ZERO-SUM MAGIC GRAPHS AND THEIR NULL SETS

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ABSTRACT. For any $h \in \mathbb{N}$, a graph $G = (V, E)$ is said to be h-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. When this constant is 0 we call G a zero-sum h-magic graph. The *null set* of G is the set of all natural numbers $h \in \mathbb{N}$ for which G admits a zero-sum h-magic labeling. In this paper we will identify several classes of zero sum magic graphs and will determine their null sets.

Key Words: magic, non-magic, zero-sum, null set.
AMS Subject Classification: 05C15 05C78

1. INTRODUCTION

For an abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be A-*magic* if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic; for example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \to A - \{0\}$, the *inverse labeling* of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it is A-magic for every abelian group A. For example, every regular graph is fully magic. A graph $G = (V, E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious example of a non-magic graph is P_n $(n \geq 3)$, the path of order n. As a result, any graph with a path pendant of length $n \geq 3$ would be non-magic. Here is another example of a non-magic graph: Consider the graph H Figure 1. Given any abelian group A, a typical magic labeling of H is illustrated in that figure. Since $l^+(u) = x \neq 0$ and $l^+(v) = 0$, H is not A-magic. This fact can be generalized as follows:

Lemma 1.1. *Every even cycle C_n with $2k+1\,(< n)$ consecutive pendants is non-magic.*

Lemma 1.2. *Every odd cycle C_n with $2k\,(< n)$ consecutive pendants is non-magic.*

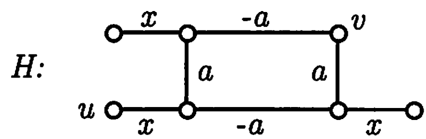

FIGURE 1. An example of non-magic graph.

Certain classes of non-magic graphs are presented in [1].
The original concept of A-magic graph is due to J. Sedlacek [11, 12], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and
(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [4] proved that a graph G is magic if and only if every edge of G is contained in a (1-2)-factor. $\mathbb{Z}$-magic graphs were considered by Stanley [13, 14], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [3, 15]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as h-magic graphs. Clearly, if a graph is h-magic, it is not necessarily k-magic ($h \neq k$).

Definition 1.3. *For a given graph G the set of all positive integers h for which G is h-magic is called the* integer-magic spectrum *of G and is denoted by $IM(G)$.*

Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H, Figure 1, is non-magic, hence $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs resulted by the amalgamation of cycles and stars have already been identified [5], and in [6] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of some other graphs have been studied in [7, 8, 9, 10].

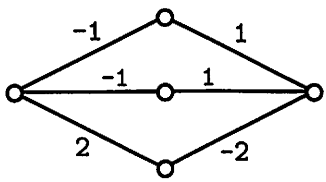

FIGURE 2. The graph $K(2,3)$ is h-magic ($\forall h \geq 3$).

Definition 1.4. *An h-magic graph G is said to be h-zero-sum (or just zero-sum) if there is a magic labeling of G in $\mathbb{Z}_h$ that induces an edge labeling with sum* 0. *The graph G is said to be strictly zero-sum if any magic labeling of G induces* 0 *sum.*

Clearly, a graph that has an edge pendant is not zero-sum. Here is an example of strictly zero-sum graph:

Lemma 1.5. *The complete bipartite graph $K(2,3)$ is strictly zero-sum magic graph.*

Proof. Since the degree set of $K(2,3)$ is $\{2,3\}$, it is not 2-magic. On the other hand, the labeling presented in Figure 2 indicates that the integer-magic spectrum of $K(2,3)$ is $\mathbb{N} - \{2\}$ with sum being 0. Now we wish to show that 0 is the only possible sum. Consider an arbitrary labeling of $K(2,3)$, as illustrated in Figure 3.

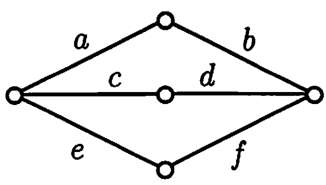

FIGURE 3. An arbitrary labeling of $K(2,3)$.

For $K(2,3)$ to be h-magic, we require that

$$\begin{aligned} a+b &\equiv e+f \pmod{h}; \\ a+c+e &\equiv a+b \pmod{h}; \\ b+d+f &\equiv a+b \pmod{h}. \end{aligned}$$

If we add these equations, we get $c+d=0$. Hence, the induced sum cannot be nonzero. □

Definition 1.6. *The null set of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G is h-magic and admits a zero-sum labeling in $\mathbb{Z}_h$.*

One can introduce a number of operations among zero-sum graphs which produce magic graphs. Frucht and Harary [2] introduced the *corona* of two G and H, denoted by $G©H$, to be the graph with base G such that each vertex $v \in V(G)$ is joined to all vertices of a separate copy of H.

Observation 1.7. *If G has zero-sum in $\mathbb{Z}_h$, then $G©K_1$ is h-magic.*

A graph G with a fixed vertex $u \in V(G)$ will be denoted by the order pair (G, u). Given two ordered pair (G, u) and (H, v), one can construct another graph by linking these two graphs through identifying the vertices u and v. We will use the notation $(G, u) \diamond (H, v)$ for this constrauction or simply $G \diamond H$ if there is no ambiguity about the choices of u and v.

Definition 1.8. *Given n graphs G_i $i = 1, 2 \cdots, n$, the chain $G_1 \diamond G_2 \diamond \cdots \diamond G_n$ is the graph in which one of the vertices of G_i is identified with one of the vertices of G_{i+1}. If $G_i = G$, we use the notation $\diamond G^n$ for the n-link chain all of whose links are G.*

Observation 1.9. *If graphs G_i have zero sum, so does the chain $G_1 \diamond G_2 \diamond \cdots \diamond G_n$, hence it is a magic graph. Moreover, if $G_i = G$, then the null set of the chain $\diamond G^n$ is the same as $N(G)$.*

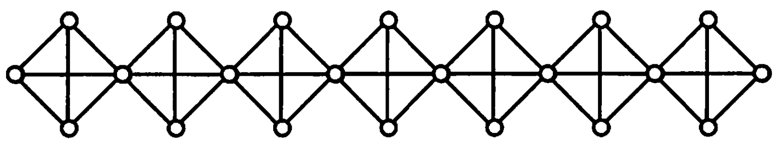

FIGURE 4. A 7-link chain whose links are K_4

With the notation in 1.8, if we further identify one of the vertices of G_n by another vertex of G_1, the resulting graph is a necklace. Similarly, all the bids of this necklace can be the same graph G, for which we have the following observation:

Observation 1.10. *If the graphs G_i have zero sum, so does the necklace formed by these graphs. Moreover, if $G_i = G$, then the null set of this necklace is the same as $N(G)$.*

These are just a few operations among the graphs that preserve the magic property, when the graphs are zero-sum. In magic labeling of graphs, knowing the components of a graph and the null sets of the components will be extremely helpful. For example, consider the graph G illustrated in Figure 5. This graph is constructed by five copies of K_4. In the next section,

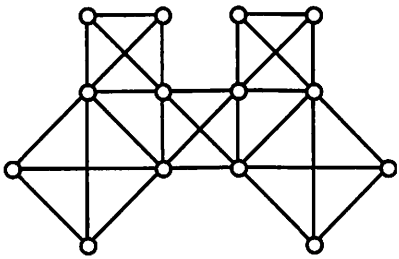

FIGURE 5. Find the integer-magic spectrum of this graph!!!

(theorem 2.1), it is shown that the null set of K_4 is $\mathbb{N} - \{2\}$. With this information and the fact that the applied construction preserves the zero-sum property, one can easily see that $N(G) = IM(G) = \mathbb{N} - \{2\}$.
In the following sections the null sets of a few well known classes of graphs will be characterized.

2. Null Sets of Complete Graphs

Complete graphs being regular are fully magic, hence their integer-magic spectrum is $\mathbb{N}$. In this section we will determine the null sets of these graphs. Note that $K_3 \equiv C_3$ and $N(K_3) = 2\mathbb{N}$. In what follows we will assume that $n \geq 4$.

Theorem 2.1. *If $n \geq 4$, then* $N(K_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is odd;} \\ \mathbb{N} - \{2\} & \text{if } n \text{ is even.} \end{cases}$

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of K_n and assume that they are arranged counterclockwise around a circle. If n is even, then $\deg(u_i)$ is odd and K_n cannot have zero sum in $\mathbb{Z}_2$. Also, with the following convention, we will use u_j as one of the vertices even if $j \neq 1, 2, \cdots, n$: Let $u_j = \begin{cases} u_{j-n} & \text{if } j > n\,; \\ u_{j+n} & \text{if } j \leq 0. \end{cases}$ To prove the theorem, we will consider the following five cases, in each case we will introduce an appropriate labeling $l : E(K_n) \to \mathbb{Z}_3$ with sum 0.
Case 1. n is odd and $n = 4p + 1$. In this case, labeling of the edges are done by

$$l(u_iu_j) = \begin{cases} 1 & \text{if } j = i \pm r \ (1 \leq r \leq p); \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 4p$, there are $4p$ edges that are incident with vertex u_i, half of which are labeled 1 and the other half -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$.

Case 2. n is odd and $n = 4p + 3$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } j = i + 1; \\ 1 & \text{if } j = i \pm r \ (2 \leq r \leq p); \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 4p + 2$, there are $4p + 2$ edges incident with this vertex, two edges are labeled 2, $2p-2$ edges are labeled 1, and the remaining $2p + 2$ edges are labeled -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$. This labeling is illustrated in table (2.1).

(2.1)

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	*	2	−1	−1	−1	−1	2
u_2	2	*	2	−1	−1	−1	−1
u_3	−1	2	*	2	−1	−1	−1
u_4	−1	−1	2	*	2	−1	−1
u_5	−1	−1	−1	2	*	2	−1
u_6	−1	−1	−1	−1	2	*	2
u_7	2	−1	−1	−1	−1	2	*

Case 3. n is even and $n = 6p + 4$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } j - i = 3p + 2; \\ 2 & \text{if } j = i \pm r \ (1 \leq r \leq p); \\ -1 & \text{otherwise.} \end{cases}$$

Note that u_iu_j $(j - i = 3p + 2)$ are the opposite vertices, u_iu_j $(j = i + r)$ are on the left of u_i, and u_iu_j $(j = i - r)$ are on the right of u_i. Since the $\deg(u_i) = n - 1 = 6p + 3$, there are $6p + 3$ edges incident with this vertex. We label $2p+1$ of them by 2 (opposite, p on the left, and p on the right) and the remaining $4p + 2$ by -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$.
Case 4. n is even and $n = 6p + 2$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } j - i = 3p + 2; \\ 2 & \text{if } j = i \pm r \ (2 \leq r \leq p); \\ 1 & \text{if } j = i \pm 1; \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 6p + 1$, there are $6p + 1$ edges incident with this vertex. We label $2p - 1$ of them by 2 (opposite, $p - 1$ on the left, and $p - 1$ on the right), two edges by 1 (immediate left and right), and the remaining $4p$ by -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$.
Case 5. n is even and $n = 6p$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } j - i = 3p + 2; \\ 2 & \text{if } j = i \pm r \ (2 \le r \le p); \\ 1 & \text{if } j = i + 1 \ (i = 2r - 1); \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 6p - 1$, there are $6p - 1$ edges incident with this vertex. We label $2p - 1$ of them by 2 (opposite, $p - 1$ on the left, and $p - 1$ on the right), one edges by 1, and the remaining $4p$ by -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$. This labeling is illustrated in table (2.2).

(2.2)

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	*	1	−1	2	−1	−1
u_2	1	*	−1	−1	2	−1
u_3	−1	−1	*	1	−1	2
u_4	2	−1	1	*	−1	2
u_5	−1	2	−1	−1	*	1
u_6	−1	−1	2	−1	1	*

□

3. Null Sets of Complete Bipartite Graphs

Theorem 3.1. *Let $m, n \ge 2$. Then*

$$N(K(m,n)) = \begin{cases} I\!N & \text{if } m + n \text{ is even;} \\ I\!N - \{2\} & \text{if } m + n \text{ is odd.} \end{cases}$$

Proof. Let $S = \{u_1, u_2, \cdots, u_m\}$ and $T = \{v_1, v_2, \cdots, v_n\}$ be the two partite sets. In labeling of edges $u_i v_j$, with elements of $\mathbb{Z}_h$ ($h \ge 3$), we will consider three cases:

Case I. m, n are both even. We label the edges by $l(u_iv_j) = (-1)^{i+j}$. This will result in $l^+ \equiv 0$.

Case II. m is even and n is odd. We label the edges by

$$l(u_iv_j) = \begin{cases} 2(-1)^{i-1} & \text{if } j = 1 \\ (-1)^i & \text{if } j = 2, 3 \\ (-1)^{i+j} & \text{otherwise} \end{cases}$$

This labeling is illustrated in table (3.1).

(3.1)

	v_1	v_2	v_3	v_4	$\dots$	v_n
u_1	2	−1	−1	−1	$\dots$	1
u_2	−2	1	1	1	$\dots$	−1
u_3	2	−1	−1	−1	$\dots$	1
u_4	−2	1	1	1	$\dots$	−1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_{m-1}	2	−1	−1	−1	$\dots$	1
u_m	−2	1	1	1	$\dots$	−1

Case III. m, n are both odd. We label the edges by using the following table (8):

(3.2)

	v_1	v_2	v_3	v_4	v_5	$\dots$	v_n
u_1	2	−1	−1	2	−2	$\dots$	−2
u_2	−1	2	−1	−1	1	$\dots$	1
u_3	−1	−1	2	−1	1	$\dots$	1
u_4	2	−1	−1	$\ddots$			
u_5	−2	1	1				
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$(-1)^{i+j}$		
u_m	2	−1	−1				$\ddots$

Finally, we observe that if m, n have different parity, the graph would not be 2-magic. □

4. Null Sets of Cycle Related Graphs

There are different classes of cycle related graphs that have been studied for variety of labeling purposes. J. Gallian [3] has a nice collection of such graphs. In this section, the null sets of some of the cycle related graphs are investigated. First, one useful observation:

Observation 4.1. *In any magic labeling of of a cycle the edges should alternatively be labeled the same elements of the group.*

Proof. Let u_1, u_2, u_3, and u_4 be the four consecutive vertices of a cycle. The requirement of $l(u_1u_2) + l(u_2u_3) = l(u_2u_3) + l(u_3u_4)$ implies that $l(u_1u_2) = l(u_3u_4)$. □

Since a cycle is a 2-regular graph, it is fully magic. Therefore, its integer-magic spectrum is $\mathbb{N}$. For the null-set of C_n we have the following theorem:

Theorem 4.2. $N(C_n) = \begin{cases} \mathbb{N} & \textit{if } n \textit{ is even;} \\ 2\mathbb{N} & \textit{if } n \textit{ is odd}. \end{cases}$

Proof. If n is even, then there are even number of edges and we label every other edge by 1 and -1. If n is odd, then in any magic labeling of C_n, all the edges are labeled the same element of $x \in \mathbb{Z}_h$. As a result, for C_{2k+1} to be zero-sum, one needs $2x \equiv 0 \pmod{h}$ or $2|h$. On the other hand, if $h = 2r$, then the choice of $x = r$ will result to the zero-sum magic labeling of C_{2k+1}. □

A *cycles with a P_k chord* is a cycle with the path P_k joining two nonconsecutive vertices of the cycle. Since the degree set of these graphs is $\{2, 3\}$, they are not 2-magic. Based on Observation 4.1, it is enough to consider the cases when $k = 2, 3$. The chord P_k splits C_n into two subcycles. Depending on the number of edges of these subcycles, we will have different results for the null set. The next lemma is about cycles with a P_2 chord:

Lemma 4.3. *Let $G_{n,2}$ be the cycle C_n with a P_2 chord. Then*

$$N(G_{n,2}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \textit{otherwise .} \end{cases}$$

Proof. Since the degree set of $G_{n,2}$ is $\{2, 3\}$, the graph is not 2-magic. Now based on the observation 4.1, it is enough to consider C_3 and C_4 as the two subcycles.

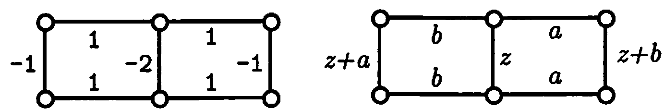

FIGURE 6. $G_{n,2}$ consists of two even subcycles.

Case I. Both subcycles are even. The labeling illustrated in Figure 6, proves that the integer-magic spectrum of $G_{n,2}$ is the same as its null set; that is, $N(G_{n,2}) = IM(G_{n,2}) = \mathbb{N} - \{2\}$.

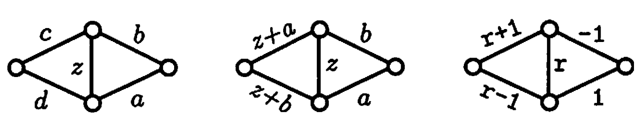

FIGURE 7. $G_{n,2}$ consists of two odd subcycles.

Case II. Both subcycles are odd. The typical labeling of $G_{n,2}$ in $\mathbb{Z}_h$ is illustrated in Figure 7. The requirement $a+z+d = c+d$ and $b+z+c = c+d$ imply that $c = a + z$ and $d = b + z$. Also, $a + b = c + d$ will result to $2z \equiv 0 \pmod{h}$ or $2|h$. On the other hand, if $h = 2r$, then the choice

of $z = r$, $a = 1$, and $b = -1$ provides a zero sum result. Therefore, $IM(G_{n,2}) = N(G_{n,2}) = 2\mathbb{N} - \{2\}$.
Case III. Subcycles have different parities. The typical labeling of $G_{n,2}$ in $\mathbb{Z}_h$ is illustrated in Figure 8. The condition $a + x + z = a + y + z$ implies $x = y$. Also, the requirements $a + z + x = 2x$ will result to $z = x - a$ and $b = 2x - a$. Therefore, given $x \in \mathbb{Z}_h - \{0\}$, we need another nonzero element $a \neq x$, $2x$, hence $h \geq 4$. Therefore, the integer-magic spectrum of such graphs would be $\mathbb{N} - \{2, 3\}$, while the null set is $2\mathbb{N} - \{2\}$. □

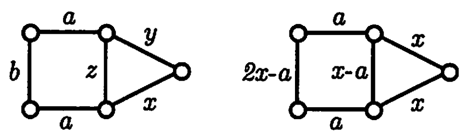

FIGURE 8. $G_{n,2}$ consists of one odd and one even subcycles.

Corollary 4.4. *C_n with a P_2 chord is not uniformly null.*

Lemma 4.5. *Let $G_{n,3}$ be the cycle C_n with a P_3 chord. Then*

$$N(G_{n,3}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \textit{otherwise .} \end{cases}$$

Proof. Based on the observation 4.1, it is enough to consider C_4 and C_5 as the two subcycles.

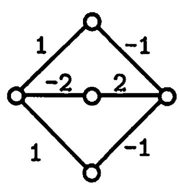

FIGURE 9. $G_{n,3}$ consists of two even subcycles.

Case I. Both subcycles are C_4. The labeling illustrated in Figure 9, shows that the integer-magic spectrum of $G_{n,3}$ is the same as its null set; that is, $N(G_{n,3}) = IM(G_{n,3}) = \mathbb{N} - \{2\}$.
Case II. Both subcycles are C_5. The typical magic labeling of $G_{n,3}$ in $\mathbb{Z}_h$ is illustrated in Figure 10, which has sum $2x$. Here, given $x \in \mathbb{Z}_h$, one needs another nonzero element $a \neq x, -x$. Hence, the graph cannot be 3-magic, and its integer-magic spectrum is $\mathbb{N} - \{2, 3\}$. However, for the graph to have zero sum, we need $2x \equiv 0 \pmod h$; that is, h has to be

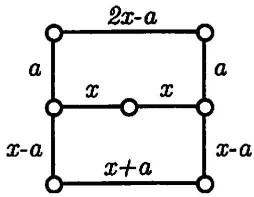

FIGURE 10. $G_{n,3}$ consists of two odd subcycles.

even. Therefore, its null set in contained in $2\mathbb{N} - \{2\}$. On the other hand, if $h = 2r$, then the choices of $x = r$ and $a = 1$ provide a zero sum result. Therefore, $N(G_{n,3}) = 2\mathbb{N} - \{2\}$.

Case III. Subcycles have different parities. The typical magic labeling of $G_{n,3}$ in $\mathbb{Z}_h$ is illustrated in Figure 8. For the graph to be magic, we need $3a + c + x = a + c + x$ or $2a \equiv 0 \pmod h$; that is, h is even and the integer-magic spectrum of the graph would be $2\mathbb{N} - \{2\}$. For the graph to have zero sum, we need the additional condition $a + c + x \equiv 0 \pmod h$, that is always possible. One such labeling has been provided in Figure 11. Thus $IM(G_{n,3}) = N(G_{n,3}) = 2\mathbb{N} - \{2\}$. □

Corollary 4.6. *C_n with a P_3 chord is not uniformly null.*

We summarize the above 4.3 and 4.5 in the following theorem:

Theorem 4.7. *Let $G_{n,k}$ be the cycle C_n with a P_k chord. Then*

$$N(G_{n,k}) = \begin{cases} \mathbb{N} - \{2\} & \text{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \text{otherwise .} \end{cases}$$

Moreover, $G_{n,k}$ is not a uniformly null graph.

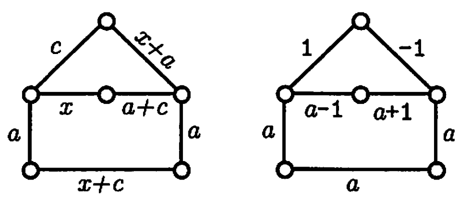

FIGURE 11. $G_{n,3}$ consists of one odd and one even subcycles.

When k copies of C_n share a common edge, it will form an *n-gon book of k pages* and is denoted by $B(n, k)$.

Theorem 4.8. $N(B(n,k)) = \begin{cases} \mathbb{N} & n \text{ is even, } k \text{ is odd;} \\ \mathbb{N}-\{2\} & n \text{ and } k \text{ are both even;} \\ 2\mathbb{N}-\{2\} & n \text{ is odd, } k \text{ is even;} \\ 2\mathbb{N} & n \text{ and } k \text{ both are odd.} \end{cases}$

Proof. Depending on whether n is even or odd it will be enough to consider C_4 and C_3, respectively.
If n is even and k is odd, we will label the common edge by -1 and top edges 1, -1 alternatively. This is a zero sum magic labeling.
If n and k are both even, we will label the common edge by -1 and one top edge by 2 the remaining top edges -1, 1 alternatively. This provides a zero sum magic labeling. Note that, in this case, the degrees of vertices do not have the same parity and the book is not 2-magic.
Suppose n is odd (C_3). We label the common edge by z and the i^{th} cycle edges by a_i, $-a_i$ as illustrated in Figure 12.

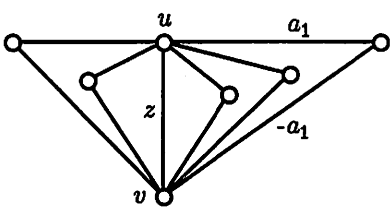

FIGURE 12. A typical zero sum magic labeling of $B(3,k)$.

The requirements $l^+(u) = l^+(v) = 0$ will lead us to the equations $z+\sum a_i = z-\sum a_i = 0$ or $2z \equiv 0 \pmod h$, which implies that h is even ($z \neq 0$). On the other hand if $h = 2r$ is even, then we consider two cases:
case I. If k is odd, we label all the edges by r which results in a zero sum magic labeling.

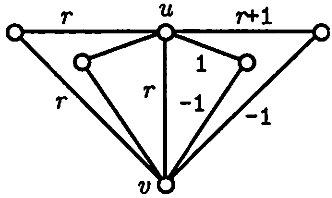

FIGURE 13. A zero sum magic labeling of $B(3,2k)$.

case II. If k is even, we choose $a_1 = r-1$, $a_2 = 1$, and $z = a_i = r$ ($i \geq 3$), as illustrated in Figure 13.

Finally, we observe that when n is odd and k is even, the book cannot be 2-magic. Therefore, the null space would be $2\mathbb{N} - \{2\}$. □

There are many other classes of cycle related graphs. *Wheels* $W_n = C_n + K_1$ and Fans (also known as Shells) are among them. When $n - 3$ chords in cycle C_n share a common vertex, the resulting graph is called *Fan (or Shell)* and is denoted by F_n, which is isomorphic to $P_{n-1} + K_1$. We conclude this paper by the following problems:

Problem 4.9. *Find the null sets of W_n and F_n.*

Problem 4.10. *In 1.5 it was shown that $K(2,3)$ is strictly zero-sum graph. Identify a class of graphs whose elements are strictly zero-sum.*

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Determining a permutation from its set of reductions

by John Ginsburg

ABSTRACT For any positive integer n, let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. We think of a permutation just as an ordered list. For any p in S_n and for any $i \leq n$, let $p \downarrow i$ be the permutation on the set $\{1, 2, ..., n-1\}$ obtained from p as follows: delete i from p and then subtract 1 in place from each of the remaining entries of p which are larger than i. For any p in S_n we let $R(p) = \{q \in S_{n-1} : q = p \downarrow i$ for some $i \leq n\}$, the set of reductions of p. It is shown that, for $n > 4$, any p in S_n is determined by its set of reductions $R(p)$.

Key words and phrases: permutation, reduction, set of reductions, reconstruction

For any positive integer n, let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. We think of a permutation just as an ordered list, and a permutation is displayed simply by listing its entries in order, sometimes with commas between them for clarity.

Let $n \geq 2$, and let $p \in S_n$. For any $i \leq n$, let $p \downarrow i$ be the permutation on the set $\{1, 2, ..., n-1\}$ obtained from p as follows: delete i from p and then subtract 1 in place from each of the remaining entries of p which are larger than i. Thus $p \downarrow i$ is an element of S_{n-1}, which we call the i'th *reduction* of p.

To illustrate, let $n = 5$ and let $p = 53412$. We then have
$p \downarrow 1 = 4231$, $p \downarrow 2 = 4231$, $p \downarrow 3 = 4312$, $p \downarrow 4 = 4312$, $p \downarrow 5 = 3412$.

This example shows that the reductions of a permutation are not necessarily all distinct.

For any p in S_n we let $R(p) = \{q \in S_{n-1} : q = p \downarrow i$ for some $i \leq n\}$. The set $R(p)$ is called *the set of reductions of p*.

For our example above, with $p = 53412$, we have $R(p) = \{4231, 4312, 3412\}$.

The main question we consider in this paper is the following:

Is a permutation determined by its set of reductions?

We will show that the answer is yes for $n > 4$, and we will describe a simple procedure for determining p from its set of reductions $R(p)$.

We note that this result fails for $n = 4$. If we let $p = 3142$ and $q = 2413$, then p and q are two different permutations with the same set of reductions $R(p) = R(q) = \{213, 231, 312, 132\}$.

Deleting one entry i, in all possible ways, from a permutation on $\{1, 2, ..., n\}$, to create various $n-1$-permutations, is of course a very commonly used idea. Recent papers in coding theory [7] and permutation graphs [4], [5] use these one-element deletions. By subtracting 1 from each of the entries which are larger than i, we are just creating a standardized version of the one-element deletion, so that it becomes an $n-1$-permutation on the "standard $n-1$-element set" $\{1, 2, ..., n-1\}$. The n reductions of a permutation on $\{1, 2, ..., n\}$ can be thus be thought of as being the n one-element deletions, up to isomorphism. This form of reduction is employed in [11], pages 85-86, in an inductive description of the Schensted correspondence.

The problem we are considering here can be viewed as a simple type of *reconstruction problem*, in which one attempts to reconstruct an object from its one-element deleted sub-objects. While this type of problem is perhaps most familiar for graphs(see [1]) and ordered sets(see [10]), a recent paper on reconstructing subsets of the plane [9] includes references to reconstructing codes, sets of real numbers, sequences and geometries. We refer the reader to [2], [6] and [8] for interesting recent work on reconstructing sequences from subsequences.

Before proceeding further, we emphasize that we are not considering here the *multiset* of reductions of a permutation p, in which each reduction would be included as many times as it occurs in the list $p \downarrow 1, p \downarrow 1, ..., p \downarrow n$.

In our proof that a permutation p is determined by its set of reductions $R(p)$, there are two basic steps: we show that the position of the entry n in p is determined by the set $R(p)$, and, letting $p-n$ denote the element of S_{n-1} obtained by deleting n from p, we show that the set of reductions of $p-n$ is also determined by $R(p)$. The result then follows by induction. We will establish these facts by means of a number of lemmas. In connection with part (vi) of Lemma 1 below, we note that, for any p in S_n and for any $i \leq n$, $p \downarrow i$ is an element of S_{n-1} and so $(p \downarrow i) \downarrow j$ is defined for all $j \leq n-1$. We will usually omit the brackets in referring to this iterated reduction, denoting it simply by $p \downarrow i \downarrow j$.

As basic notation for exhibiting a permutation $p \in S_n$ we will write $p(1), p(2), ..., p(n)$ or alternately $p_1 p_2 \cdots p_n$ to indicate the entries of p. In using such notation, we thus write $p(i) = k$ or $p^{-1}(k) = i$ to express the fact that the integer k occurs in the i'th position of p. We will also let p^{opp} denote the permutation obtained by listing the entries in the opposite order from which they are listed in p. Thus, for $p = 35124$ in S_5, we have $p^{opp} = 42153$.

Lemma 1. Let $n \geq 2$ and let $p \in S_n$.

(i) Let $1 \leq i \leq n$. Then we have $p^{opp} \downarrow i = (p \downarrow i)^{opp}$.

(ii) $p \downarrow n = p - n$.

(iii) Let $1 \leq i \leq n$ and suppose that i and $i+1$ occur consecutively in p. Then $p \downarrow i = p \downarrow (i+1)$.

(iv) Let i and j be positive integers such that $i, j \leq n$. Then $p \downarrow i = p \downarrow j$ if and only if the segment of p from i to j (including i and j) is either an increasing sequence of consecutive integers or a decreasing sequence of consecutive integers.

(v) Let $c = |\{k : p(k)$ and $p(k+1)$ are consecutive integers$\}|$. Then $|R(p)| = n - c$.

(vi) For any positive integers i and j with $i < j \leq n$, we have

$$p \downarrow j \downarrow i = p \downarrow i \downarrow (j-1).$$

Proof: (i) and (ii) are obvious. To verify (iii), note that, if i and $i+1$ occur consecutively in p, then both $p \downarrow i = p \downarrow (i+1)$ can be described as follows: replace the pair of entries $\{i, i+1\}$ by the single entry i and then subtract 1 from all other entries which are larger than i.

The implication from right to left in (iv) follows from (iii). For the converse, assume that $i < j$ and that $p \downarrow i = p \downarrow j$. By part (i), it is sufficient to consider the case when i is to the left of j in p. Suppose $i = p(k)$ and $j = p(l)$ where $k < l$. Note that the k'th entry of $p \downarrow j$ is i. Therefore the k'th entry of $p \downarrow i$ is i. But this latter entry is either $p(k+1)$ or $p(k+1)-1$, depending on whether or not $p(k+1)$ is larger than i. Since $p(k+1)$ is not i, we must have $i = p(k+1) - 1$, and we see that the entry immediately following i in p is $i+1$. Continuing in this way (or, equivalently, using induction on the number of entries of p between i and j), we see that the segment of p from i to j consists of an increasing sequence of consecutive integers.

To verify (v), consider the equivalence relation $\sim$ defined on the set $\{1, 2, ..., n\}$ by $i \sim j \leftrightarrow$ the segment of p from i to j is either an increasing sequence of consecutive integers or a decreasing sequence of consecutive integers. Suppose there are exactly t different equivalence classes $C_1, C_2, ..., C_t$. By (iv) we have $|R(p)| = t$. Let $S = \{k : p(k)$ and $p(k+1)$ are consecutive integers $\}$. For any r, the class C_r contains exactly $|C_r| - 1$ elements of S. Summing over r gives the size of S, namely $n - t$.

(vi) Suppose $i < j$. By part (i), it is sufficient to consider the case when i is to the left of j in p. In the following illustrations, we will let

x_1, x_2 and x_3 denote integers which are $> j$, and we will let y_1, y_2 and y_3 denote integers which are between i and j. When the operations $p \downarrow i$ and $p \downarrow j$ are applied, it is only integers which are larger than $> j$ and integers which are between i and j which are reduced. In illustrating the result of applying two operations successively below, we consider the three segments into which p is divided by i and j. In each segment, we illustrate only entries x for which $x > j$ and entries y for which $i < y < j$. Note that any of the three segments of p may contain none or one or both types of entries. Also note that, in each segment, any x's and y's which appear can be in any relative order. For our purposes, the order in which these entries appear is not important – it is how these entries change in place. All the other entries of p are left in place unchanged. We will use the symbol $\circ$ to indicate an empty spot from which an entry has been deleted. In the following illustration, we show p in the top row and then the result of first applying $\downarrow j$ and then $\downarrow i$.

$$\cdots x_1 \quad \cdots y_1 \quad \cdots i \quad \cdots x_2 \quad \cdots y_2 \quad \cdots j \quad \cdots x_3 \quad \cdots y_3 \quad \cdots$$

$$\downarrow j$$

$$\cdots x_1 - 1 \quad \cdots y_1 \quad \cdots i \quad \cdots x_2 - 1 \quad \cdots y_2 \quad \cdots \circ \quad \cdots x_3 - 1 \quad \cdots y_3 \quad \cdots$$

$$\downarrow i$$

$$\cdots x_1 - 2 \quad \cdots y_1 - 1 \quad \cdots \circ \quad \cdots x_2 - 2 \quad \cdots y_2 - 1 \quad \cdots \circ \quad \cdots x_3 - 2 \quad \cdots y_3 - 1 \quad \cdots$$

Similarly we next illustrate the result of first applying $\downarrow i$ to p and then $\downarrow (j-1)$. As we see, the result is the same.

$$\cdots x_1 \quad \cdots y_1 \quad \cdots i \quad \cdots x_2 \quad \cdots y_2 \quad \cdots j \quad \cdots x_3 \quad \cdots y_3 \quad \cdots$$

$$\downarrow i$$

$$\cdots x_1 - 1 \quad \cdots y_1 - 1 \quad \cdots \circ \quad \cdots x_2 - 1 \quad \cdots y_2 - 1 \quad \cdots j - 1 \quad \cdots x_3 - 1 \quad \cdots y_3 - 1 \quad \cdots$$

$$\downarrow j - 1$$

$$\cdots x_1 - 2 \quad \cdots y_1 - 1 \quad \cdots \circ \quad \cdots x_2 - 2 \quad \cdots y_2 - 1 \quad \cdots \circ \quad \cdots x_3 - 2 \quad \cdots y_3 - 1 \quad \cdots$$

Lemma 2. Let $n \geq 5$ and let p and q be elements of S_n such that $R(p) = R(q)$. Then the entry n occurs in the same position in p as it does in q.

Proof: Equivalently, we show that the position of n in p can be determined from the set $R(p)$. Let us first make some preliminary observations on the possible positions which the entry $n-1$ might occupy among the various members of $R(p)$. We will let $Z_p = \{q^{-1}(n-1) : q \in R(p)\}$. Obviously the set Z_p is determined by the set $R(p)$.

If $p^{-1}(n) = 1$ then $n-1$ is the first entry of $p \downarrow i$ for $i = 1, 2, ..., n-1$, and, in $p \downarrow n$, the position of $n-1$ is one smaller than its position in p. So in this case we either have $Z_p = \{1\}$ (when $n-1$ is the second entry of p), or $Z_p = \{1, j\}$ for some $j > 1$(when $n-1$ is not the second entry of p). Similarly, when n is the last entry of p, we either have $Z_p = \{n-1\}$ or $Z_p = \{j, n-1\}$ for some $j < n-1$.

Suppose n is neither the first nor last entry of p, but occurs in position r for some integer r such that $1 < r < n$. If i is any integer which lies to the left of n in p, then $n-1$ is in position $r-1$ in the reduction $p \downarrow i$. If i is any integer which lies to the right of n in p, then $n-1$ is in position r in the reduction $p \downarrow i$. The position of $n-1$ in $p \downarrow n$ is either the same as the position of $n-1$ in p (when $n-1$ is to the left of n in p), or one less than that position (when $n-1$ is to the right of n in p). So in this case, we either have $Z_p = \{r-1, r\}$ for some r with $1 < r < n$, or $Z_p = \{j, r-1, r\}$ for some r with $1 < r < n$ and for some j distinct from both r and $r-1$.

Thus for any p in S_n, Z_p must be one of the following sets: $\{1\}, \{n-1\}, \{1, n-1\}, \{1, j\}$ for some j with $1 < j < n-1$, $\{j, n-1\}$ for some j with $1 < j < n-1$, $\{r-1, r\}$ for some r such that $2 < r < n-1$, $\{j, r-1, r\}$ for some r with $1 < r < n$ and for some j distinct from both r and $r-1$. These possibilities are listed so as to be mutually exclusive. To show that $p^{-1}(n)$ is determined by the set $R(p)$, we proceed as follows: given the set $R(p)$, we find the corresponding set Z_p. It will be one of the 7 kinds of sets just listed. We will show that the position of n in p can be determined in each case.

The case when Z_p is a one-element set requires very little thought. If $Z_p = \{1\}$ then n must be the first entry of p - if n was in any other position, the above remarks show that Z_p would have to be one of the other 6 kinds of sets. Similarly, if $Z_p = \{n-1\}$ then n must be the last entry of p.

Next, suppose Z_p has two elements. We distinguish several possibilities here. If $Z_p = \{1, j\}$ for some j with $2 < j < n-1$, then n must be the first entry of p : n could not be the last entry in p because $n-1$ is not an element of Z_p and n could not occupy a position r with $1 < r < n$, since, as remarked above, in this eventuality the set Z_p would contain two consecutive integers.

If $Z_p = \{1, 2\}$ then n could not be in any position r in p with $r > 2$. This is clear for $r = n$, because $n-1$ is not an element of Z_p. If n were in position r for some r such that $2 < r < n$ then r would be an element of Z_p. This isn't possible, since $r > 2$. So n must be the first or second element of p. We have to show that one of these two positions is ruled out. Inspecting the elements of set $R(p)$, we determine in how many of these the entry $n-1$ is in the first position. Let s denote the number of elements of $R(p)$ in which the entry $n-1$ is in the first position. Note that $s \geq 1$.

If $s = 1$, then n must be in position 2 of p. To see this, we just need to show that it could not be in position 1 in p: if it were, then $n-1$ would be the first entry in every one of the reductions $p \downarrow i$, for $i < n$. Since $s = 1$, these reductions must all be equal to one another, which implies, by part (iii) of Lemma 1, that the last $n-1$ entries of p are either in consecutive increasing order or consecutive decreasing order. But $n-1$ could not be the last entry of p, because $n-1$ is not an element of Z_p, and $n-1$ could not be the second entry of p, because this would imply that $Z_p = \{1\}$.

Now suppose that $s > 1$. Now, let us show that n must be the first entry of p. To see this, we only need to show that n could not be the second entry of p. Suppose it was. Well, then $n-1$ could not be the first entry of p: if $p = n-1, n, p_3, \ldots$ then, for any $i \notin \{n-1, n\}$, $n-1$ is not the first entry of $p \downarrow i$, and so the only elements of $R(p)$ in which $n-1$ is the first entry are $p \downarrow n$ and $p \downarrow n-1$. But $p \downarrow n = p \downarrow n-1$, since n and $n-1$ are consecutive in p. So there is only one element of $R(p)$ with this property. This is contrary to $s > 1$. So the first entry of p would be some integer m with $m \neq n-1$. But then the only integer i for which $n-1$ is the first entry of $p \downarrow i$ is $i = m$, which is again contrary to $s > 1$.

By applying the preceding arguments to p^{opp} we see that the position of n in p is also determined when Z_p is $\{j, n-1\}$ for some j such that $1 < j < n-2$ and when $Z_p = \{n-2, n-1\}$.

Let us next consider the case when $Z_p = \{1, n-1\}$. In this case, n cannot occupy any position r in p for which $1 < r < n$, since the set Z_p does not contain two consecutive integers $r, r-1$. So n is either the first or last entry of p. This accounts for one element of the set Z_p. The second element of Z_p is the position of $n-1$ in $p \downarrow n = p - n$. So, in order to have $Z_p = \{1, n-1\}$, we must either have n first in p and $n-1$ last in p, or vice-versa. We need to show that one of these(and therefore the position of n in p) is determined by the set $R(p)$. Clearly $|R(p)| \geq 2$. If $|R(p)| > 2$ then either there are 2 elements of $R(p)$ in which $n-1$ is the first entry or 2 elements of $R(p)$ in which $n-1$ is the last entry. If it is the former, then n must be the first entry of p: the only other possibility is to have $n-1$ first and n last in p. But this would imply that the only element of the set $R(p)$ in which $n-1$ is the first entry is $p \downarrow n$, contrary to our assumption that two such elements exist. Similarly, if there are 2 elements of the set $R(p)$ in which $n-1$ is the last entry then n must be the last entry of p.

On the other hand, what if $|R(p)| = 2$? In this case, Lemma 1(v) implies that $n-2$ consecutive pairs of integers occur in the permutation. We know that $\{n-1, n\}$ is not one of these pairs, and so the integers $\{1, 2, ..., n-1\}$ must all be consecutive. So p must be either $n, 1, 2, 3, ..., n-1$ or $n-1, n-2, ..., 3, 2, 1, n$. We can easily determine which(and therefore the position of n) from the set $R(p)$: if $R(p)$ has an element in which the entry $n-2$ is last, then p must be $n, 1, 2, 3, ..., n-1$.

The last remaining possibility to consider when Z_p is a two-element set is when $Z_p = \{r-1, r\}$ for some r such that $2 < r < n-1$. In this case, n must be in position r in p. For any position other than r, it is clear that Z_p would be a set different from $\{r-1, r\}$.

Finally, we consider the case when Z_p is a three-element set. As we saw above, in this case we have $Z_p = \{j, r-1, r\}$ for some r with $1 < r < n$ and for some j distinct from both r and $r-1$. Here r is the position of n in p and j is either the position of $n-1$ in p (when $n-1$ is to the left of n in p), or one less than that position (when $n-1$ is to the right of n in p). So Z_p is a three-element set which contains two consecutive integers and a third integer which may or may not be consecutive with the other two. If Z_p does *not* consist of three consecutive integers, then the position of n in p is easily determined: it is the larger of the two consecutive integers in Z_p. So let us assume that $Z_p = \{i-1, i, i+1\}$ for some integer i. The position of n in p must either be i or $i+1$. We need to show that one of these is determined by the set $R(p)$. If the position of n is i, p would have the form

$$(1) \qquad \cdots, x, n, y, n-1, \cdots$$

If the position of n is $i+1$, p would have the form

$$(2) \qquad \cdots, n-1, x, n, y, \cdots$$

In both cases the displayed elements denote four distinct, consecutive elements of p.

In **(1)**, there are at most two elements in the set $R(p)$ in which $n-2$ is to the left of $n-1$, possibly $p \downarrow n$ and $p \downarrow n-1$. Similarly, in **(2)**, there are at most two elements in the set $R(p)$ in which $n-2$ is to the right of $n-1$. So, if we inspect the set $R(p)$ and find there are at least 3 elements in $R(p)$ in which $n-2$ is to the left of $n-1$, the position of n in p is determined to be $i+1$, the largest element of Z_p. And if we inspect the set $R(p)$ and find there are at least 3 elements in $R(p)$ in which $n-2$ is to the right of $n-1$, the position of n in p is determined to be i, the middle element of Z_p. If $|R(p)| > 4$, one of these two will apply by the pigeonhole principle. So we may assume that $|R(p)| \leq 4$. Now $|R(p)| \geq 3$, since $|Z_p| = 3$. Therefore either $|R(p)| = 3$ or $|R(p)| = 4$.

If $|R(p)| = 3$ then in **(1)** we must have $y = n-2$, since otherwise the four reductions $p \downarrow x, p \downarrow n, p \downarrow y, p \downarrow n-1$ would be four distinct elements of $R(p)$ by part (iii) of Lemma 1. Similarly, in **(2)**, we would have $x = n-2$. So we need to see that, in this case, the set $R(p)$ distinguishes between

(1′) $\cdots, x, n, n-2, n-1, \cdots$ and

(2′) $\cdots, n-1, n-2, n, y, \cdots$

In **(1′)**, in two of the three elements of $R(p)$ we have $n-1$ to the left of $n-2$ ($p \downarrow x$ and $p \downarrow n-2$), whereas in **(2′)**, in two of the three elements of $R(p)$ we have $n-1$ to the right of $n-2$ ($p \downarrow y$ and $p \downarrow n-2$). So $R(p)$ determines the position of n accordingly.

Finally, we consider the case when $|R(p)| = 4$. As noted above, if $R(p)$ has three elements having $n-1$ and $n-2$ in the same relative order, then the position of n can be determined. So we may as well suppose that $n-1$ is to the left of $n-2$ in two of the elements of $R(p)$, and to the right of $n-2$ in the other two elements of $R(p)$. This implies that **(1)** above could only occur with $y \neq n-2$. For, if y were equal to $n-2$, then $p \downarrow n$ would be the one and only element of $R(p)$ in which $n-1$ is to the right of $n-2$. Similarly, **(2)** above can occur only with $x \neq n-2$. Thus we have either

(1) $\cdots, x, n, y, n-1, \cdots$ with $y \neq n-2$, or

(2) $\cdots, n-1, x, n, y, \cdots$ with $x \neq n-2$.

Since $R(p)$ has exactly 4 elements, in both cases we have $R(p) = \{p \downarrow x, p \downarrow n, p \downarrow y, p \downarrow n-1\}$. Reading from left to right, the positions occupied by $n-1$ in each of these four elements of $R(p)$ are $i-1, i+1, i, i$ in **(1)** and $i, i-1, i+1, i$. Note that $n-1$ has the smallest position only once, and the largest position only once.

In **(1)** the two elements of $R(p)$ where $n-1$ is to the right of $n-2$ must be $p \downarrow n$ and $p \downarrow n-1$. Since $y \neq n-2$, this implies that $n-2$ must be to the left of n. Now, since $|R(p)| = 4$, Lemma 1(v) implies that p consists of 4 segments of consecutive integers. Since we know that n, y and $n-1$ are not adjacent to any integers consecutive to themselves, these individual integers constitute 3 of the segments. Thus the remaining integers are consecutive and y must be 1. It follows that there are two configurations possible for p in **(1)**:

(1a) $[2,3,\cdots,n-2,n,1,n-1]$ or **(1b)** $[n-2,n-3,\cdots,2,n,1,n-1]$

In a similar way we see there are two configurations possible for p in **(2)**:

(2a) $[n-1,1,n,2,3,\cdots,n-2]$ or **(2b)** $[n-1,1,n,n-2,n-3,\cdots,2]$.

We only need to show that $R(p)$ distinguishes **(1)** from **(2)**. To see this, note that, since $n > 4$, $n-1$ could never occur as the first entry in any of the elements of $R(p)$ if **(1a)** or **(1b)** applied, whereas it clearly does in both **(2a)** and **(2b)**. So in this case, we simply check whether 1 does or does not belong to Z_p. □

Lemma 3. Let $n \geq 3$ and let p and q be elements of S_n. Let $p' = p - n$ and let $q' = q - n$. If $R(p) = R(q)$ then $R(p') = R(q')$.

Proof: Using formula (vi) of Lemma 1 we have
$R(p') = \{t \in S_{n-2} : t = p' \downarrow i \text{ for some } i \leq n-1\}$
$= \{t \in S_{n-2} : t = (p \downarrow n) \downarrow i \text{ for some } i \leq n-1\}$
$= \{t \in S_{n-2} : t = (p \downarrow i) \downarrow n-1 \text{ for some } i \leq n-1\}$.
Since (again using formula (vi) in Lemma 1) $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$, we have
$R(p') = \{t \in S_{n-2} : t = (p \downarrow i) \downarrow n-1 \text{ for some } i \leq n\}$
$= \{t \in S_{n-2} : t = s \downarrow n-1 \text{ for some } s \in R(p)\}$
$= \{t \in S_{n-2} : t = s \downarrow n-1 \text{ for some } s \in R(q)\} = R(q')$. □

Lemma 4. Let p and q be elements of S_5 such that $R(p) = R(q)$. Then $p = q$.

Proof: One can, of course, execute a simple computer program to verify this statement, which we have done using GAP[3]. We can also argue directly. First of all, by Lemma 2, we can assume that 5 occurs in the same position in p and q. And secondly, by (i) of Lemma 1, we can assume that this common position is either first, second or third. Similar arguments can be made in all three cases. We will include the details for the first two of these cases and leave the third to the reader.

In the first case, we would have $p = 5p_2p_3p_4p_5$ and $q = 5q_2q_3q_4q_5$, and $R(p) = R(q)$. This implies that $p_2p_3p_4p_5 \in R(q)$ and so $p_2p_3p_4p_5 = q \downarrow i$ for some $i \leq 5$. If this occurs for $i = 5$ we are done. So we can suppose $i \neq 5$. This implies that $q \downarrow i$ begins with 4 and so $p_2 = 4$. Thus $p = 54p_3p_4p_5$.

So every element of $R(p)$ begins with 4. Since the same must be true for $R(q)$, we see that $q_2 = 4$, and so $q = 54q_3q_4q_5$. If $4p_3p_4p_5$ is equal to $q \downarrow i$ for $i = 4$ or $i = 5$, this would clearly imply that $p = q$, so we can suppose that $4p_3p_4p_5 = q \downarrow i$ for some $i \leq 3$. But, for any $i \leq 3$, the first two entries of $q \downarrow i$ are 43, and so this implies that $p_3 = 3$, and so $p = 543p_4p_5$. This implies that every element of $R(p)$ begins with 43. Since the same must be true of the elements in $R(q)$, we must also have $q_3 = 3$. Thus $q = 543q_4q_5$. Finally, since $R(54321) \neq R(54312)$, we infer that $p = q$.

In the second case, we would have $p = p_15p_3p_4p_5$ and $q = q_15q_3q_4q_5$, and $R(p) = R(q)$. As in the first case, we can assume that $p_1p_3p_4p_5 = q \downarrow i$ for some $i \neq 5$. For any such i, either the first or second element of $q \downarrow i$ is 4, so either $p_1 = 4$ or $p_3 = 4$. We consider two subcases:

(i) $p = 45p_3p_4p_5$, and (ii) $p = p_154p_4p_5$.

In subcase (i), every element of $R(p)$ begins with either 3 or 4, so the same holds for the elements of $R(q)$. Therefore we must have $q_1 = 4$, and so $q = 45q_3q_4q_5$. Now we have $4p_3p_4p_5 = q \downarrow i$ for some i. Since $q \downarrow i$ does not begin with 4 for $i < 4$, we must have $4p_3p_4p_5 = q \downarrow i$ for $i = 4$ or 5. This implies that $p = q$. In the second subcase, $p = p_154p_4p_5$. We cannot have $q_1 = 4$, because this would imply that every element of $R(q)$ begins with 4 or 3, which is clearly not the case for all elements of $R(p)$. Now $R(p)$ has an element which begins 43. Therefore $R(q)$ does too, and, since $q_1 \neq 4$, this implies that $q_3 = 4$. Thus $q = q_154q_4q_5$. Now we must have $p_14p_4p_5 = q \downarrow i$ for some i. If $i = 4$ or 5, this implies $p = q$ as desired. Clearly $p_14p_4p_5 \neq q \downarrow q_1$, and so either $p_14p_4p_5 = q \downarrow q_4$ or $p_14p_4p_5 = q \downarrow q_5$. Either way, we must have $p_4 = 3$. Thus $p = p_1543p_5$. This implies that, in every element of $R(p)$, 3 is either second or third. Therefore the same is true of the elements of $R(q)$. This implies that 3 cannot be the first or the last element of q, and so $q_4 = 3$. Thus $q = q_1543q_5$. And since $R(15432) \neq R(25431)$, we infer that $p = q$. □

Our theorem now follows directly from the preceding lemmas by induction on n.

Theorem. Let $n \geq 5$. Then any $p \in S_n$ is determined by its set of reductions $R(p)$. Equivalently, if p and q are elements of S_n and $R(p) = R(q)$, then $p = q$.

The proof of our theorem leads to a straightforward recursive procedure for reconstructing a permutation p from its set of reductions $R(p)$. At the bottom, we tabulate the 120 different sets $R(p)$ corresponding to the elements p in S_5. The GAP program [3] is very well-suited to this task.

For any $n > 5$, if we are given the set $R(p)$ for some $p \in S_n$, we first apply the method used in the proof of Lemma 2 to find the position of n in p. Then, as in the proof of Lemma 3, letting $q = p - n$, we find the set $R(q) = R(p - n) = \{t \in S_{n-2} : t = s \downarrow n - 1 \text{ for some } s \in R(p)\}$. From this set we reconstruct q. We then insert n into q so that it occupies the position it must occupy.

Example. Here is an illustration for $n = 7$. Suppose we are given that the set of reductions for p is $R = \{536412, 542631, 543612, 546312, 653412\}$. The set of positions of $n - 1 = 6$ in the reductions is then $Z_p = \{1, 3, 4\}$. Since the three elements in Z_p are not all consecutive, the proof of Lemma 2 shows that the position of 7 in p must be the larger of the consecutive pair in Z_p, namely 4. We now let q be $p - 7$. The set of reductions of q is the set $\{t \in S_5 : t = s \downarrow 6 \text{ for some } s \in R(p)\} = \{s - 6 : s \in R(p)\}$. So we simply delete 6 from each of the permutations belonging to R. We get the set $R' = \{54231, 53412, 54312\}$. To find the position of the entry 6 in q, we apply the method in the proof of Lemma 2 to the set R'. The set of positions of $6 - 1 = 5$ in the reductions belonging to R' is $Z_q = \{1\}$. Thus (as in the proof of Lemma 2) 6 must be the first entry of q. Now, let $r = q - 6$. The set of reductions of r is found simply by deleting 5 from each of the permutations belonging to R'. We get the set $R'' = \{3412, 4231, 4312\}$. From the example at the beginning of this paper, r is given by $r = 53412$. We insert 6 into r to form q so that 6 is the first entry. We get $q = 653412$. Finally, we insert 7 into q to find p, so that 7 is the fourth entry. We get $p = 6537412$. □

To conclude, we would like to suggest two possible directions for future work related to our theorem. In recent work on reconstructing an n-sequence s from its multiset of k-subsequences, significant progress has been made in finding a function $f(n)$, having as small an order as possible, so that s can be reconstructed from its k-subsequences as long as $k \geq f(n)$. We refer the reader to [2], [6] and [8]. Analogously, it would be interesting to know how many reductions of a permutation are needed to reconstruct it. Can we find a non-trivial function $f(n)$ so that, for n sufficiently large, if I is any subset of $\{1, 2, ..., n\}$ with $|I| \geq f(n)$, then any permutation p in S_n can be reconstructed from the set of reductions relative to I, that is, from the set $R^I(p) = \{q \in S_{n-1} : q = p \downarrow i \text{ for some } i \in I\}$? We have made no useful progress on this question.

Secondly, one can consider variations on the notion of "reduction". After one entry i of a permutation on $\{1, 2, ..., n\}$ is deleted from it, there are many natural ways to view the result as a permutation on $\{1, 2, ..., n - 1\}$.

Instead of reducing by 1, in place, all entries of p which are larger than i, we could instead, for $i \neq n$, replace the entry n by i. This gives another type of "reduction", and one can again consider the set of all such reductions over all $i \leq n$. Can p be reconstructed from this set of reductions? A more general inquiry may be useful. One might attempt to define a general notion of "reduction" along the following lines. Let P_{n-1} denote the set of all $n-1$-permutations of the n-element set $\{1, 2, ..., n\}$. For any $p \in S_n$ and $i \leq n$, let $p - i$ be the element of P_{n-1} obtained by deleting i from p. Then any function $F : P_{n-1} \longrightarrow S_{n-1}$ gives rise to a type of "reduction set" for p, namely the set $R_F(p) = \{F(p - i) : i = 1, 2, ..., n\}$. Is there a large, natural class of functions F for which p can always be reconstructed from $R_F(p)$?

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