Finding The Chromatic Polynomial Of Cayley Graphs Using The Tutte Polynomial

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Abstract

In Algebraic Graph Theory, Biggs[2] gives a method for finding the chromatic polynomial of any connected graph by computing the Tutte polynomial. It is used by Biggs[2] to compute the chromatic polynomial of Peterson's graph. In 1972 Sands[4] developed a computer algorithm using matrix operations on the incidence matrix to compute the Tutte Polynomial. In [1], Anthony finds the worst-case time-complexity of computing the Tutte Polynomial. This paper shows a method using group-theoretical properties to compute the Tutte polynomial for Cayley graphs which improves the time-complexity.

1 Introduction

Suppose G is a finite group, and $S \subset G$ is a symmetric set of generators for the group. We define $\Gamma = \Gamma(G, S)$ to be a Cayley Graph if for vertices $g, h \in G$ the edge $(g, h) \in \Gamma$ if $g^{-1}h \in S$. We wish to find the chromatic polynomial of Γ by computing the Tutte polynomial. We define the chromatic polynomial

$$C(\Gamma, u) = \sum m_r(\Gamma) u_r$$

where $m_r(\Gamma)$ is the number of distinct color partitions into r color classes of $V\Gamma$, the vertex set of Γ and u_r is the complex number $u(u-1)(u-2)...(u-r+1)$.

Now we can, with the aid of a computer, find the chromatic polynomial using the Tutte polynomial. Biggs[2] shows the development leading to the result. To understand it however we need some more definitions which will be shown in section 2. Next we will see Biggs[2] contructions of the basic circuits in section 3. Section 4 will show the construction of the Tutte polynomial for Cayley graphs. Section 5 will show the time complexity analysis.

2 Definitions and constructions for the Tutte Polynomial

This paper assumes readers know the definitions of Cayley graph, circuit, tree, spanning tree and directed graph. We will use V as an abbreviation of $V\Gamma$, the vertex set of Γ, and E as an abbreviation of $E\Gamma$, the edge set of Γ.

An incidence matrix, D, of a directed graph Γ where $|E| = m$ and $|V| = n$ is an $n \times m$ matrix where

$$D[i,j] = \begin{cases} -1 \text{ if } v_i \in V \text{ is the negative end of edge } j \\ 1 \text{ if } v_i \in V \text{ is the positive end of edge } j \end{cases}$$

If $\Gamma = (V, E)$ and if $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, then a cutset of Γ is the set of edges connecting V_1 and V_2.

We call the circuit formed by adding the edge $b \in E\Gamma - ET$, to the spanning tree $T, cir(T, b)$. We call the cutset of $a \in ET$ with respect to the spanning tree $T, cut(T, a)$. We notice that for $b \notin ET, a \in ET$,

$$a \in cir(T, b) \Leftrightarrow b \in cut(T, a).$$

If a connected graph Γ has n vertices, m edges and 1 component, the rank, $r(\Gamma)$, is $n-1$ and the corank, $s(\Gamma)$, is $m - n + 1$. If $S \subseteq \Gamma$ we denote by $< S >$ the edge-induced subgraph S of Γ.

We now impose some total ordering $\leq$ on the edges of Γ. An edge is externally (internally) active with respect to Γ if it is the first edge, with respect to $\leq$ of the fundamental cycle (cutset) containing it. The number of edges externally (internally) active to T is the external (internal) activity of $< T >$. Denote by $T^e(T^i)$ the set of edges externally (internally) active to $< T >$.

Define $t_{i,j}$ to be the number of spanning trees with internal activity i and external activity j. The Tutte polynomial of Γ, with respect to $\leq$ is the two-variable polynomial

$$T(\Gamma, \leq; x, y) = \sum t_{i,j} x^i y^j$$

The Tutte polynomial is related to the chromatic polynomial and we obtain

Theorem 2.1 *(Biggs [2]) If Γ is connected with n vertices, then*

$$C(\Gamma; u) = (-1)^{n-1} u \sum_{i=1}^{n-1} t_{i0} (1-u)^i$$

where t_{i0} is the number of spanning trees with internal activity i and external activity 0.

For the development behind this result and the proof of it, see Biggs[2].

Now, we see to compute the Tutte polynomial we must compute the internal and external activities of Γ. To do so, we need the circuits and cutsets of the graph. The next section shows Biggs[2] method for doing so.

3 Biggs[2] Construction of Basic Circuits

If Γ has n vertices and m edges where the edges are ordered so that $e_1, e_2, .., e_{n-1}$ are the edges of a spanning tree T, we can partition the incidence matrix of $\Gamma$by

$$D = \begin{pmatrix} D_T & & D_N \\ 1 & d_n & \end{pmatrix}$$

where D_T is the incidence matrix of the spanning tree T and d_n is the last row of D.

Let C denote the matrix whose columns are the vectors representing the elements of the circuit subspace of Γ. Then C can be written as

$$C = \begin{pmatrix} C_T \\ I_{m-n+1} \end{pmatrix}$$

Since every column represents a circuit and thus belongs to the kernel of D, we have $DC = 0$. Thus

$$C_T = -D_T^{-1} D_N$$

Similarly, K whose columns represent the elements of the cutset subspace can be written as

$$K = \begin{pmatrix} I_{n-1} \\ K_T \end{pmatrix}$$

Since every column of K belongs to the orthogonal complement of the circuit-subspace, we have $CK^T = 0$, that is $C_T + K_T^t = 0$. Thus

$$K_T = (D_T^{-1} D_N)^t$$

We now have a method for computing the chromatic number for a graph. However, this method involved many matrix operations of large matrices for graphs with many vertices and edges. The next section shows a way to cut down on the matrix operations by using properties of the Cayley graphs.

4 Finding the Tutte Polynomial for Cayley Graphs

Lemma 4.1 *If in a Cayley graph, Γ, we define a mapping*

$$\pi : V\Gamma \to V\Gamma, \text{ where } \pi(g) = gk, \text{ for } g, k \in V\Gamma, k \text{ fixed}$$

then π maps spanning trees to spanning trees.

Proof.
By the properties of graph isomorphism, we need only show that π is a graph isomorphism. We first see that this is a permutation of the vertex set since $gk = \overline{g}k \Leftrightarrow g = \overline{g}$. Elementary group theory properties tell us that for $s \in S$

$$(g, h) \in E\Gamma \Leftrightarrow g = sh \Leftrightarrow$$

$$gk = shk \Leftrightarrow (gk, hk) \in E\Gamma$$

From now on we will call the circuits of a particular spanning tree, its *associated circuits*. We will call the cutsets of a particular spanning tree the *associated cutsets* .

Lemma 4.2 *π maps the associated circuits of a spanning tree, T, to the associated circuits of the spanning tree, $\pi(T)$.*

Proof.
Assume T has an associated circuit of length l. Let (g_1, g_2), $(g_2, g_3), \ldots, (g_{l-1}, g_l)$
be the $l - 1$ edges of this circuit in $E\Gamma$ and then (g_l, g_1) is the remaining edge which is not in $E\Gamma$. Now let $\pi(g) = gk$. Then $(g_l, g_1) \in E\Gamma$ but $(g_l, g_1) \notin ET$. For $s \in S$,

$$g_l = sg_1 \Leftrightarrow g_l k = sg_1 k$$

So $(\pi(g_l), \pi(g_1)) \in E\Gamma$ but $(\pi(g_l), \pi(g_1)) \notin \pi(E(T))$ by Lemma 4.1. So (g_l, g_1) is an edge of an associated circuit in T and $(\pi(g_l), \pi(g_1))$ is an edge of an associated circuit in $\pi(T)$. To complete the proof, we need see that for $1 \leq i \leq l$

$$(g_{i+i}, g_i) \in T \Leftrightarrow \pi(g_{i+1}, \pi(g_i)) \in \pi(T)$$

which is true by the Lemma 4.1.

Lemma 4.3 *π maps the associated cutsets of a spanning tree, T, to the associated cutsets of the spanning tree, $\pi(T)$*

Proof.

We need to show that for

$$e \notin T, f \in T, \pi(e) \notin \pi(T), \pi(f) \in \pi(T)$$

$$e \in cut(T, f) \Leftrightarrow \pi(e) \in cut(\pi(T), \pi(f)).$$

$$e \in cut(T, f) \Leftrightarrow f \in cir(T, e) \Leftrightarrow$$
$$\pi(f) \in cir(\pi(T), \pi(e)) \Leftrightarrow \pi(e) \in cut(\pi(T), \pi(f)).$$

Lemma 4.4 *We can find at most $|V(\Gamma)| - 1$ isomorphic spanning trees by the π mappings.*

Proof.

This is clear since there are $|V(\Gamma)| - 1$ non-identity elements in G.

We will define the *canonical spanning trees* as the spanning trees obtained from the original spanning trees by replacing edges in the cutsets. We will call the spanning trees obtained by π mappings, *mapped.*

Theorem 4.5 *In a Cayley graph, to use the Tutte polynomial to color the vertices, we need only do matrix operations to find associated circuits and cutsets of canonical spanning trees. The others we can get by mapping.*

Proof.

This result follows from Lemmas 4.1,4.2,4.3 and 4.4.

5 Analysis

The major change is how the algorithm finds the circuit and cutset spaces for all the spanning trees. Without the edge mappings, it does a matrix inverse of D_T and then a matrix multiply by D_N. D_T is of dimension $n-1 \times n-1$ and D_N is of dimension $n-1 \times m-(n-1)$. To find the circuit subspace the product is negated and to find the cutset subspace it is transposed. The change is to instead map edges from one spanning tree to another and likewise map edges to find the circuits. So there is a decrease by a factor of $n-1$ in the number of computations on spanning trees. Also there is still the time to compute the internal and external activities.

It would seem that this time savings is offset by the mapping of vertices to vertices. Since there are n vertices, we need to calculate the product of each vertex by each other for a total of n^2 operations. However this computation was already performed when computing the edge set of the graph since an edge is in the Cayley Graph if for $g, h \in G, g^{-1}h \in S$.

Anthony[1] shows that the original algorithm can be performed in worst-case $c\binom{m}{n-1}n^2m$ operations for c some constant. He shows that the time to find a spanning tree can be done in $2(n-1)^3$ operations. The time to compute a fundamental cycle associated with a spanning tree can be done in $(n-1)^2(m-n+1)$ operations. And the time to find the external and internal activities can be done in $4(m-n+1)(n-1)$ operations.

We notice that $(m-n+1)(n-1) < (n-1)m < \frac{n^2}{n-1}m$ Since the change shown in this paper affects finding spanning trees and cycles, that part of the complexity in Anthony's argument is decreased by a factor of $n-1$. By the above inequality and considering a constant c, the time-complexity is $c\binom{m}{n-1}\frac{n^2}{n-1}m$.

References

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[4] D.A. Sands, Dichromatic Polynomials of Linear Graphs, PhD thesis, University of London.