THE FORCING DOMINATION NUMBERS OF SOME GRAPHS

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ABSTRACT. In this paper the forcing domination numbers of the graphs $P_n \times P_3$ and $C_n \times P_3$ are completely determined. This improves the previous results on the forcing domination numbers of $P_n \times P_2$ and $C_n \times P_2$.

1. INTRODUCTION

A vertex v in a graph G is said to *dominate* all the vertices in its closed neighborhood $N[v]$. A subset S of $V(G)$ is a *dominating set* of G if $\cup_{v\in S} N[v] = V(G)$; that is, every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality among the dominating sets of G. A *minimum dominating set* of G is a dominating set of cardinality $\gamma(G)$. Consult [4] for domination in graphs. For graph theory we follow the notation and terminology of [3].

Let S be a minimum dominating set of a graph G. A subset T of S is called a *forcing subset* for S if S is the unique minimum dominating set containing T. The forcing domination number $f(S,\gamma)$ of S is the minimum cardinality of a forcing subset for S. The *forcing domination number* $f(G,\gamma)$ of G is the smallest forcing number of a minimum dominating set of G. It is clear that $0 \le f(G,\gamma) \le \gamma(G)$.

Consider the graphs $H = P_n \times G$ and $K = C_n \times G$, where G is a graph, P_n is a path of length $n-1$, and C_n is a cycle of length n. Let S be

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a dominating set in H or K. Suppose c_i, for $1 \leq i \leq n$, is the number of common vertices between S and the ith copy of G in H or K. By a *component* of S we mean a subsequence of the sequence $c_1, c_2, c_3, \ldots, c_n$ with consecutive terms, say $c_i, c_{i+1}, c_{i+2}, \ldots, c_{i+k}$, such that $c_i = c_{i+2} = c_{i+4} = \ldots = c_{i+k} = 0$ and $0 \notin \{c_{i+1}, c_{i+3}, c_{i+5}, \ldots, c_{i+k-1}\}$. (Note that a component of S starts with zero and ends at zero.) A component of S is *maximal* if it is not a proper subsequence of a component of S. An *odd (even) component* of S is a component of S with an odd (even) number of zero terms. For example, consider $P_9 \times P_3$ shown in Figure 1. Let $S = \{v_1, u_3, w_3, v_5, u_7, w_7, v_9\}$. Then $c_2 = c_4 = c_6 = c_8 = 0$, $c_1 = c_5 = c_9 = 1$ and $c_3 = c_7 = 2$. Moreover, $c_2, c_3, \ldots, c_8$ is a maximum even component of S. Throughout this paper we use a labeling for the vertices of $P_n \times G$ and $C_n \times G$, where $G = P_3$ or $G = C_3$, similar to that shown in Figure 1.

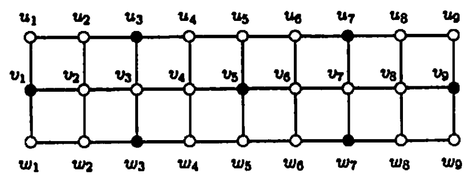

FIGURE 1. The graph $P_9 \times P_3$

In [1], the forcing domination numbers of several classes of graphs are determined, including paths, cycles, ladders and prisms. The authors of [1] prove the following results.

Lemma 1. *For a graph G, the forcing domination number $f(G, \gamma) = 0$ if and only if G has a unique minimum dominating set. Moreover, $f(G, \gamma) = 1$ if and only if G does not have a unique minimum dominating set but some vertex of G belongs to exactly one minimum dominating set.*

Corollary 2. *For a graph G, the forcing domination number $f(G, \gamma) > 1$ if and only if every vertex of each minimum dominating set belongs to at least two minimum dominating sets.*

Proposition 3. *For every integer $n \geq 2$,*

$$f(P_n \times P_2, \gamma) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proposition 4. *For every integer $n \geq 3$,*

$$f(C_n \times P_2, \gamma) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ 3 & \text{if } n \equiv 2 \pmod 4 \\ 2 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

In Section 2 we deal with domination numbers for graphs. We prove $\gamma(P_n \times P_3) = \gamma(P_n \times C_3) = \lfloor \frac{3n+4}{4} \rfloor$. (The domination number for the graph $P_n \times P_3$ was first found in [2]). We also prove $\gamma(C_n \times P_3) = \gamma(C_n \times C_3) = \lceil \frac{3n}{4} \rceil$ for every integer $n \geq 3$. In Sections 3 and 4 we find the forcing numbers of $P_n \times P_3$ and $C_n \times P_3$, respectively.

2. Minimum dominating sets

In this section first we find a lower bound for the cardinality of a dominating set of $P_n \times G$ and $C_n \times G$, where G is a non-empty simple connected graph. Then we apply these results to determine the minimum dominating number for the graphs $P_n \times P_3$, $P_n \times C_3$, $C_n \times P_3$ and $C_n \times C_3$. We use a labeling for the vertices of these graphs similar to that shown in Figure 1. Then the vertices of the ith copy of P_3 (C_3) in $P_n \times P_3$ or $P_n \times C_3$ ($C_n \times P_3$ or $C_n \times C_3$) become u_i, v_i, w_i.

Lemma 5. *Let G be a non-empty simple connected graph. Then there exists a minimum dominating set in $H = P_n \times G$ ($K = C_n \times G$) such that*

(1) *it intersects the first copy of G in H (K) and the last copy of G in H;*
(2) *if it does not intersect the ith copy of G it intersects the $(i+1)$th copy of G.*

Proof. 1) Let S be a minimum dominating set in H (or K). Assume that G_i, for $1 \leq i \leq n$, is the ith copy of G in H (or K). Define $c_i = \mid V(G_i) \cap S \mid$. If S is a minimum dominating set in K with $c_1 = 0$ then by relabeling the copies of G in K we can have $c_1 \neq 0$. If S is a minimum dominating set in H with $c_1 = 0$ then since every vertex in G_1 is dominated by precisely one vertex of G_2 we have $c_2 = \mid V(G_2) \mid$. Suppose that $u_2v_2 \in E(G_2)$. Define $S' = (S \setminus \{u_2\}) \cup u_1$ if $n = 2$ and $S' = (S \setminus \{u_2, v_2\}) \cup \{u_1, v_3\}$ if $n \geq 3$, where u_1 and v_3 are the vertices corresponding to u_2 and v_2 in G_1 and G_3, respectively. Now S' is the required minimum dominating set in H. In a similar fashion we use S' to construct a minimum dominating set in H which intersects both G_1 and G_n.
2) Let S be a minimum dominating set in H (or K) and $c_1 \neq 0$. If $c_i = c_{i+1} = 0$ for some $2 \leq i \leq n-1$, then, since every vertex of G_i is dominated by precisely one vertex of G_{i-1}, we have $c_{i-1} = \mid V(G) \mid$. Let $\{u_{i-1}, v_{i-1}\}$ be an edge in G_{i-1}. Define $S' = (S \setminus \{u_{i-1}\}) \cup \{u_i\}$ if $i = 2$ and $S' = (S \setminus \{u_{i-1}, v_{i-1}\}) \cup \{u_i, v_{i-2}\}$ if $i \geq 3$. Then S' is the required minimum dominating set in H (or K). □

Lemma 6. *Let S be a dominating set in $H = P_n \times G$ ($K = C_n \times G$), where $G = P_3$ or $G = C_3$. Suppose that precisely r copies of G in H (K) do not intersect S. If there are r_1 maximal odd components in S then $\mid S \mid \geq n + (r_1 - r)/2$.*

Proof. Let G_i be the ith copy of G in H (K) and $c_i = | S \cap V(G_i) |$ for $1 \leq i \leq n$. Without loss of generality (see Lemma 5) we can assume $c_1 \neq 0$ and there is no i in $\{1, 2, \ldots, n-1\}$ such that $c_i = c_{i+1} = 0$. First let S be a dominating set in H. By Lemma 5 we may also assume $c_n \neq 0$. Now let $c_i = 0$ for some i. (Then we note that c_{i-1} and c_{i+1} cannot both be one.) Since every vertex of G_i is dominated by precisely one vertex of G_{i-1} or G_{i+1} it follows that $c_{i-1} + c_{i+1} \geq 3$. So for a maximal odd component in S, say $c_i, c_{i+1}, c_{i+2}, \ldots, c_{i+k}$, with $\alpha = (k+2)/2$ zero terms we obtain

$$2c_{i-1} + 2c_{i+1} + 2c_{i+3} + \ldots + 2c_{i+k+1} \geq 3(\alpha + 1).$$

Similarly, for a maximal even component in S, say $c_j, c_{j+1}, c_{j+2}, \ldots, c_{j+m}$, with $\beta = (m+2)/2$ zero terms we have

$$c_{j-1} + 2c_{j+1} + 2c_{j+3} + \ldots + 2c_{j+m-1} + c_{j+m+1} \geq 3\beta.$$

Now since $c_{j-1} \neq 0$ and $c_{j+m+1} \neq 0$ we obtain

$$2c_{j-1} + 2c_{j+1} + 2c_{j+3} + \ldots + 2c_{j+m+1} \geq 3\beta + 2.$$

Now let there be precisely r_1 maximal odd components in S with $o_1, \ldots, o_{r_1}$ zero terms, respectively. Moreover, let there be precisely r_2 maximal even components in S with $e_1, \ldots, e_{r_2}$ zero terms, respectively. Define $A =$ $\{i \mid 2 \leq i \leq n-1 \text{ and } c_i \neq 0, c_{i-1}.c_{i+1} = 0\} \cup \{1 \text{ if } c_2 = 0\} \cup \{n \text{ if } c_{n-1} = 0\}$ and $B = \{i \mid c_i \neq 0\} \setminus A$. Then

$$\begin{aligned}
2 \mid S \mid &= 2\textstyle\sum_{i=1}^{n} c_i \\
&= 2\textstyle\sum_{i \in B} c_i + 2\sum_{i \in A} c_i \\
&\geq 2[n - r - ((o_1 + 1) + (o_2 + 1) \ldots + (o_{r_1} + 1)) - \\
&\quad ((e_1 + 1) + (e_2 + 1) + \ldots + (e_{r_2} + 1))] + \\
&\quad [3(o_1 + \ldots + o_{r_1} + e_1 + \ldots + e_{r_2}) + 3r_1 + 2r_2] \\
&\geq 2(n - r - r - r_1 - r_2) + 3r + 3r_1 + 2r_2 \\
&\geq 2n - r + r_1.
\end{aligned}$$

Now let S be a dominating set in K. If there is an i such that $c_i, c_{i+1} \neq 0$ then by relabeling the copies of G we may assume $c_1, c_n \neq 0$. Now an argument similar to that described above proves that $| S | \geq n + (r_1 - r)/2$. Finally suppose that there is no i such that $c_i, c_{i+1} \neq 0$. Then n is even and we have $c_{2i-1} \neq 0$ and $c_{2i} = 0$ for $i = 1, 2, \ldots, n/2$. So $c_{2i-1} + c_{2i+1} \geq 3$ for $i = 1, 2, \ldots, n/2$. This leads to $2c_1 + 2c_3 + \ldots + 2c_{n-1} \geq 3(n/2)$. So

$$2 \mid S \mid = 2c_1 + 2c_3 + \ldots + 2c_{n-1} \geq 3(n/2).$$

Now if $n \equiv 0 \pmod 4$ then $r_1 = 0$ and if $n \equiv 2 \pmod 4$ then $r_1 = 1$. So $| S | \geq n + (r_1 - r)/2$ as required. □

The proof of the following corollary can also be found in [2].

Corollary 7. *Let $G = P_3$ or $G = C_3$. Then $\gamma(P_n \times G) = \lfloor \frac{3n+4}{4} \rfloor$ for every positive integer n.*

Proof. Let S be a dominating set in $P_n \times G$. By Lemma 6 we have $| S | \geq n + (r_1 - r)/2$, where r and r_1 are as in Lemma 6 and $0 \leq r \leq \lceil (n-2)/2 \rceil$. If $\lceil (n-2)/2 \rceil$ is odd (even) then the minimum of r_1 is one (zero). Now it is straightforward to see that $| S | \geq \lfloor \frac{3n+4}{4} \rfloor$.
Now we define a dominating set in $P_n \times G$ with $\lfloor \frac{3n+4}{4} \rfloor$ vertices. Let $\{u_i, v_i, w_i\}$ be the vertices of the ith copy of G in $P_n \times G$ (see Figure 1). Define

$$\begin{array}{ll} S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k}\} & \text{if } n = 4k \\ S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k+1}\} & \text{if } n = 4k+1 \\ S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}\} & \text{if } n = 4k+2 \\ S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k\} & \text{if } n = 4k+3. \end{array}$$

Then S is a dominating set in $P_n \times G$ with $\lfloor \frac{3n+4}{4} \rfloor$ vertices. □

Corollary 8. *Let $G = P_3$ or $G = C_3$. Then $\gamma(C_n \times G) = \lceil \frac{3n}{4} \rceil$ for every integer $n \geq 3$.*

Proof. Let S be a dominating set in $C_n \times G$. Using Lemma 6 it is easy to see that $| S | \geq \lceil \frac{3n}{4} \rceil$.
Now we define a dominating set in $C_n \times G$ with $\lceil \frac{3n}{4} \rceil$ vertices. Let $\{u_i, v_i, w_i\}$ be the vertices of the ith copy of G in $C_n \times G$ (see Figure 1). Define

$$\begin{array}{ll} S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} & \text{if } n = 4k \\ S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k+1}\} & \text{if } n = 4k+1 \\ S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}\} & \text{if } n = 4k+2 \\ S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k\} & \text{if } n = 4k+3. \end{array}$$

Then S is a dominating set in $C_n \times G$ with $\lceil \frac{3n}{4} \rceil$ vertices. □

3. Forcing domination numbers for $P_n \times P_3$

Throughout this section S is a minimum dominating set in $P_n \times P_3$. So by Corollary 7 we have $|S| = \lfloor \frac{3n+4}{4} \rfloor$. We also assume that S has precisely c_i vertices in the ith copy of P_3 in $P_n \times P_3$ for $i = 1, 2, \ldots, n$.

Lemma 9. *There is no i in $\{1, 2, \ldots, n\}$ such that $c_i = c_{i+1} = 0$.*

Proof. Since the vertices of the first copy of P_3 can only be dominated by the vertices of the second copy of P_3 it is impossible to have $c_1 = c_2 = 0$. Similarly, it is impossible to have $c_{n-1} = c_n = 0$. The reader can verify that $c_2 = c_3 = 0$ ($c_{n-2} = c_{n-1} = 0$) is not possible either. So let $3 \leq i \leq n-3$ and $c_i = c_{i+1} = 0$. Then $c_{i-1} = c_{i+2} = 3$. Moreover,

$$P_n \times P_3 \setminus N[u_{i-1}, v_{i-1}, w_{i-1}, u_{i+2}, v_{i+2}, w_{i+2}] = (P_{i-3} \times P_3) \cup (P_{n-i-3} \times P_3),$$

where u_j, v_j, w_j are the vertices of the jth copy of P_3 in $P_n \times P_3$. Now by Corollary 7 we have

$$\begin{aligned} |S| &\geq \lfloor\frac{3(i-3)+4}{4}\rfloor + \lfloor\frac{3(n-i-3)+4}{4}\rfloor + 6 \\ &> \frac{3i-5}{4} - 1 + \frac{3n-3i-5}{4} - 1 + 6 \\ &= \frac{3n+6}{4}. \end{aligned}$$

This is a contradiction. □

Lemma 10. *$c_1 \neq 3$ and $c_n \neq 3$.*

Proof. Let $c_1 = 3$. Consider the graph $(P_n \times P_3) \setminus N[u_1, v_1, w_1] = P_{n-2} \times P_3$. By Corollary 7 we have

$$\begin{aligned} |S| &\geq \lfloor\frac{3(n-2)+4}{4}\rfloor + 3 \\ &> (\frac{3n-2}{4} - 1) + 3 \\ &= \frac{3n+6}{4}. \end{aligned}$$

This is a contradiction. Similarly, we can prove $c_n \neq 3$. □

Lemma 11. *If $n \not\equiv 0 \pmod 4$ then $c_1 \neq 0$ and $c_n \neq 0$.*

Proof. If $c_1 = 0$ then $c_2 = 3$. Consider the graph $P_n \times P_3 \setminus N[u_2, v_2, w_2] = P_{n-3} \times P_3$, where u_2, v_2, w_2 are the vertices of the second copy of P_3 in $P_n \times P_3$. Now by Corollary 7 we have

$$\begin{aligned} |S| &\geq \lfloor\frac{3(n-3)+4}{4}\rfloor + 3 \\ &> (\frac{3n-5}{4} - 1) + 3 \\ &= \frac{3n+3}{4}. \end{aligned}$$

Since $n \not\equiv 0 \pmod 4$ this is a contradiction. Similarly, one can prove $c_n \neq 0$. □

Lemma 12. *If $n \not\equiv 0 \pmod 4$ then $c_i \neq 3$ for $i \in \{1, 2, \ldots, n\}$.*

Proof. For $i = 1$ or n we apply Lemma 10. Let $2 \leq i \leq n-1$ and $c_i = 3$. Consider the graph

$$(P_n \times P_3) \setminus N[u_i, v_i, w_i] = (P_{i-2} \times P_3) \cup (P_{n-(i+1)} \times P_3).$$

Now by Corollary 7 we have

$$\begin{aligned} |S| &\geq \lfloor\frac{3(i-2)+4}{4}\rfloor + \lfloor\frac{3(n-i-1)+4}{4}\rfloor + 3 \\ &> (\frac{3i-2}{4}-1)+(\frac{3n-3i+1}{4}-1)+3 \\ &= \frac{3n+3}{4}. \end{aligned}$$

Since $n \not\equiv 0 \pmod 4$ this is a contradiction. □

We are now ready to determine the forcing number for $P_n \times P_3$.

Lemma 13. $f(P_{4k+1} \times P_3, \gamma) = 0$.

Proof. Let S be a minimum dominating set in $P_{4k+1} \times P_3$. By Corollary 7 we have $|S| = 3k+1$. Applying Lemmas 6 and 9 we see that precisely $r = 2k$ of the c_is must be zeros. Now by Lemmas 9 and 11 we have $c_2 = c_4 = \ldots = c_{4k} = 0$. We claim that $c_1 = 1$. If $c_1 > 1$ then $(P_{4k+1} \times P_3) \setminus N[u_1, v_1, w_1] = P_{4(k-1)+3} \times P_3$. By Lemma 6 any dominating set for this graph has at least $3k$ vertices. So $|S| \geq 3k+2$ which is a contradiction. Therefore $c_1 = 1$. Similarly, $c_{4k+1} = 1$. By Lemma 12 we have $0 \leq c_i \leq 2$ for $i \in \{1, 2, \ldots, 4k+1\}$. Now since $c_1 = 1$ and $c_2 = 0$ it follows that $v_1 \in S$. Moreover, to dominate the vertices u_2 and w_2 we must have $\{u_3, w_3\} \in S$. So $v_3 \notin S$ since $c_3 \leq 2$. Now since $c_4 = 0$ it follows that $v_5 \in S$. It is also easy to see that if $c_5 = 2$ then $|S| \geq 3k+2$ which is not possible. So $c_5 = 1$. Now a simple induction argument leads to

$$S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k+1}\}.$$

So there is a unique minimum dominating set in $P_{4k+1} \times P_3$. Now by Lemma 1 the result follows. □

Lemma 14. $f(P_{4k+2} \times P_3, \gamma) = 1$.

Proof. First we note that

$$S_1 = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}\}$$

and

$$S_2 = \{u_{4i+4}, v_{4i+2}, w_{4i+4} \mid 0 \leq i \leq k-1\} \cup \{v_1, v_{4k+2}\}$$

are two minimum dominating sets for $P_{4k+2} \times P_3$. So $f(P_{4k+2} \times P_3, \gamma) > 0$ by Lemma 1. We prove that $f(S_2, \gamma) = 1$. Hence $f(P_{4k+2} \times P_3, \gamma) = 1$. We claim that $\{v_2\}$ is a forcing subset for S_2. Let S_3 be a minimum dominating set in $P_{4k+2} \times P_3$ containing v_2. Let c_i be the number of vertices of S_3 in the ith copy of P_3 for $i = 1, 2, \ldots, 4k+2$. By Lemma 11 we have $c_1, c_n \neq 0$. It is easy to see that $c_1 + c_2 = 2$. So $v_1 \in S_3$. Now by Lemmas 6, 9 and 11 we have $c_3 = c_5 = \ldots = c_{4k+1} = 0$. By Lemma 12 we have $0 \leq c_i \leq 2$

for $i \in \{1, 2, \ldots, 4k+2\}$. Since $\{v_1, v_2\} \subseteq S_3$, $c_2 = 1$ and $c_3 = 0$ it follows that $\{u_4, w_4\} \subseteq S_3$ and hence $c_4 = 2$. Now consider the following graph:

$$P_{4k+2} \times P_3 \setminus N[\{u_i, v_i, w_i \mid i = 1, 2, 3, 4\}] = P_{4(k-1)+1} \times P_3.$$

By Corollary 7 we need at least $3k - 2$ vertices to dominate all the vertices of the above graph. So by Lemma 13, $S_4 = \{u_{4i+4}, v_{4i+2}, w_{4i+4} \mid 1 \leq i \leq k-1\} \cup \{v_{4k+2}\}$ is the unique minimum dominating set for this graph. This forces $S_4 \subseteq S_3$ and hence $S_3 = S_2$. Therefore $\{v_1\}$ is a forcing subset for S_2. This completes the proof.

□

Lemma 15. $f(P_{4k+3} \times P_3, \gamma) = \begin{cases} 2 & \text{if} \quad k = 0, 1 \\ 1 & \text{if} \quad k \geq 2. \end{cases}$

Proof. It is straightforward to see that every vertex of $P_3 \times P_3$ and $P_7 \times P_3$ belongs to at least two minimum dominating sets. So $f(P_3 \times P_3, \gamma) > 1$ and $f(P_7 \times P_3, \gamma) > 1$. Now define $S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k\}$, where $k = 0, 1$. Obviously, S is a minimum dominating set for $P_{4k+3} \times P_3$. We show that $\{u_{4k+3}, w_{4k+3}\}$ is a forcing subset in S. Consider

$$P_{4k+3} \times P_3 \setminus N[u_{4k+3}, v_{4k+3}, w_{4k+3}] = P_{4k+1} \times P_3.$$

Since $f(P_{4k+1} \times P_3, \gamma) = 0$ by Lemma 13, it follows that S is the unique minimum dominating set containing $\{u_{4k+3}, w_{4k+3}\}$. Hence, for $k = 0, 1$, we have $f(P_{4k+1} \times P_3, \gamma) = 2$.
Now let $k \geq 2$. It is easy to see that

$$S = \{w_1, u_2, v_3\} \cup \{u_{4i+1}, v_{4i+3}, w_{4i+1} \mid 1 \leq i \leq k\}$$

is a minimum dominating set in $P_{4k+3} \times P_3$. We prove that $\{u_2\}$ is a forcing subset in S. Let S_1 be a minimum dominating set in $P_{4k+3} \times P_3$ containing $\{u_2\}$. As before, let c_i be the number of vertices of S_1 in the ith copy of P_3. By Lemma 12 we have $0 \leq c_i \leq 2$ for $i \in \{1, 2, \ldots, 4k+3\}$. By Lemma 11 we have $c_1, c_{4k+3} \neq 0$ and by Lemma 9 there is no i such that $c_i = c_{i+1} = 0$. It is also easy to see that $c_1 + c_2 \leq 2$ and $c_{4k+2} + c_{4k+3} \leq 2$. So $c_1 = c_2 = 1$ and $c_{4k+3} = 1$ or 2. We claim that $c_{4k+3} = 1$. Let $c_{4k+3} = 2$. Then $c_{4k+2} = 0$. Consider

$$P_{4k+3} \times P_3 \setminus N[u_{4k+3}, v_{4k+3}, w_{4k+3}] = P_{4k+1} \times P_3.$$

Since $f(P_{4k+1} \times P_3, \gamma) = 0$ by Lemma 13, we see that $c_2 = 0$. This is a contradiction. So $c_{4k+3} = 1$. Now by Lemmas 6 and 9 we either have $c_3 = c_5 = \ldots = c_{4k+1} = 0$ or $c_4 = c_6 = \ldots = c_{4k+2} = 0$.
Case 1. $c_3 = c_5 = \ldots = c_{4k+1} = 0$. In this case it is easy to see that $\{w_1, v_4, w_4\} \subseteq S_1$. Moreover, since $c_5 = c_7 = 0$ we must have $c_6 = 2$. So $\sum_{i=1}^{6} c_i = 6$. Consider

$$P_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid 1 \leq i \leq 6\}] = P_{4(k-1)} \times P_3.$$

This forces S_1 to have at least $3k+4$ vertices which is a contradiction.
Case 2. $c_4 = c_6 = \ldots = c_{4k+2} = 0$. Since $c_{4k+3} = 1$ and $c_{4k+2} = 0$ we must have $v_{4k+3} \in S_1$. We also need to have $u_{4k+1}, w_{4k+1} \in S_1$ to dominate the vertices u_{4k+2} and w_{4k+2}. Now S_1 dominates v_{4k} only if $v_{4k-1} \in S_1$. One can prove that $c_{4k-1} = 1$. In a similar manner we can prove that $\{v_3\} \cup \{u_{4i+1}, v_{4i+3}, w_{4i+1} \mid 1 \leq i \leq k\} \subseteq S_1$. Now to dominate v_1 and w_2 we must have $w_1 \in S_1$. So $S_1 = S$. This completes the proof. □

Lemma 16. $f(P_{4k} \times P_3, \gamma) = 2$.

Proof. It is easy to see that every vertex of $P_4 \times P_3$ is in at least two minimum dominating sets. So $f(P_4 \times P_3, \gamma) \geq 2$. For $K \geq 2$ we consider the following minimum dominating sets for $P_{4k} \times P_3$.

$$
\begin{aligned}
S_1 &= \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\} \cup \{v_{4k}\}\\
S_2 &= \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-2\} \cup \{v_{4k-j} \mid j = 0,1,2,3\}\\
S_3 &= \{u_{4i+1}, v_{4i+3}, w_{4i+1} \mid 0 \leq i \leq k-1\} \cup \{v_{4k}\}\\
S_4 &= \{u_{4i+1}, v_{4i+3}, w_{4i+1} \mid 0 \leq i \leq k-2\} \cup \{v_{4k-4}, v_{4k}, u_{4k-2}, w_{4k-2}\}\\
S_5 &= \{v_1, u_{4i+4}, v_{4i+2}, w_{4i+4} \mid 0 \leq i \leq k-2\} \cup \{v_{4k-2}, v_{4k-1}, v_{4k}\}\\
S_6 &= \{v_1, u_{4i+4}, v_{4i+2}, w_{4i+4} \mid 0 \leq i \leq k-2\} \cup \{v_{4k-2}, u_{4k-1}, w_{4k}\}\\
S_7 &= \{v_1, u_{4i+2}, v_{4i+4}, w_{4i+2} \mid 0 \leq i \leq k-1\}\\
S_8 &= \{v_1, w_2, u_3, v_4\} \cup \{u_{4i+2}, v_{4i+4}, w_{4i+2} \mid 1 \leq i \leq k-1\}\\
S_9 &= \{v_1, u_2, w_3, v_4\} \cup \{u_{4i+2}, v_{4i+4}, w_{4i+2} \mid 1 \leq i \leq k-1\}\\
S_i' &= \{u_{4k+1-m} \mid u_m \in S_i\} \cup \{v_{4k+1-m} \mid v_m \in S_i\} \cup\\
&\quad \{w_{4k+1-m} \mid w_m \in S_i\} \quad i = 1,2,3,\ldots,9.
\end{aligned}
$$

Using these minimum dominating sets it is now straightforward to see that every vertex of $P_{4k} \times P_3$ is in at least two minimum dominating sets. So $f(P_{4k} \times P_3) \geq 2$. Now we prove $f(P_{4k} \times P_3, \gamma) = 2$. Consider the minimum dominating set

$$S = \{u_2, v_2, w_2, v_4\} \cup \{u_{4i+2}, v_{4i+4}, w_{4i+2} \mid 1 \leq i \leq k-1\}$$

in $P_{4k} \times P_3$. We show that $\{u_2, v_2\}$ is a forcing subset in S. Let S_1 be a minimum dominating set in $P_{4k} \times P_3$ containing $\{u_2, v_2\}$. As before, let c_i be the number of vertices of S_1 in the ith copy of P_3. First note that v_1, w_1 or w_2 is in S_1 otherwise w_1 is not dominated by S_1. We show that $w_2 \in S_1$. Consider

$$P_{4k} \times P_3 \setminus N[u_2, v_2, w_2] = P_{4(k-1)+1} \times P_3.$$

Since $f(P_{4(k-1)+1} \times P_3, \gamma) = 0$ by Lemma 13, it follows that

$$S_2 = \{u_{4i+2}, v_{4i}, w_{4i+2} \mid 1 \leq i \leq k-1\} \cup \{v_{4k}\} \subseteq S_1.$$

Now the facts that $|S_1| = 4k+1$ and $|S_2 \cup \{u_2, v_2\}| = 4k$ force $w_2 \in S_1$. Therefore $S_1 = S$. This completes the proof. □

Now we state the Main Theorem of this section.

Theorem 17. *For every integer $n \geq 1$ and $n \neq 3, 7$,*

$$f(P_n \times P_3, \gamma) = \begin{cases} 0 & \text{if} \quad n \equiv 1 \pmod 4 \\ 1 & \text{if} \quad n \equiv 2 \pmod 4 \\ 1 & \text{if} \quad n \equiv 3 \pmod 4 \\ 2 & \text{if} \quad n \equiv 0 \pmod 4). \end{cases}$$

Moreover, $f(P_3 \times P_3, \gamma) = f(P_7 \times P_3, \gamma) = 2$.

4. Forcing domination numbers for $C_n \times P_3$

Throughout this section S is a minimum dominating set in $C_n \times P_3$. So by Corollary 8 we have $|S| = \lceil \frac{3n}{4} \rceil$. We also assume that S has precisely c_i vertices in the ith copy of P_3 in $C_n \times P_3$ for $i = 1, 2, \ldots, n$.

Lemma 18. $c_i \neq 3$ *for* $i \in \{1, 2, \ldots, n\}$.

Proof. Let $c_i = 3$ for some i. Consider the graph

$$C_n \times P_3 \setminus N[u_i, v_i, w_i] = P_{n-3} \times P_3.$$

By Corollary 7 we have

$$\begin{aligned} |S| &\geq \lfloor \frac{3(n-3)+4}{4} \rfloor + 3 \\ &= \lfloor \frac{3n-5}{4} \rfloor + 3. \\ &> \lceil \frac{3n}{4} \rceil. \end{aligned}$$

But this is a contradiction by Corollary 8. □

Lemma 19. *There is no i in $\{1, 2, \ldots, n\}$ such that $c_i = c_{i+1} = 0$.*

Proof. If $c_i = c_{i+1} = 0$ for some i then $c_{i-1} = 3$ which is a contradiction by Lemma 18. □

Lemma 20. *Let $n \equiv 0$ or $1 \pmod 4$. Then there is no i in $\{1, 2, \ldots, n\}$ such that $c_i + c_{i+2} \geq 4$.*

Proof. Let $c_i + c_{i+2} \geq 4$ for some i. Consider the graph

$$C_n \times P_3 \setminus N[\{u_j, v_j, w_j \mid j \in \{i-1, i, i+1, i+2, i+3\}\}] = P_{n-5} \times P_3.$$

By Corollary 7 we have

$$\begin{aligned} |S| &\geq \lfloor \frac{3(n-5)+4}{4} \rfloor + 4 \\ &= \lfloor \frac{3n-11}{4} \rfloor + 4. \\ &> \lceil \frac{3n}{4} \rceil \text{ since } n \equiv 0 \text{ or } 1 \pmod 4. \end{aligned}$$

But this is a contradiction by Corollary 8.

□

Lemma 21. $f(C_n \times P_3, \gamma) > 0$ *for* $n \geq 3$.

Proof. Let S_1 be a minimum dominating set for $C_n \times P_3$. define

$$S_2 = \{u_i \mid u_{i-1} \in S_1\} \cup \{v_i \mid v_{i-1} \in S_1\} \cup \{w_i \mid w_{i-1} \in S_1\}.$$

It is easy to see that $S_1 \neq S_2$ and S_2 is also a minimum dominating set for $C_n \times P_3$. So by Lemma 1 we have $f(C_n \times P_3, \gamma) > 0$. □

We are now ready to determine the forcing number for $C_n \times P_3$.

Lemma 22. $f(C_{4k} \times P_3, \gamma) = 1$ *for* $k \geq 1$.

Proof. By Lemma 21 we have $f(C_{4k} \times P_3, \gamma) > 0$. Now consider the minimum dominating set $S = \{u_{4i+3}, v_{4i+1}, w_{4i+3} \mid 0 \leq i \leq k-1\}$. We prove that $\{v_1\}$ is a forcing subset in S. Let S_1 be a minimum dominating set in $C_{4k} \times P_3$ containing v_1. Let c_i be the number of vertices of S_1 in the ith copy of P_3. By Lemma 18 we have $c_1 \neq 3$. Let $c_1 = 2$ and without loss of generality let $\{u_1, v_1\} \in S_1$. Since $|S_1| = 3k$, by Lemmas 6 and 19 precisely $2k$ of the c_is are zeros. So we must have $c_2 = c_4 = \ldots = c_{4k} = 0$. This forces $w_3 \in S_1$. Moreover, either $u_3 \in S_1$ or $v_3 \in S_1$, otherwise S_1 does not dominate u_3. Now we have $c_1 + c_3 \geq 4$ which is a contradiction by Lemma 20. Therefore $c_1 = 1$. This forces $u_3, w_3, v_5 \in S_1$. Now by Lemma 20 we have $c_5 = 1$. Repeating this procedure leads to $S = S_1$. So S is the unique minimum dominating set containing $\{v_1\}$. This completes the proof. □

Lemma 23. $f(C_{4k+1} \times P_3, \gamma) = \begin{cases} 1 & \text{if} \quad k = 1 \\ 2 & \text{if} \quad k \geq 2 \end{cases}.$

Proof. First let $k = 1$. By Lemma 21 we have $f(C_5 \times P_3, \gamma) > 0$. It is easy to see that $\{u_3\}$ is contained in the unique minimum dominating set $\{v_1, u_3, w_3, v_5\}$. So $f(C_5 \times P_3, \gamma) = 1$.
Now let $k \geq 2$. Consider the following minimum dominating sets.

$$S_i = \{v_i\} \cup \{u_{i+4j+3}, v_{i+4j+1}, w_{i+4j+3} \mid 0 \leq j \leq k-1\},$$

for $i = 1, 2, \ldots, 4k+1$. It is straightforward to see that every vertex of $C_{4k+1} \times P_3$ belongs to at least two of these sets. Therefore $f(C_{4k+1} \times P_3, \gamma) \geq 2$ by Corollary 2. Now we prove $\{v_1, v_2\}$ is a forcing subset in S_1. Let S' be a minimum dominating set in $C_{4k+1} \times P_3$ containing v_1 and v_2. Let c_i be the number of vertices of S' in the ith copy of P_3. By Lemma 18 we have $c_i \neq 3$ for $i = 1, 2, \ldots, 4k+1$. And by Lemmas 6 and 19 we obtain $c_3 = c_5 = \ldots = c_{4k+1} = 0$. We claim that $c_1 + c_2 \leq 3$. Consider the graph

$$C_{4k+1} \times P_3 \setminus N[u_1, v_1, w_1, u_2, v_2, w_2] = P_{4(k-1)+1} \times P_3.$$

If $c_1 + c_2 \geq 4$ by Corollary 6 we see that $|S'| \geq 3k+2$ which is a contradiction by Corollary 8. So $c_1 + c_2 \leq 3$. Without loss of generality we can assume $c_2 = 1$. Now by Lemma 20 and the fact that S' is a minimum dominating set we see that $u_4, w_4 \in S'$ and $c_4 = 2$. So $c_6 = 1$ by Lemma 20 and

$v_6 \in S'$. Continuing this procedure leads to $c_{4i+2} = 1$ for $i = 0, 1, \ldots, k-1$ and $c_{4j} = 2$ for $j = 1, \ldots, k$. Moreover, $S \setminus \{v_1\} \subseteq S' \setminus \{v_1\}$. Now since $|S| = |S'|$ and $v_1 \in S \cap S'$ we must have $S = S'$. This completes the proof. □

Lemma 24. *$f(C_{4k+2} \times P_3, \gamma) = 2$ for every $k \geq 1$.*

Proof. First let $k = 1$. It is straightforward to see that every vertex of $C_6 \times P_3$ is in at least two minimum dominating sets. So by Corollary 2 we have $f(C_6 \times P_3, \gamma) \geq 2$.
Now let $k \geq 2$. Consider the following minimum dominating sets.

$$S_i = \{v_i, v_{i+1}\} \cup \{u_{i+4(j+1)}, v_{i+4j+2}, w_{i+4(j+1)} \mid 0 \leq j \leq k-1\},$$

for $i = 1, 2, \ldots, 4k+2$. It is straightforward to see that every vertex of $C_{4k+2} \times P_3$ belongs to at least two S_is. Therefore $f(C_{4k+2} \times P_3, \gamma) \geq 2$ by Corollary 2. Now we prove $\{v_1, v_3\}$ is a forcing subset in S_1. Let S' be a minimum dominating set in $C_{4k+2} \times P_3$ containing v_1 and v_3. Let c_i be the number of vertices of S' in the ith copy of P_3. We claim that $c_2 \neq 0$. Suppose that $c_2 = 0$. By Lemma 18 we have $c_1, c_3 \neq 3$. So without loss of generality we may assume $\{v_1, v_3, u_1, w_3\} \subseteq S'$. Consider the graph

$$C_{4k+2} \times P_3 \setminus N[u_1, v_1, w_1, u_3, v_3, w_3] = P_{4(k-1)+1} \times P_3.$$

By Corollary 6 to cover the vertices of this graph we need at least $3k-2$ vertices. Since $|S'| = 3k+2$ this forces $c_4 = 0$. Now by Lemma 13 we have $f(P_{4(k-1)+1} \times P_3, \gamma) = 0$. So $v_5 \in S'$ and $c_5 = 1$. Now u_4 is not dominated by S' which is a contradiction. So we must have $c_2 \neq 1$. Similarly, we cannot have $c_1 + c_2 + c_3 \geq 4$. So $c_2 = 1$ and $v_2 \in S'$. By Lemma 6 precisely either $2k$ or $2k+1$ of the c_is are zeros. By Lemma 19 and the fact that $c_1 = c_2 = c_3 = 1$ we must have $c_4 = c_6 = \ldots = c_{4k+2} = 0$. Now to dominate the vertices of the 4th and $(4k+2)$th copies we must have $u_5, w_5, u_{4k+1}, w_{4k+1} \in S'$. If $k = 1$ then we obtain $S = S'$. For $k \geq 2$ we consider

$$C_{4k+2} \times P_3 \setminus N[\{u_{4k+3-i}, v_{4k+3-i}, w_{4k+3-i}, \\ u_j, v_j, w_j \mid 1 \leq i \leq 2,\ 1 \leq j \leq 5\}] = P_{4(k-2)+1} \times P_3.$$

Since $f(P_{4(k-2)+1} \times P_3, \gamma) = 0$ by Lemma 13, we see that $S' = S_1$. This completes the proof. □

Lemma 25. *$f(C_{4k+3} \times P_3, \gamma) = 2$ for $k \geq 0$.*

Proof. It is easy to see that every vertex of $C_3 \times P_3$ is in at least two minimum dominating sets. Moreover, $\{v_1, v_2\}$ is a forcing subset for the minimum dominating set $\{v_1, v_2, v_3\}$. So $f(C_3 \times P_3, \gamma) = 2$. Now let $k \geq 1$ and define the minimum dominating set

$$S_i = \{u_{i+4j+2}, v_{i+4j}, w_{i+4j+2} \mid 0 \leq j \leq k\},$$

for $i = 1, \ldots, 4k+3$. One can easily show that each vertex of $C_{4k+3} \times P_3$ is in at least two S_is. So $f(C_{4k+3} \times P_3, \gamma) \geq 2$ by Corollary 2. For $k = 1$ it is easy to see that $\{u_1, u_2\}$ is a forcing subset in $S = \{u_1, u_2, w_3, u_5, v_5, w_7\}$. So $f(C_7 \times P_3, \gamma) = 2$. For $k \geq 2$ we prove $\{u_1, w_2\}$ is a forcing subset in the minimum dominating set

$$S = \{u_1, w_2, v_{4k+3}\} \cup \{u_{4i+5}, v_{4i+3}, w_{4i+5} \mid 0 \leq i \leq k-1\}.$$

Let S' be a minimum dominating set in $C_{4k+3} \times P_3$ containing u_1 and w_2. Let c_i be the number of vertices of S' in the ith copy of P_3. By Lemma 18 we have $c_i \neq 3$. Moreover, since S' is a minimum dominating set, by Lemmas 6 and 19 precisely either $2k$ or $2k+1$ of the c_is are zeros. First assume precisely $2k+1$ of the c_is are zeros. Then $c_3 = c_5 = \ldots = c_{4k+3} = 0$ by Lemma 19. Obviously, $c_1 + c_2 \leq 3$ otherwise, S' has at least $3k+4$ vertices which is a contradiction. We claim that $c_1 + c_2 \neq 3$. Let $c_1 = 1$ and $c_2 = 2$. Then $w_{4k+2}, v_{4k+2} \in S'$ and by 18 we have $c_{4k+2} = 2$. This forces $c_{4k} = 2$. Now consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid 1 \leq i \leq 2 \text{ or } 4k \leq i \leq 4k+3\}] = P_{4(k-2)+3} \times P_3.$$

Then we must have $|S'| \geq 3(k-2)+3+7 = 3k+4$ which is a contradiction. Similarly, the Case $c_1 = 2$ and $c_2 = 1$ leads to a contradiction. So $c_1 = c_2 = 1$. This forces $c_4 = c_6 = c_{4k} = c_{4k+2} = 2$. Hence, $|S| \geq 10$ if $k = 2$ and $|S| \geq 3k+4$ if $k \geq 3$. This is a contradiction. Therefore precisely $2k$ of the c_is are zeros. Note that in this case there is no minimal odd component in S' by Lemma 6. Consider two cases.

Case 1. $c_3 = 0$. Then $c_5 = 0$ otherwise S' has a minimal odd component. Now we consider two subcases.

Subcase 1.1 $c_{4k+3} \neq 0$. Obviously $c_1 + c_2 + c_{4k+3} \leq 4$ otherwise S' has at least $3k+4$ vertices which is impossible. Let $c_1 + c_2 + c_{4k+3} = 4$. We claim that $c_2 = 2$. If $c_2 = 1$ then $c_4 = 2$. Consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid = 1, 2, 3, 4, 4k+3\}] = P_{4(k-1)} \times P_3.$$

By Corollary 6 to cover the vertices of this graph we need at least $3k-2$ vertices. So we must have $|S'| \geq 3k+4$ which is a contradiction. Therefore $c_2 = 2$ and $c_{4k+3} = 1$. In a similar fashion it is proved that $c_{4k+2} = 1$. This forces $c_3 = c_5 = \ldots = c_{4k+1} = 0$. Now to dominate the vertices of the $(4k+1)$th copy of P_3 we must have $c_{4k} = 2$. Consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid 1 \leq i \leq 2 \text{ or } 4k \leq i \leq 4k+3\}] = P_{4(k-2)+3} \times P_3.$$

Then $|S'| \geq 3(k-2)+3+7 = 3k+4$ which is a contradiction. Therefore $c_1 + c_2 + c_{4k+3} = 3$. So $c_1 = c_2 = c_{4k+3} = 1$. We claim that $c_{4k+2} = 0$. If $c_{4k+2} \neq 0$ then $c_3 = c_5 = \ldots = c_{4k+1} = 0$. This forces $c_4 = c_6 = 2$ and $c_{4k} + c_{4k+2} \geq 3$. Now it is easy to see that $|S'| \geq 3k+4$ which is a contradiction. So $c_{4k+2} = 0$. Since S' has no minimal odd component we obtain $c_{4k} = 0$. It is now easy to see that these conditions lead to

$|S'| \geq 10$ when $k = 2$ which is impossible. Now assume $k \geq 3$. Since $c_3 = c_5 = c_{4k} = c_{4k+2} = 0$ and $c_2 = c_{4k+3} = 1$ we must have $c_4 = c_{4k+1} = 2$ and $c_6, c_{4k-1} \neq 0$. Consider

$$C_{4k+3} \times P_3 \backslash N[\{u_i, v_i, w_i \mid 1 \leq i \leq 4 \text{ or } 4k+1 \leq i \leq 4k+3\}] = P_{4(k-2)+2} \times P_3.$$

By Lemma 6 and Corollary 7 it is necessary that $2(k-2)$ copies of P_3 in $P_{4(k-2)+2} \times P_3$ be zeros. Therefore we must have $c_7 = c_9 = \ldots = c_{4k-3} = 0$ or $c_8 = c_{10} = \ldots = c_{4k-2} = 0$. First let $c_7 = c_9 = \ldots = c_{4k-3} = 0$. Then we must have $c_4 = c_6 = c_{4k+1} = 2$. Consider

$$C_{4k+3} \times P_3 \backslash N[\{u_i, v_i, w_i \mid 1 \leq i \leq 6 \text{ or } 4k+1 \leq i \leq 4k+3\}] = P_{4(k-2)} \times P_3.$$

By Corollary 7 we need at least $3k-5$ vertices to cover all the vertices of this graph. So $|S'| \geq 3k+4$ which is a contradiction. Secondly, let $c_8 = c_{10} = \ldots = c_{4k-2} = 0$. Then we have $c_4 = c_{4k+1} = 2$ and $c_6 \geq 1$. Consider

$$C_{4k+3} \times P_3 \backslash N[\{u_i, v_i, w_i \mid 1 \leq i \leq 5 \text{ or } 4k+1 \leq i \leq 4k+3\}] = P_{4(k-2)+1} \times P_3$$

and apply Lemma 13 to obtain $v_7 \in S'$. This implies $c_6 + c_7 \geq 3$. So $|S'| \geq 3k+4$ which is impossible.

Subcase 1.2 $c_{4k+3} = 0$. Then $c_{4k+1} = 0$, otherwise S' has a minimal odd component. Obviously $c_1 + c_2 \leq 3$, otherwise S' has at least $3k+4$ elements which is impossible. We claim that $c_1 = c_2 = 1$. Let $c_1 = 1$ and $c_2 = 2$. Then $v_{4k+2}, w_{4k+2} \in S'$. Consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid i = 1, 2, 4k+2, 4k+3\}] = P_{4(k-1)+1} \times P_3.$$

Let $z \in \{u_2, v_2\}$. By Corollary 7 we see that $S' \backslash \{u_1, w_2, z, v_{4k+2}, w_{4k+2}\}$ is a minimum dominating set for this graph. So $v_4, v_{4k} \in S'$ and $c_4 = c_{4k} = 1$ by Lemma 13. But in this case the vertex u_{4k+1} is not covered by S' which is a contradiction. If $c_1 = 2$ and $c_2 = 1$ then $c_4 = 2$. Now consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid i = 1, 2, 3, 4\}] = P_{4(k-1)+1} \times P_3$$

and proceed as above to show that S' cannot be a dominating set which is a contradiction. So $c_1 = c_2 = 1$. This forces $u_4, v_4, v_{4k+2}, w_{4k+2} \in S'$. Consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid i = 1, 2, 3, 4, 4k+2, 4k+3\}] = P_{4(k-2)+3} \times P_3.$$

By Corollary 7 we see that $S' \setminus \{u_1, w_2, u_4, v_4, v_{4k+2}, w_{4k+2}\}$ is a minimum dominating set for this graph. So by Lemma 11 we have $c_6 \neq 0$ and $c_{4k} \neq 0$. If $c_6 + c_{4k} \geq 4$ then $|S'| \geq 3k+4$ which is not possible. So $c_6 + c_{4k} \leq 3$. Let $c_6 = 1$. Then $w_6 \in S'$ and so $u_7 \in S'$. Now consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid i = 1, 2, 3, 4, 4k+2, 4k+3\}] = P_{4(k-2)+3} \times P_3.$$

By Lemma 15, $\{u_7\}$ is a forcing subset for $S' \backslash \{u_1, w_2, u_4, v_4, v_{4k+2}, w_{4k+2}\}$. Moreover, $v_{4k} \in S'$ and $c_{4k} = 1$. Now we see that vertex u_{4k+1} is not covered by S' which is a contradiction. Similarly, the case $c_{4k} = 1$ leads to a contradiction. This completes Case 1.

Case 2. $c_3 \neq 0$. A method similar to that described in Case 1 shows that $c_{4k+3} \neq 0$. Now since precisely $2k$ of the c_is are zeros we must have $c_4 = c_6 = \ldots = c_{4k+2} = 0$. If $c_1 + c_2 + c_3 + c_{4k+3} \geq 6$ then $|S'| \geq 3k+4$ which is not possible. If $c_1 + c_2 + c_3 + c_{4k+3} = 5$ we consider

$$C_{4k+3} \times P_3 \setminus N[\{u_i, v_i, w_i \mid i = 1, 2, 3, 4k+3\}] = P_{4(k-1)+1} \times P_3$$

and apply Lemma 13 to obtain $c_5 = c_{4k+1} = 1$. But this forces $c_3 = c_{4k+3} = 2$ and so $c_1 + c_2 + c_3 + c_{4k+3} \geq 6$ which is not possible. So $c_1 = c_2 = c_3 = c_{4k+3} = 1$. Now if $u_3 \in S'$ then since $c_4 = 0$ we must have $v_5, w_5 \in S'$ and so $c_7 = 2$. Now this implies $|S'| \geq 3k+4$ which is impossible. So $v_3 \in S'$. Similarly, $v_{4k+3} \in S'$. Now since $v_3 \in S'$ and $c_4 = 0$ we have $\{u_5, w_5\} \in S'$. So $v_5 \notin S'$ since $c_5 \leq 2$ by Lemma 18. Therefore $v_7 \in S'$ since $c_6 = 0$. It is straightforward to see that $c_7 = 1$. Now one can apply a simple induction argument to prove that $S' = S$. This completes the proof. □

Now we state the Main Theorem of this section.

Theorem 26. *For every integer $n \geq 3$ and $n \neq 5$,*

$$f(C_n \times P_3, \gamma) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 4 \\ 2 & \text{if } n \equiv 2 \pmod 4 \\ 2 & \text{if } n \equiv 3 \pmod 4 \\ 1 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Moreover, $f(C_5 \times P_3, \gamma) = 1$.

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