3), (0,4), \dots, (0, \frac{s-2}{2}), (0, s-3), (0, s-2), (0, s-1)\}$, $(2,0) \to (1, s-1) \to \{(0,0), (0,1), (0,2)\}$ and $(2,0) \to (1, \frac{s-2}{2}) \to \{(0, \frac{s}{2}), (0, \frac{s+2}{2}), \dots, (0, s-4)\}$ in D guarantees that $d_D((2,0), X_0) \leq 2$. We next show that $d_D((2,0), X_1) \leq 2$. The existence of the paths $(2,0) \to \{(1,2), (1,3), \dots, (1, \frac{s-2}{2}), (1, s-1)\}$, $(2,0) \to (0, s-1) \to \{(1,0), (1,1)\}$ and $(2,0) \to (0, \frac{s-2}{2}) \to \{(1, \frac{s}{2}), (1, \frac{s+2}{2}), \dots, (1, s-2)\}$ in D proves that $d_D((2,0), X_1) \leq 2$. Finally, we show that $d_D((2,0), X_2) \leq 2$. The existence of the paths $(2,0) \to (1, s-1) \to \{(2,1), (2,2) \dots, (2, \frac{s-2}{2})\}$, $(2,0) \to (1, \frac{s-4}{2}) \to \{(2, \frac{s}{2}), (2, \frac{s+2}{2}), \dots, (2, s-2)\}$ and $(2,0) \to (1, \frac{s-2}{2}) \to (2, s-1)$ in D proves that $d_D((2,0), X_2 \backslash \{(2,0)\})) \leq 2$, and hence $d_D((2,0), X_2) \leq 2$. ■

Lemma 2.9. *For any odd integer* $s \geq 5$, $\rho(K_4 \times K_s) = 0$.

Proof. We consider the case $s = 5$ at the end of the proof. For any odd integer $s \geq 7$, we orient $K_4 \times K_s$ so that for any $j \in V(K_s)$,

$$\begin{aligned}
(0,j) &\to \{(1, j+1), (1, j+2), \dots, (1, j + \tfrac{s-1}{2}), \\
&\quad (2, j+3), (2, j+4), \dots, (2, j + \tfrac{s-1}{2}), (2, j-2), (2, j-1), \\
&\quad (3, j+2), (3, j+3), \dots, (3, j + \tfrac{s-1}{2}), (3, j-1)\}, \\
(1,j) &\to \{(0, j+1), (0, j+2), \dots, (0, j + \tfrac{s-1}{2}), \\
&\quad (2, j+2), (2, j+3), \dots, (2, j + \tfrac{s-1}{2}), (2, j-1), \\
&\quad (3, j+3), (3, j+4), \dots, (3, j + \tfrac{s-1}{2}), (3, j-2), (3, j-1)\}, \\
(2,j) &\to \{(0, j+3), (0, j+4), \dots, (0, j + \tfrac{s-1}{2}), (0, j-2), (0, j-1), \\
&\quad (1, j+2), (1, j+3), \dots, (1, j + \tfrac{s-1}{2}), (1, j-1), \\
&\quad (3, j+1), (3, j+2), \dots, (3, j + \tfrac{s-1}{2})\}, \\
(3,j) &\to \{(0, j+2), (0, j+3), \dots, (0, j + \tfrac{s-1}{2}), (0, j-1), \\
&\quad (1, j+3), (1, j+4), \dots, (1, j + \tfrac{s-1}{2}), (1, j-2), (1, j-1), \\
&\quad (2, j+1), (2, j+2), \dots, (2, j + \tfrac{s-1}{2})\}.
\end{aligned}$$

Let D be the resulting digraph. Clearly, D is $\frac{3(s-1)}{2}$-regular and vertex-transitive (see (2) of Appendix). We complete the proof by showing $e_D((0,0)) =$

2.

First we prove that $d_D((0,0),X_0) \leq 2$. The existence of the paths $(0,0) \to (1,1) \to \{(0,2),(0,3),\ldots,(0,\frac{s+1}{2})\}$, $(0,0) \to (1,\frac{s-1}{2}) \to \{(0,\frac{s+3}{2}),(0,\frac{s+5}{2}),\ldots,(0,s-1)\}$ and $(0,0) \to (3,2) \to (0,1)$ in D proves that $d_D((0,0),X_0\backslash\{(0,0)\}) \leq 2$, and hence $d_D((0,0),X_0) \leq 2$.

To show that $d_D((0,0),X_1) \leq 2$, we consider two cases, namely, $s = 7$ and $s \geq 9$. If $s = 7$, the existence of the paths $(0,0) \to \{(1,1),(1,2),(1,3)\}$, $(0,0) \to (2,3) \to (1,6)$, $(0,0) \to (3,2) \to (1,0)$ and $(0,0) \to (3,6) \to \{(1,4),(1,5)\}$ in D shows that $d_D((0,0),X_1) \leq 2$. If $s \geq 9$, the existence of the paths $(0,0) \to \{(1,1),(1,2),\ldots,(1,\frac{s-1}{2})\}$, $(0,0) \to (3,2) \to (1,0)$, $(0,0) \to (2,\frac{s-1}{2}) \to (1,s-1)$ and $(0,0) \to (2,\frac{s-3}{2}) \to \{(1,\frac{s+1}{2}),(1,\frac{s+3}{2}),\ldots,(1,s-2)\}$ in D guarantees that $d_D((0,0),X_1) \leq 2$.

We next show that $d_D((0,0),X_2) \leq 2$. The existence of the paths $(0,0) \to \{(2,3),(2,4),\ldots,(2,\frac{s-1}{2}),(2,s-2),(2,s-1)\}$, $(0,0) \to (3,s-1) \to \{(2,0),(2,1),(2,2)\}$ and $(0,0) \to (3,\frac{s-1}{2}) \to \{(2,\frac{s+1}{2}),(2,\frac{s+3}{2}),\ldots,(2,s-3)\}$ in D shows that $d_D((0,0),X_2) \leq 2$.

Next we prove that $d_D((0,0),X_3) \leq 2$. The existence of the paths $(0,0) \to \{(3,2),(3,3),\ldots,(3,\frac{s-1}{2}),(3,s-1)\}$, $(0,0) \to (2,s-1) \to \{(3,0),(3,1)\}$ and $(0,0) \to (2,\frac{s-1}{2}) \to \{(3,\frac{s+1}{2}),(3,\frac{s+3}{2}),\ldots,(3,s-2)\}$ in D proves that $d_D((0,0),X_3) \leq 2$. Thus if $s \geq 7$ is odd, then $\rho(K_4 \times K_s) = 0$.

Finally, we consider the case when $s = 5$. The following orientation of $K_4 \times K_5$ has diameter 2. Orient $K_4 \times K_5$ so that for any $j \in V(K_5)$,

$(0,j) \to \{(1,j+1),(1,j+3),(2,j+3),(2,j+4),(3,j+3),(3,j+4)\}$,
$(1,j) \to \{(0,j+1),(0,j+3),(2,j+1),(2,j+3),(3,j+3),(3,j+4)\}$,
$(2,j) \to \{(0,j+3),(0,j+4),(1,j+1),(1,j+3),(3,j+1),(3,j+3)\}$,
$(3,j) \to \{(0,j+3),(0,j+4),(1,j+3),(1,j+4),(2,j+1),(2,j+3)\}$.

Let D be the resulting digraph. To verify that $d(D) = 2$, it is enough to verify that $e_D((0,0)) = 2$ and $e_D((1,0)) = 2$, because of the symmetric nature of the orientation (see (3) of Appendix). As the verification is easy, it is omitted. ■

Lemma 2.10. $\rho(K_4 \times K_4) \leq 1$.

Proof. We orient $K_4 \times K_4$ so that for any $j \in V(K_4)$,

$(0,j) \to \{(1,j+1),(1,j+2),(2,j+1),(3,j+1),(3,j+2)\}$,
$(1,j) \to \{(0,j+1),(2,j+1),(2,j+2),(3,j+1)\}$,
$(2,j) \to \{(0,j+1),(0,j+2),(1,j+1),(3,j+1),(3,j+2)\}$,
$(3,j) \to \{(0,j+1),(1,j+1),(1,j+2),(2,j+1)\}$.

Let D be the resulting digraph. To verify that $\rho(K_4 \times K_4) \leq 1$, it is enough to verify that $e_D((i,0)) \leq 3$, $i \in V(K_4)$. The verification is easy and hence it is omitted. ■

Lemma 2.11. $\rho(K_4 \times K_6) = 0$.

Proof. We orient $K_4 \times K_6$ so that for any $j \in V(K_6)$,

$$\begin{aligned}
(0,j) &\to \{(1,j+1),(1,j+4),(2,j+3),(2,j+4),(2,j+5),(3,j+4),(3,j+5)\},\\
(1,j) &\to \{(0,j+1),(0,j+3),(0,j+4),(2,j+1),(2,j+4),(3,j+4),(3,j+5)\},\\
(2,j) &\to \{(0,j+4),(0,j+5),(1,j+1),(1,j+3),(1,j+4),(3,j+1),(3,j+4)\},\\
(3,j) &\to \{(0,j+3),(0,j+4),(0,j+5),(1,j+3),(1,j+4),(1,j+5),(2,j+1),\\
&\qquad (2,j+3),(2,j+4)\}.
\end{aligned}$$

Let D be the resulting digraph. To verify that $d(D) = 2$, it is enough to verify that $e_D((i,0)) = 2$, $i \in V(K_4)$. The verification is easy and hence it is omitted. ■

Lemma 2.12. $\rho(K_4 \times K_8) = 0$.

Proof. We orient $K_4 \times K_8$ so that for any $j \in V(K_8)$,

$$\begin{aligned}
(0,j) &\to \{(1,j+1),(1,j+2),(1,j+3),(1,j+4),(2,j+1),(2,j+3),\\
&\qquad (2,j+5),(2,j+6),(3,j+1),(3,j+4),(3,j+6)\},\\
(1,j) &\to \{(0,j+1),(0,j+2),(0,j+3),(2,j+1),(2,j+4),(2,j+6),\\
&\qquad (3,j+1),(3,j+3),(3,j+5),(3,j+6)\},\\
(2,j) &\to \{(0,j+1),(0,j+4),(0,j+6),(1,j+1),(1,j+3),(1,j+5),\\
&\qquad (1,j+6),(3,j+1),(3,j+2),(3,j+3),(3,j+4)\},\\
(3,j) &\to \{(0,j+1),(0,j+3),(0,j+5),(0,j+6),(1,j+1),(1,j+4),\\
&\qquad (1,j+6),(2,j+1),(2,j+2),(2,j+3)\}.
\end{aligned}$$

Let D be the resulting digraph. To verify that $d(D) = 2$, it is enough to verify that $e_D((i,0)) = 2$, $i \in V(K_4)$. The verification is easy and hence it is omitted. ■

Lemma 2.13. *If $s \geq 10$ is even, then* $\rho(K_4 \times K_s) = 0$.

Proof. Consider the subset $E_1 = \{(i,j)(k,j+\frac{s}{2}) : i \in V(K_4),\ j \in V(K_s),\ k \in V(K_4)\backslash\{i\}\}$ of $E(K_4 \times K_s)$. Clearly, the subgraph induced by E_1 is 3-regular. Delete the edges of E_1 from $K_4 \times K_s$ and denote the resulting subgraph by H. We orient H so that for any $j \in V(K_s)$,

$$\begin{aligned}
(0,j) &\to \{(1,j+2),(1,j+3),\ldots,(1,j+\tfrac{s}{2}-2),(1,j+\tfrac{s}{2}+1),(1,j-1),\\
&\qquad (2,j+1),(2,j+2),\ldots,(2,j+\tfrac{s}{2}-2),(2,j+\tfrac{s}{2}+1),\\
&\qquad (3,j+2),(3,j+3),\ldots,(3,j+\tfrac{s}{2}-1),(3,j-1)\},\\
(1,j) &\to \{(0,j+2),(0,j+3),\ldots,(0,j+\tfrac{s}{2}-2),(0,j+\tfrac{s}{2}+1),(0,j-1),\\
&\qquad (2,j+2),(2,j+3),\ldots,(2,j+\tfrac{s}{2}-1),(2,j-1),\\
&\qquad (3,j+1),(3,j+2),\ldots,(3,j+\tfrac{s}{2}-2),(3,j+\tfrac{s}{2}+1)\},\\
(2,j) &\to \{(0,j+1),(0,j+2),\ldots,(0,j+\tfrac{s}{2}-2),(0,j+\tfrac{s}{2}+1),\\
&\qquad (1,j+2),(1,j+3),\ldots,(1,j+\tfrac{s}{2}-1),(1,j-1),\\
&\qquad (3,j+2),(3,j+3),\ldots,(3,j+\tfrac{s}{2}-2),(3,j+\tfrac{s}{2}+1),(3,j-1)\},\\
(3,j) &\to \{(0,j+2),(0,j+3),\ldots,(0,j+\tfrac{s}{2}-1),(0,j-1),\\
&\qquad (1,j+1),(1,j+2),\ldots,(1,j+\tfrac{s}{2}-2),(1,j+\tfrac{s}{2}+1),\\
&\qquad (2,j+2),(2,j+3),\ldots,(2,j+\tfrac{s}{2}-2),(2,j+\tfrac{s}{2}+1),(2,j-1)\}.
\end{aligned}$$

Let D be the resulting orientation of H. Clearly, D is vertex-transitive (see (2) of Appendix) and $3(\frac{s}{2}-1)$-regular. We shall show that $d(D) = 2$, and it follows that $\vec{d}(K_4 \times K_s) = 2$. To show that $d(D) = 2$, it is enough to show that $e_D((0,0)) = 2$.

First we prove that $d((0,0), X_0) \leq 2$. The existence of the paths $(0,0) \to (3, s-1) \to \{(0,1),(0,2),\ldots,(0,\frac{s}{2}-2)\}$, $(0,0) \to (2,\frac{s}{2}-2) \to \{(0,\frac{s}{2}-1),(0,\frac{s}{2}),\ldots,(0,s-4)\}$ and $(0,0) \to (2,\frac{s}{2}+1) \to \{(0,s-3),(0,s-2),(0,s-1)\}$ in D proves that $d_D((0,0), X_0\backslash\{(0,0)\}) \leq 2$, and hence $d_D((0,0), X_0) \leq 2$.

We next show that $d_D((0,0), X_1) \leq 2$. The existence of the paths $(0,0) \to \{(1,2),(1,3),\ldots,(1,\frac{s}{2}-2),(1,\frac{s}{2}+1),(1,s-1)\}$, $(0,0) \to (3,\frac{s}{2}-1) \to \{(1,\frac{s}{2}),(1,\frac{s}{2}+2),(1,\frac{s}{2}+3),\ldots,(1,s-3),(1,0)\}$, $(0,0) \to (2,2) \to \{(1,1),(1,\frac{s}{2}-1)\}$ and $(0,0) \to (2,\frac{s}{2}+1) \to (1,s-2)$ in D shows that $d_D((0,0), X_1) \leq 2$.

We now prove that $d_D((0,0), X_2) \leq 2$. The existence of the paths $(0,0) \to \{(2,1),(2,2),\ldots,(2,\frac{s}{2}-2),(2,\frac{s}{2}+1)\}$, $(0,0) \to (1,\frac{s}{2}-3) \to \{(2,\frac{s}{2}-1),(2,\frac{s}{2})\}$, $(0,0) \to (1,\frac{s}{2}-2) \to \{(2,\frac{s}{2}+2),(2,\frac{s}{2}+3),\ldots,(2,s-3)\}$ and $(0,0) \to (1,\frac{s}{2}+1) \to \{(2,s-2),(2,s-1),(2,0)\}$ in D proves that $d_D((0,0), X_2) \leq 2$.

Finally, we show that $d_D((0,0), X_3) \leq 2$. The existence of the paths $(0,0) \to \{(3,2),(3,3),\ldots,(3,\frac{s}{2}-1),(3,s-1)\}$, $(0,0) \to (1,\frac{s}{2}-2) \to \{(3,\frac{s}{2}),(3,\frac{s}{2}+1),\ldots,(3,s-4)\}$, $(0,0) \to (1,\frac{s}{2}+1) \to \{(3,s-3),(3,s-2)\}$ and $(0,0) \to (1,s-1) \to \{(3,0),(3,1)\}$ in D guarantees that $d_D((0,0), X_3) \leq 2$. We have verified that $e_D((0,0)) = 2$. As D is vertex-transitive, $d(D) = 2$. ■

Lemma 2.14. $\rho(K_5 \times K_6) = 0$.

Proof. We orient $K_5 \times K_6$ so that for any $i \in V(K_5)$ and $j \in V(K_6)$,

$$(i,j) \to \{(i+1,j+1),(i+1,j+4),(i+2,j+3),(i+2,j+4),(i+2,j+5),(i+3,j+4),(i+3,j+5),(i+4,j+1),(i+4,j+3),(i+4,j+4)\}.$$

Let D be the resulting digraph. Clearly D is vertex-transitive (see (1) of Appendix) and hence to verify that $d(D) = 2$, it is enough to verify that $e_D((0,0)) = 2$. The verification is easy and hence it is omitted. ■

Lemma 2.15. $\rho(K_5 \times K_8) = 0$.

Proof. Consider the subset $E_1 = \{(i,j)(i+l,j+4) : i \in V(K_5),\ j \in V(K_8)$ and $l \in V(K_5)\backslash\{0\}\}$ of $E(K_5 \times K_8)$. Clearly, the subgraph induced by E_1 is 4-regular. Delete the edges of E_1 from $K_5 \times K_8$ and denote the resulting graph by H. We orient H so that for any $i \in V(K_5)$ and $j \in V(K_8)$,

$$(i,j) \to \{(i \pm 1,j+1),(i \pm 1,j+2),(i \pm 1,j+3),(i \pm 2,j+1),(i \pm 2,j+3),(i \pm 2,j+6)\}.$$

Let D be the resulting orientation of H. We shall show that $d(D) = 2$, and it follows that $\vec{d}(K_5 \times K_8) = 2$. Clearly D is vertex-transitive (see (1) of Appendix) and hence to verify that $d(D) = 2$, it is enough to verify that $e_D((0,0)) = 2$. The verification is easy and hence it is omitted. ■

Lemma 2.16. *If $s \geq 10$ is even, then $\rho(K_5 \times K_s) = 0$.*

Proof. Consider the subset $E_1 = \{(i,j)(k, j+\frac{s}{2}) : i \in V(K_5),\ j \in V(K_s)$ and $k \in V(K_5)\backslash\{i\}\}$ of $E(K_5 \times K_s)$. Clearly, the subgraph induced by E_1 is 4-regular. Delete the edges of E_1 from $K_5 \times K_s$ and denote the resulting graph by H. We orient H so that for any $i \in V(K_5)$ and $j \in V(K_s)$,

$$(i,j) \to \{(i\pm 1, j+2), (i\pm 1, j+3), \ldots, (i\pm 1, j+\tfrac{s}{2}-2), (i\pm 1, j+\tfrac{s}{2}+1), (i\pm 1, j-1), (i\pm 2, j+1), (i\pm 2, j+2), \ldots, (i\pm 2, j+\tfrac{s}{2}-2), (i\pm 2, j+\tfrac{s}{2}+1)\}.$$

Let D be the resulting orientation of H. Clearly, D is vertex-transitive (see (1) of Appendix) and $(2s-4)$-regular. We shall show that $d(D) = 2$, and it follows that $\rho(K_5 \times K_s) = 0$.

First we prove that $d_D((0,0), X_0) \leq 2$. The existence of the paths $(0,0) \to (1,2) \to \{(0,4), (0,5), \ldots, (0, \frac{s}{2})\}$, $(0,0) \to (1, \frac{s}{2}-2) \to \{(0, \frac{s}{2}+1), (0, \frac{s}{2}+2), \ldots, (0, s-4)\}$, $(0,0) \to (2, \frac{s}{2}+1) \to \{(0, s-3), (0, s-2), (0, s-1)\}$, $(0,0) \to (1, s-1) \to \{(0,1), (0,2)\}$ and $(0,0) \to (2,1) \to (0,3)$ in D proves that $d_D((0,0), X_0\backslash\{(0,0)\}) \leq 2$, and hence $d_D((0,0), X_0) \leq 2$.

We next show that $d_D((0,0), X_1) \leq 2$. The existence of the paths $(0,0) \to \{(1,2), (1,3), \ldots, (1, \frac{s}{2}-2), (1, \frac{s}{2}+1), (1, s-1)\}$, $(0,0) \to (2,1) \to (1,0)$, $(0,0) \to (2,2) \to (1,1)$, $(0,0) \to (3, \frac{s}{2}-2) \to \{(1, \frac{s}{2}-1), (1, \frac{s}{2})\}$ and $(0,0) \to (3, \frac{s}{2}+1) \to \{(1, \frac{s}{2}+2), (1, \frac{s}{2}+3), \ldots, (1, s-2)\}$ in D shows that $d_D((0,0), X_1) \leq 2$. Similarly, $d_D((0,0), X_4) \leq 2$.

Next we prove that $d_D((0,0), X_2) \leq 2$. $(0,0) \to \{(2,1), (2,2), \ldots, (2, \frac{s}{2}-2), (2, \frac{s}{2}+1)\}$, $(0,0) \to (3,1) \to (2,0)$, $(0,0) \to (4, \frac{s}{2}-2) \to \{(2, \frac{s}{2}-1), (2, \frac{s}{2})\}$ and $(0,0) \to (4, \frac{s}{2}+1) \to \{(2, \frac{s}{2}+2), (2, \frac{s}{2}+3), \ldots, (2, s-1)\}$ in D proves that $d_D((0,0), X_2) \leq 2$. Similarly, $d_D((0,0), X_3) \leq 2$. This completes the proof for $e_D((0,0)) = 2$. As D is vertex-transitive, $d(D) = 2$. ■

Theorem 2.1. *Let $r \leq s$, then for $(r,s) \notin \{(3,5), (3,6), (4,4)\}$,*

$$\rho(K_r \times K_s) = \begin{cases} 2 & \text{if } (r,s) \in \{(2,3),(2,4)\}, \\ 1 & \text{if } (r,s) \in \{(3,3),(3,4)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for $(r,s) \in \{(3,5), (3,6), (4,4)\}$, $\rho(K_r \times K_s) \leq 1$.

Proof. $\rho(K_2 \times K_3) = 2 = \rho(K_2 \times K_4)$ follows from Lemma 2.3, $\rho(K_3 \times K_3) = 1$ follows from $\rho(K_3 \Box K_3) = 1$ [11], as $K_3 \times K_3$ is isomorphic to $K_3 \Box K_3$ (see p.183 of [8]), $\rho(K_3 \times K_4) = 1$ follows from Lemma 2.6 and for $(r,s) \notin \{(2,3), (2,4), (3,3), (3,4), (3,5), (3,6), (4,4)\}$, $\rho(K_r \times K_s) = 0$ follows from Lemmas 2.1, 2.2, 2.3, 2.5, 2.8, 2.9, 2.11, 2.12, 2.13, 2.14, 2.15 and 2.16.

For the exceptional values, namely, $(r,s) \in \{(3,5), (3,6), (4,4)\}$, $\rho(K_r \times K_s) \leq 1$ follows from Lemmas 2.4, 2.7 and 2.10. ■

Note that, for any r and s, $K_r(s)$ is a supergraph of $K_r \times K_s$. Also $d(K_r(s)) = d(K_r \times K_s)$ if $\min\{r,s\} \geq 3$. So, for $r \geq 3$ and $s \geq 3$, $\rho(K_r \times K_s) = 0$ implies that $\rho(K_r(s)) = 0$. Thus we have

Corollary 2.1. *For $r \geq 3$, $s \geq 3$ and $(r,s) \neq (3,3),(3,4),(3,5),(3,6),(4,3),(4,4),(5,3)$ and $(6,3)$, $\rho(K_r(s)) = 0$.*

In fact, it is known [6] that for $s \geq 3$, $\vec{d}(K_r(s)) = 2$ except when $(r,s) = (4,1)$.

Remark 2.1. *All the optimal orientations obtained in this paper result in digraphs with the difference of the indegree and outdegree of each vertex is at most one. This type of orientation is considered to be significant (for example, see pp. 750 – 751 of [11]).*

Appendix

1. For any two vertices (a,b) and (c,d), the automorphism mapping (a,b) to (c,d), denoted by $f_{(a,b)}^{(c,d)}$, is given by $f_{(a,b)}^{(c,d)}((u,v)) = (u+c-a, v+d-b)$, in all the digraphs obtained by the orientations described on $K_r \times K_s$, in Lemmas 2.1, 2.2, 2.3 ($s \geq 7$ is odd), 2.14, 2.15 and 2.16. This proves the vertex-transitivity of the respective digraphs.
2. For any two vertices (a,b) and (c,d), the automorphism mapping (a,b) to (c,d), denoted by $f_{(a,b)}^{(c,d)}$, is defined as follows:

   If $a = c$, $f_{(a,b)}^{(c,d)}((u,v)) = (u, v+d-b)$;

   If $a \neq c$, let $V(K_4)\backslash\{a,c\} = \{p,q\}$, and consider the permutation $\phi = (a,c)(p,q)$ of $V(K_4)$, and define $f_{(a,b)}^{(c,d)}((u,v)) = (\phi(u), v+d-b)$, in all the digraphs obtained by the orientations described on $K_r \times K_s$, in Lemmas 2.9 ($s \geq 7$) and 2.13. This proves the vertex-transitivity of the respective digraphs.
3. Consider the digraph D obtained by the orientation defined on $K_4 \times K_5$ in Lemma 2.9. If we apply the permutation $(0,3)(1,2)$ of $V(K_4)$ to the first co-ordinates of the vertices of D, then the resulting digraph is isomorphic to D. Because of this, we have verified the eccentricity only for the vertices $(0,0)$ and $(1,0)$.

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