Domination Graphs of Extended Rotational Tournaments: Chords and Cycles

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Abstract

Only the rotational tournament U_n for odd $n \geq 5$, has the cycle C_n as its domination graph. To include an internal chord in C_n, it is necessary for one or more arcs to be added to U_n in order to create the extended tournament U_n^+. From this, the domination graph of U_n^+, $dom(U_n^+)$, may be constructed where C_k, $3 \leq k \leq n$, is a subgraph of $dom(U_n^+)$. This paper explores the characteristics of the arcs added to U_n that are required to create an internal chord in C_n.

Keywords: *tournament, regular tournament, rotational tournament, domination graph, loopless semi-complete digraph, extended tournament, m-tie chord, cycle, tie arc*

1 Introduction

A *tournament* T_n is a digraph on n vertices where there exists exactly one arc between every pair of vertices in the vertex set of T_n, $V(T_n)$. For $u, v \in V(T_n)$, $u \neq v$, u is said to *dominate*, or *beat*, a vertex v, denoted by $u \rightarrow v$, if the arc $(u, v) \in A(T_n)$, the set of all arcs in T_n. The *out-set* of a vertex u, $O(u)$, is the set of all vertices that u beats. The *out-degree* of a vertex u is $d^+(u) = |O(u)|$. The *domination graph of* T_n, $dom(T_n)$, has $V(dom(T_n)) = V(T_n)$, with an edge between every pair of vertices in T_n that together beat all other vertices. Fisher, et al. [8], [9] introduced the domination graph of a tournament. This initial work led to the study of domination graphs with nontrivial components [5] and with isolated vertices [6]. Other research addressed tournaments with connected domination graphs [7], [10]. Of particular interest to this paper, Cho, et al. [1], [2] presented results on the domination graphs of regular tournaments. More recently, work on the domination graphs of subdigraphs of tournaments [4] has been presented.

The concepts of *extended tournaments* and *partial domination graphs* were introduced in [3]. A partial domination graph is a domination graph in the classical sense. An *extended tournament* T_n^+ is a tournament with at least one tie arc. An extended tournament is a loopless semi-complete digraph. A *tie* between two vertices $u, v \in V(T_n^+)$ is represented by arcs $(u, v), (v, u) \in A(T_n^+)$. For the purpose of discussing results involving tournaments with ties T_n^+, its domination graph is denoted as $dom\,(T_n^+)$.

Note that adding an arc to a tournament T_n may or may not generate a new edge in $dom\,(T_n^+)$. It could still be true that $E\,(dom\,(T_n)) = E\,(dom\,(T_n^+))$. So, in general, $E\,(dom\,(T_n)) \subseteq E\,(dom\,(T_n^+))$. For example, let T_4 be a tournament where $A\,(T_4) = \{(1,2),(2,3),(2,4),(3,1),(3,4),(4,1)\}$ and $V\,(T_4) = \{1,2,3,4\}$. See Figure 1 where $A\left(T_4^+\right) = A\,(T_4) \cup (2,1)$ and $E(dom\,(T_4)) = E\left(Pdom\,(T_4^+)\right)$.

Figure 1: Extended tournament T_4^+ where $dom(T_4) = dom\left(T_4^+\right)$

In Factor and Factor [3], extended tournaments T_n^+ where $dom\,(T_n^+) = K_n$, the complete graph on n vertices, were completely characterized. They proved the following theorem and corollaries:

Theorem 1.1 [3] *Let T_n^+ be an extended tournament. If $dom\,(T_n^+) = K_n$, then T_n^+ has at least $\binom{n}{2} - n$ ties.*

Corollary 1.2 [3] *Let T_n^+ be an extended tournament. If $dom\,(T_n^+) = K_n$, then T_n^+ has at most n non-tie arcs.*

Corollary 1.3 [3] *C_4 is the induced subgraph of an extended tournament.*

This last corollary is an interesting contrast to the fact that there does not exist any tournament T_n that generates an even cycle as a subgraph in $dom\,(T_n)$, proven by Cho, Kim, and Lundgren [2].

In view of the above results, what can be said about the nature of the arcs added to T_n, creating T_n^+, in regard to how they generate internal chords and hence subgraphs in $dom\,(T_n^+)$? Here we explore this question in the context of rotational tournaments.

2 The Rotational Tournament U_n

A tournament T_n is *regular* if n is an odd integer and for all $u \in V(T_n)$, $d^+(u) = \frac{n-1}{2}$. The domination graph of a regular tournament ($n \geq 3$) is either an odd cycle or a forest of paths [2]. Regular tournaments whose domination graphs are odd cycles will be considered here. A regular tournament $T(S)$ can be defined as the *rotational tournament* with symbol S whose vertices are labeled by elements of $\mathbb{Z}_n$(the integers mod n), for odd integer $n \geq 3$, with arc (i, j) if $j - i \equiv s$, where $s \in S$ and S is a $\frac{n-1}{2}$-set contained in $\mathbb{Z}_n$ where $0 \notin S$ and $s_1 + s_2 \not\equiv 0$ for all $s_1, s_2 \in S$. Specifically, $S' = \{1, 3, 5, ..., n-2\}$, where odd n satisfies $0 \notin S'$ and $s_1 + s_2 \not\equiv 0$ for all $s_1, s_2 \in S'$. Define the rotational tournament $U_n = T(S')$ for odd $n \geq 3$. U_n has vertices labeled by consecutive numbers $\{0, 1, 2, ..., n-1\}$ and has arcs (i, j) if $j - i \equiv s(\text{mod}\, n)$ where $s \in S'$ and (j, i) otherwise. Consequently, $d^+(i) = \frac{n-1}{2}$ for all $i \in V(U_n)$.

In the following discussion, vertices will be labeled $0, 1, ..., n-1$. Without loss of generality, it will be assumed to be a consecutive labeling in the clockwise direction in U_n. An example of a rotational tournament for $n = 5$ using this labeling is shown in Figure 2, where $V(U_5) = \{0, 1, 2, 3, 4\}$, $S' = \{1, 3\}$, and $d^+(i) = 2$ for all $i \in V(U_5)$.

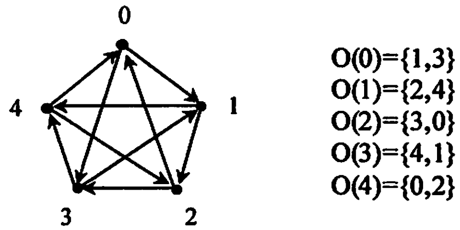

Figure 2: The rotational tournament U_5

Throughout the remainder of this paper, it will be assumed in all proofs and discussions that $i + p$ means $(i + p) \bmod n$ for odd $n \geq 3$, $p \in \mathbb{Z}$.

The following lemma describes the vertices that are not beaten by the pair of vertices $i, i+k$ in U_n when k is odd, and gives the number of vertices they do not dominate. These characteristics are significant for the rest of the paper.

Lemma 2.1 *Given $i \in V(U_n)$, $n \geq 3$, and k odd where $1 \leq k \leq n-2$, the vertices not beaten by either i or $i+k$ are of the form $i+d$, where $d = 2, 4, ..., k-1$. The number of vertices not beaten by the pair is $\frac{k-1}{2}$.*

Proof: By definition of U_n, $i \to i+k$ only for $k \in \{1, 3, ..., n-2\}$, and $i+k$ does not beat any vertex of the form $i+2q$ where $0 \le q \le \frac{k-1}{2}$, for each k. Therefore, the only vertices not beaten by i and $i+k$ are of the form $i+d$ where $d = 2, 4, ..., k-1$, and there exist $\frac{k-1}{2}$ of these. □

Note that given $i \in V(U_n)$, only vertices i and $i+k$, where k is odd, are considered in Lemma 2.1. Consider the following proposition below, which addresses the vertices i and $i+2p$, where $0 \le p \le \frac{n-1}{2}$.

Proposition 2.2 *When considering all $i, i+k \in V(U_n)$, $n \ge 3$, the case where k is odd gives all possible pairings of vertices in $V(U_n)$.*

Proof: Let $i \in V(U_n)$. Consider the vertices i and $i+2p$ where $0 \le p \le \frac{n-1}{2}$. The distance clockwise from vertex i to vertex $i+2p$ is even. Since n is odd, the distance clockwise from $i+2p$ to i must be odd. Thus, when $j = i+2p$ and $j+k = i$, where k is the odd clockwise distance from $i+2p$ to i, the pair is obtained. □

From Proposition 2.2, it follows that given $i \in V(U_n)$, $n \ge 3$, it is only necessary to consider vertices i and $i+k$, where k is odd, in order to produce all possible pairs of vertices in U_n.

3 Internal Chords in $dom(U_n^+)$

Let C_n be a cycle on n vertices. For odd $n \ge 3$, all of the edges in C_n are considered *external* and come from the definition of U_n, where $dom(U_n) = C_n$ by Fisher, et al. [8]. Let U_n^+ be an extended tournament where $i, j \in V(U_n^+)$, $i \ne j$, and let m be a nonnegative integer. A *tie arc* is an arc that creates a tie in U_n^+ when added to U_n. Any new arc added to U_n creates a tie in U_n^+. Edge $\{i, j\}$ is an *m-tie chord* in $dom(U_n^+)$ if a minimum of m tie arcs must be added to U_n in order to produce the edge $\{i, j\}$ in the domination graph. This m-tie chord need not be the only chord created by the m arcs. Note if $m = 0$, then edge $\{i, j\}$ is referred to as a *0-tie chord.* This means that $\{i, j\}$ is an external chord in $dom(U_n^+)$, and is part of the original domination graph $dom(U_n)$. Thus, no new arcs must be added to U_n in order to create this graph. If $m > 0$, the m-tie chord is referred to as an *m-tie internal chord,* where the generated edge $\{i, j\}$ is an internal chord in $dom(U_n^+)$, and is not part of the original domination graph.

The following propositions follow directly from the definition of regular tournaments.

Proposition 3.1 *Let $i, j \in V(U_n)$, $i \ne j$, for $n \ge 3$. $|O(i) \cap O(j)| = m$ if and only if $n - 1 - |O(i) \cup O(j)| = m$.*

Proposition 3.2 *Let $i, j \in V(U_n)$, $i \neq j$, for $n \geq 3$. There are the same number of common vertices that i and j beat as there are the number of vertices that neither i nor j beat. Furthermore, this count does not include the vertices i and j.*

Remark 1 *Consider consecutive vertices $i, i+1 \in V(U_n)$, for $n \geq 3$, where $0 \leq i \leq n-1$. $V(U_n) = O(i) \cup O(i+1) \cup \{i\}$ where $O(i) \cap O(i+1) = \varnothing$. By Proposition 3.1, $|O(i) \cap O(i+1)| = 0$ implies $m = 0$ and $|O(i) \cup O(i+1)| = n-1$. Therefore, $\{i, i+1\}$ is a 0-tie chord. As such, it is an external chord in $dom(U_n^+)$, since no new arcs must be added to U_n.*

Proposition 3.3 *Let $i, j \in V(U_n)$, $i \neq j$, for $n \geq 3$. If $n-1-|O(i) \cup O(j)| = m$ or $|O(i) \cap O(j)| = m$, then $\{i, j\}$ is an m-tie chord in $dom(U_n^+)$ for some U_n^+.*

Proof: Let $i, j \in V(U_n)$, $i \neq j$, for $n \geq 3$ and $m \geq 0$. By Proposition 3.1, $|O(i) \cap O(j)| = m$ if and only if $n-1-|O(i) \cup O(j)| = m$. Suppose that $n-1-|O(i) \cup O(j)| = m$. This means that there exists m vertices not beaten by i or j, excluding i and j. So a minimum number of m arcs must be added to U_n to create some U_n^+ which has $\{i, j\}$ as an edge in its domination graph. Thus $\{i, j\}$ is an m-tie chord in $dom(U_n^+)$. □

Suppose U_n is a rotational tournament for $n \geq 5$ where $i \in V(U_n)$. By Lemma 2.1, if $k = 3$, then the vertices not beaten by i or $(i+3)$ are of the form $(i+2)$. Results from Factor and Factor [3] involving arcs added to U_n from vertices that beat the vertex $(i+2)$ and form a tie in U_n^+ are as follows.

Proposition 3.4 [3] $dom(U_n \cup (i+3, i+2)) = C_n \cup \{i, i+3\}$.

Proposition 3.5 [3] $dom(U_n \cup (i, i+2)) = C_n \cup \{i, i+3\}$.

In either case, we see that $\{i, i+3\}$ is a 1-tie internal chord in $dom(U_n^+)$, since U_n^+ is created by adding a minimum of one arc to U_n, creating one tie in U_n^+. Note that when either arc $(i+3, i+2)$ or arc $(i, i+2)$ or both are added to U_n, giving U_n^+, only the one internal chord $\{i, i+3\}$ is generated. This is true for each of the n vertices $i \in V(U_n^+)$.

A natural question that arises is whether an m-tie internal chord can be created without producing additional chords. If $\{i, j\}$ is an m-tie internal chord, it is easy to see that adding arcs from i to each of the m vertices not beaten by either i or j will produce the chord $\{i, j\}$ in the domination graph. However, it is not guaranteed that only the one chord will be produced in this manner. For example, let $i \in V(U_n)$ where $i = 0$ and $n = 5$. By definition of U_5, 0 beats 1, and 0 beats 3. Add arcs $(0, 2)$ and $(0, 4)$ to generate U_5^+. Since 0 beats all vertices then so will any other vertex paired with 0. Therefore, $\{0, 3\}$and $\{0, 4\}$ are generated in *dom* (U_5^+). Thus, there is a need to characterize a set of arcs whose addition to $A(U_n)$ will generate exactly the one edge $\{i, j\}$ in *dom* (U_n^+). As a first step in this process, the following results characterize the vertices comprising an m-tie internal chord in *dom* (U_n^+) and enumerate the additional vertices that must be added to the set $O(i) \cup O(j)$ in U_n with corresponding tie arcs in U_n^+.

Lemma 3.6 *Let $i \neq j \in V(U_n)$, such that $j = i + k$, for k odd and $1 \leq k \leq n-2$. Then $\{i, j\}$ is an m-tie chord in dom (U_n^+) if and only if $m = \frac{k-1}{2}$.*

Proof: ($\Rightarrow$) Let $i \neq j \in V(U_n)$, such that $j = i + k$, for k odd and $1 \leq k \leq n-2$. Let $\{i, j\}$ be an m-tie chord in *dom* (U_n^+). Therefore, a minimum of m-tie arcs must be added to U_n in order to create $\{i, j\}$ in *dom* (U_n^+). By Lemma 2.1, the only vertices not beaten in U_n by $\{i, i + k\}$, k odd, $1 \leq k \leq n-2$, are of the form $i + d$ where $d = 2, 4, ..., k - 1$, and there are $\frac{k-1}{2}$ of these vertices. Therefore, $m = \frac{k-1}{2}$.

($\Leftarrow$) Let $i \neq j \in V(U_n)$, such that $j = i + k$, for k odd and $1 \leq k \leq n-2$. By definition of U_n, i beats j and $n - 1 - |O(i) \cup O(j)|$ is the number of vertices not beaten by i or j, excluding the pair. By Lemma 2.1, the number of vertices that neither i nor $i + k$ beat is $\frac{k-1}{2}$. Let $m = \frac{k-1}{2}$. Proposition 3.3 states that $\{i, i + k\}$ is an m-tie chord in *dom* (U_n^+) for some U_n^+.□

From Lemma 3.6, it follows that there are different types of m-tie chords possible in the same domination graph of an extended rotational tournament. For example, let $n = 9$. Then $\{i, i + 1\}$ is a 0-tie chord, $\{i, i + 3\}$ is a 1-tie internal chord, $\{i, i + 5\}$ is a 2-tie internal chord, and $\{i, i + 7\}$ is a 3-tie internal chord in *dom* (U_9^+) for $i \in V(U_9^+)$. Figure 3 illustrates this, where $i = 0$.

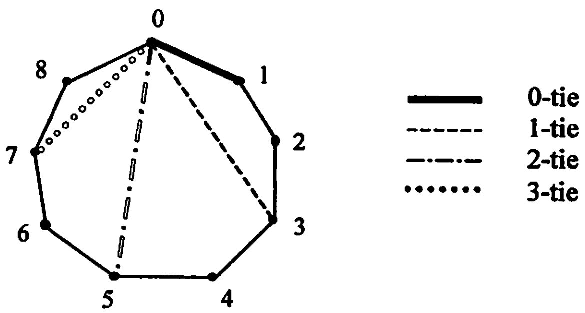

Figure 3: Different m-tie chords in *dom* (U_9^+)

Corollary 3.7 *Let $i, j \in V(U_n)$, $n \geq 3$, where $j > i$. $j = i + 2m + 1$ for $0 \leq m \leq \frac{n-3}{2}$ if and only if $\{i, j\}$ is an m-tie chord in dom (U_n^+).*

Proof: Note that $m = \frac{k-1}{2} \Leftrightarrow k = 2m + 1$. Let $i, j \in V(U_n)$, $n \geq 3$, where $j > i$. By Lemma 3.6, $j = i + 2m + 1$, $0 \leq m \leq \frac{n-3}{2}$ if and only if $\{i, j\}$ is an m-tie chord in *dom* (U_n^+). □

Theorem 3.8 *Let U_n^+ be the extended rotational tournament for $n \geq 5$. Then there are exactly n distinct m-tie internal chords in dom (U_n^+) for each m where $1 \leq m \leq \frac{n-3}{2}$.*

Proof: Let $n \geq 5$ and $1 \leq m \leq \frac{n-3}{2}$. By Corollary 3.7, $\{i, i + 2m + 1\}$ is an m-tie internal chord for each $i \in V(U_n)$. Further, if $j = i + 2m + 1$, then $i \neq j + 2q + 1$ for any integer q, since an even number must be added to j in order to obtain i when n is odd. Thus $\{i, i + 2m + 1\} \neq \{j, j + 2m + 1\}$ for all $i \neq j$, and as there are n ways to choose i, there exist exactly n distinct m-tie internal chords. □

Table 1 makes use of Corollary 3.7 and Theorem 3.8.

n	m	Number of internal chords
3	0	None
5	1	Five 1-tie internal chords
7	2	Seven each 1- and 2- tie internal chords
9	3	Nine each 1-, 2-, and 3-tie internal chords

Table 1: n distinct m-tie internal chords

The following lemma characterizes the m-tie internal chords $\{i, j\}, j = i + k$, in *dom* (U_n^+) for $n \geq 5$, where k is odd, $3 \leq k \leq n - 2$.

Lemma 3.9 *Let $i, j \in V(U_n)$, $n \geq 5$, where $j > i$ and $1 \leq m \leq \frac{n-3}{2}$. Then $\{i, j\}$ is an m-tie internal chord in dom (U_n^+) if and only if $i + d \notin O(i) \cup O(j)$ in U_n and $i + d \in O(i) \cup O(j)$ in U_n^+, for $d \in \{2, 4, ..., 2m\}$.*

Proof: ($\Rightarrow$) Let $i, j \in V(U_n)$, $n \geq 5$, where $j > i$. Let $\{i, j\}$ be an m-tie internal chord in *dom* (U_n^+). By Corollary 3.7, $j = i + 2m + 1$, $1 \leq m \leq \frac{n-3}{2}$. Letting $k = 2m + 1$ in Lemma 2.1 we see that $i + d \notin O(i) \cup O(j)$ for $d \in \{2, 4, ..., 2m\}$. By assumption, $\{i, j\}$ is an m-tie internal chord of *dom* (U_n^+), so i and j must beat these m vertices in U_n^+. Consequently, $i + d \in O(i) \cup O(j)$ in U_n^+.

($\Leftarrow$) Let $i, j \in V(U_n)$, $n \geq 5$, where $j > i$. Assume $i + d \notin O(i) \cup O(j)$ in U_n, and $i + d \in O(i) \cup O(j)$ in U_n^+ for $d \in \{2, 4, ..., 2m\}$, where $1 \leq m \leq \frac{n-3}{2}$. This exhibits m vertices not dominated by i or j and hence at least m arcs must be added to U_n to create some U_n^+ with $\{i, j\}$ an edge in $dom(U_n^+)$. Since $(i + d) \in O(i) \cup O(j)$ in U_n^+ for $d \in \{2, 4, ..., 2m\}$, then $\{i, i + 2m + 1\} = \{i, j\}$ and $\{i, j\}$ is an m-tie internal chord in $dom(U_n^+)$. □

Note that if $n = 3$ and $k = 1$, then $m = 0$, and all edges are external on $dom(U_3^+) = C_3$. Consequently, there are no internal chords (i.e., they are all 0-tie chords). By Lemma 3.9, if $i, j \in V(U_n)$, $n \geq 5$, where $j = i + k$ for each $k \in \{3, 5, ..., n - 2\}$, $\{i, i + k\}$ is an m-tie internal chord in $dom(U_n^+)$, where $m = \frac{k-1}{2}$, if and only if the vertices i and $i + k$ together beat vertices $\{i + 2, i + 4, ..., i + k - 3, i + k - 1\}$, for each $i \in V(U_n^+)$. The form of the m-tie internal chord in $dom(U_n^+)$, where $m = \frac{n-3}{2}$, is $\{i, i + n - 2\}$. Further, vertices i and $i + n - 2$ must additionally beat vertices $\{i + 2, i + 4, ..., i + n - 3\}$, for each $i \in V(U_n^+)$.

In the specific case where $n = 3$ and $m = 0$ (no internal chords), vertex i beats vertex $i + 1$ in the definition of U_3. Thus, $dom(U_3^+) = dom(U_3) = C_3$ for any number of tie arcs that might be added to U_3 in order to form U_3^+. Again, this is an example where arcs can be added to U_n that create a U_n^+ where the domination graph of U_n is the same as the domination graph of U_n^+.

Recall from Proposition 2.2, given $i \in V(U_n)$, $n \geq 3$, it is only necessary to consider vertices i and $i + k$, where k is odd, in order to generate all possible pairings in U_n. This is also true for all possible pairings of vertices in $dom(U_n^+)$, since $V(U_n) = V(U_n^+)$. This fact along with Corollary 3.7 motivates the following definition of the set X_n^m, for the specific U_n^+, for each odd $n \geq 3$, and all $0 \leq m \leq \frac{n-3}{2}$.

$$X_n^m = \{\{i, j\} \mid j = i + k \text{ where } k = 2m + 1 \text{ and } \{i, j\} \text{ is an } m\text{-tie chord in some } dom(U_n^+)\}$$

Note that for $m = 0$, X_n^0 is the set of external chords. For $m > 0$, X_n^m is the set of possible m-tie internal chords in $dom(U_n^+)$ of the form $\{i, i + 2m + 1\}$, $i \in V(U_n^+)$ for each $n \geq 3$.

Combining the preceding results and using the definition for X_n^m, the following lemma and theorem are obtained for the case where $dom(U_n^+) = K_n$, $n \geq 3$.

Lemma 3.10 *For each odd $n \geq 3$, let $M = \frac{n-3}{2}$ and $dom(U_n^+) = K_n$. Then the following are true:*

1. $|X_n^m| = n$ *for each m, where $0 \leq m \leq M$.*
2. *For $m_1 \neq m_2$, $0 \leq m_1, m_2 \leq M$, $X_n^{m_1} \cap X_n^{m_2} = \varnothing$.*

Proof: For each odd $n \geq 3$, let $M = \frac{n-3}{2}$, $dom\,(U_n^+) = K_n$, and X_n^m be defined as above, where $0 \leq m \leq \frac{n-3}{2}$. By Theorem 3.8, the number of distinct m-tie chords is n, so $|X_n^m| = n$ for each m where $0 \leq m \leq M$, giving the first result. For the second result, where $m_1 \neq m_2$, note that $X_n^{m_1}$ consists of chords of the form $\{i, i+k_1\}$ where $k_1 = 2m_1 + 1$, and $X_n^{m_2}$ consists of chords of the form $\{i, i+k_2\}$ where $k_2 = 2m_2 + 1$, for all $i \in V\,(U_n^+)$. Since $m_1 \neq m_2$, for $0 \leq m_1, m_2 \leq M$, then $k_1 \neq k_2$. Consequently, $X_n^{m_1} \cap X_n^{m_2} = \varnothing$. □

Lemma 3.10 proves that for a given odd $n \geq 3$, the set of n distinct m_1-tie internal chords has no chords in common with the set of n distinct m_2-tie internal chords in $dom\,(U_n^+)$, for all $m_1 \neq m_2$ where $0 \leq m_1, m_2 \leq \frac{n-3}{2}$.

Theorem 3.11 *For each odd $n \geq 3$, let $M = \frac{n-3}{2}$ and $dom\,(U_n^+) = K_n$. Then $\bigcup_{m=0}^{M} X_n^m$ consists of all chords in $dom\,(U_n^+)$. Further, $\binom{n}{2} = \left|\bigcup_{m=0}^{M} X_n^m\right|$.*

Proof: By Corollary 3.7 and Lemma 3.10, for each odd $n \geq 3$, every edge of $dom\,(U_n^+)$ is in X_n^i for some $1 \leq i \leq \frac{n-3}{2}$. So, $\left|\bigcup_{m=0}^{M} X_n^m\right| = |E(K_n)| = \binom{n}{2}$.
□

Given any edge in K_n, that edge must be some m-tie chord of $dom\,(U_n^+)$ for some U_n^+. This is addressed in the following corollary.

Corollary 3.12 *If $\{i, j\}$ is an edge in K_n, odd $n \geq 3$, then there exists an extended tournament U_n^+ such that for a certain m, $0 \leq m \leq \frac{n-3}{2}$, $\{i, j\} \in E\,(dom\,(U_n^+))$ is an m-tie chord. Further, there are $\frac{n-1}{2}$ distinct types of m-tie chords possible.*

Proof: The first part follows directly from Theorem 3.11. Further, there are $\frac{n-3}{2}$ types of internal chords and one type of external chord, giving a total of $\frac{n-1}{2}$ distinct types of m-tie chords possible. □

4 Tie-Arcs in U_n^+

This section characterizes how the tie arcs added to U_n are chosen in order to create an extended tournament U_n^+ whose domination graph will contain specific m-tie internal chords, where $n \geq 5$, $1 \leq m \leq \frac{n-3}{2}$, without any additional chords.

Proposition 4.1 *Let $i, j \in V\,(U_n)$ for $n \geq 5$, $1 \leq m \leq \frac{n-3}{2}$ and $j = i+2m+1$. There exist exactly 2^m possible sets of m arcs that each when added to U_n to produce the m-tie internal chord $\{i, j\}$ in $dom\,(U_n^+)$.*

Proof: Let $i, j \in V(U_n)$ for $n \geq 5$, $1 \leq m \leq \frac{n-3}{2}$ and $j = i + 2m + 1$. $\{i, j\} \notin E(dom(U_n))$ since $j \neq i + 1$ or $i + 2n + 1$. One arc from vertex i or vertex j to each of these vertices to make U_n^+ will create $\{i, j\}$ in $dom(U_n^+)$ by i or j, by Lemma 3.9. There are 2^m such collections of exactly m arcs. □

By Proposition 4.1, for $m = 1, n \geq 5$, there are 2 sets, with 1 arc each, either of which will create a 1-tie internal chord of the form $\{i, i+3\}$ in $dom(U_n^+)$. Propositions 3.4 and 3.5 show that $\{i, i+3\}$ is unique for either of the 2^1arcs $(i, i+2)$ or $(i+3, i+2)$ added to U_n for $i \in V(U_n)$.

Proposition 4.2 *Let $i, j \in V(U_n)$ for $n \geq 7$, $1 \leq m \leq \frac{n-3}{2}$ and $j = i+2m+1$. There exists at least one collection of m arcs that when added to U_n produces $\{i, j\}$ and at least one other internal chord in $dom(U_n^+)$. Furthermore, exactly m internal chords can be created that are incident with i in $dom(U_n^+)$.*

Proof: Let $i, j \in V(U_n)$ for $n \geq 7$, $1 \leq m \leq \frac{n-3}{2}$ and $j = i + 2m + 1$. By Corollary 3.7, $\{i, j\}$ is an m-tie internal chord in $dom(U_n^+)$. Consider the set constructed by allowing i to beat all m of the vertices $i + 2$, $i + 4, \ldots, i + 2m$. Thus, using Lemma 3.6 and Lemma 3.9, the 1-tie internal chord $\{i, i+3\}$ is formed when i beats vertex $i + 2$ and the 2-tie internal chord $\{i, i+5\}$ is created when i also beats vertex $i + 4$. Continuing in this manner, the m-tie internal chord $\{i, i+2m+1\}$ is created when i also beats vertex $i + 2m$. Therefore, exactly m internal chords can be created that are incident with i in $dom(U_n^+)$. Consequently, $dom(U_n^+)$ has at least one other internal chord created by a set of m arcs added to $A(U_n)$. □

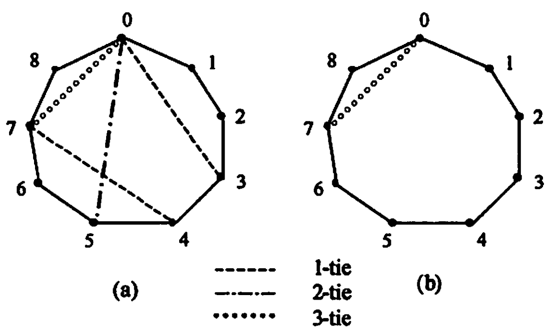

Figure 4: Different sets of 3 arcs added to U_9 produce different internal chords in $dom(U_9^+)$

From the above proposition, different sets of m arcs added to U_n can each produce a different number of internal chords in $dom\,(U_n^+)$ for $n \geq 7$. For example, let $n = 9$ and some $i \in V\,(U_9)$ and $m = 3$. By the method used in Proposition 4.2, if $(i, i+2)$, $(i, i+4)$, and $(i, i+6) \in A\,(U_9^+)$, then there are exactly 3 internal chords, each of a different type, created in $dom\,(U_9^+)$: the 3-tie internal chord $\{i, i+7\}$, the 2-tie internal chord $\{i, i+5\}$, and the 1-tie internal chord $\{i, i+3\}$. For $i = 0$, see Figure 3. Note, that it is even possible to generate more than 3 internal chords by adding only 3 arcs to U_9. Consider, if $(i, i+2)$, $(i, i+4)$, and $(i+7, i+6) \in A\,(U_9^+)$, then there are four internal chords created in $dom\,(U_9^+)$: the 3-tie internal chord $\{i, i+7\}$, the 2-tie internal chord $\{i, i+5\}$, and the two 1-tie internal chords $\{i, i+3\}$ and $\{i+4, i+7\}$. This phenomenon is shown in Figure 4(a) where $i = 0$. If, however, the three arcs $(i+7, i+2)$, $(i, i+4)$, and $(i, i+6) \in A\,(U_9^+)$, only the single 3-tie internal chord $\{i, i+7\}$ is produced in $dom\,(U_9^+)$. For $i = 0$, this is illustrated in Figure 4(b).

Proposition 4.1 states that for odd $n \geq 5$, there exist exactly 2^m possible ways to produce an m-tie internal chord using exactly m arcs in $dom\,(U_n^+)$ for any $i \in V\,(U_n^+)$. By Proposition 4.2, for odd $n \geq 7$, $1 \leq m \leq \frac{n-3}{2}$, there exists at least one collection of m arcs that when added to U_n will create the m-tie internal chord $\{i, i+2m+1\}$ and at least one additional chord in $dom\,(U_n^+)$. The following theorem shows that there is at least one set of m arcs that when added to U_n will uniquely generate an m-tie internal chord in $dom\,(U_n^+)$, creating no additional chords (i.e., $dom\,(U_n^+) = dom\,(U_n) \cup \{i, j\}$ for any given $i, j \in V\,(U_n)$).

Theorem 4.3 *Let $i, j \in V\,(U_n)$ for $n \geq 5$, $1 \leq m \leq \frac{n-3}{2}$, and $j = i + 2m + 1$. Then the following set of m arcs when added to U_n produces only the m-tie internal chord $\{i, i+2m+1\}$ in $dom\,(U_n^+)$:*

1. *For $n \geq 5$ and $m = 1$, $\{(i+3, i+2)\}$.*
2. *For $n \geq 7$ and $2 \leq m \leq \frac{n-3}{2}$, $\{(i, i+d) \mid d = 4, 6, ..., 2m\} \cup \{(i+2m+1, i+2)\}$.*

Proof: Let $i, j \in V\,(U_n)$ for $n \geq 5$, $1 \leq m \leq \frac{n-3}{2}$, and $j = i + 2m + 1$. In Case 1, $n \geq 5$ and $m = 1$. As a consequence of Proposition 3.4, only the additional chord $\{i, i+3\}$ is created in $dom\,(U_n^+)$ when $U_n^+ = U_n \cup (i+3, i+2)$.

Now consider Case 2. For odd $n \geq 7$ and $2 \leq m \leq \frac{n-3}{2}$, the set of m arcs added to U_n is $\{(i, i+d) \mid d = 4, 6, ..., 2m\} \cup \{(i+2m+1, i+2)\}$. By Lemma 3.9, $\{i, i+2m+1\}$ is an m-tie internal chord in $dom\,(U_n^+)$. Now it remains to be shown that no other chord is produced by these m arcs in $dom\,(U_n^+)$ other than $\{i, i+2m+1\}$. Suppose that there is another r-tie internal chord created in $dom\,(U_n^+)$ by the given set of m arcs. Note $1 \leq r < m$ since there are only m arcs added. By Lemma 3.6 and Lemma 3.9, letting $k = 2r + 1$, the vertices

i and $i+k$ must together beat the vertices $(i+2)$, $(i+4)$,..., $(i+k-1)$ for every $i \in V(U_n)$. Note, $(i+2)-(i+2r+1) = 2-2r-1 = -2r+1 \equiv n-2r+1 (\bmod n)$ which is even. Therefore, $(i+k, i+2) \notin A(U_n)$ and is unique in $A(U_n^+)$. Since the only arc whose head is $i+2$ which is unique to $A(U_n^+)$ is $(i+2m+1, i+2)$, then it follows that $r=m$. But r must be less than m, which gives the contradiction. Therefore, only the one internal chord $\{i, i+2m+1\}$ is generated in $dom(U_n^+)$. □

Corollary 4.4 *For odd $n \geq 5$, $m = \frac{n-3}{2}$ is the maximum number of m-tie arcs that must be added to U_n to create only a specific internal chord in $dom(U_n^+)$.*

Consider for $n \geq 7$, $i \in V(U_n)$, if the arcs $\{(i, i+d) \mid d = 4, 6, ..., n-3\}$ are added to U_n, then the only vertex not beaten by i is $i+2$. By Theorem 4.3, for any $1 \leq m \leq \frac{n-3}{2}$, if $i+2m+1$ beats $i+2$ and the arc $(i+2m+1, i+2)$ is added to the above collection of arcs creating U_n^+, then the edge $\{i, i+2m+1\}$ is an m-tie internal chord in $dom(U_n^+)$. Note $M = \frac{n-3}{2}$ is the number of arcs added to U_n to create U_n^+ and this set of M arcs generates $dom(U_n^+) = C_n \cup \{i, i+2m+1\}$ for any $1 \leq m \leq M$ and $i \in V(U_n)$. Recall from Proposition 2.2, it is only necessary to consider vertices i, $i+k \in V(U_n)$, where k is odd, in order to produce all possible pairs in U_n and consequently in $dom(U_n^+)$, since $V(U_n) = V(U_n^+)$. Because all $i+k \in V(U_n)$, $k = 2m+1$, are defined by $1 \leq m \leq \frac{n-3}{2}$, then any chord can be created without producing any additional chords in $dom(U_n^+)$ with M arcs. This result is stated in the following corollary.

Corollary 4.5 *For odd $n \geq 5$, there exists exactly $\frac{n-3}{2}$ arcs that can be added to U_n to produce only a specific internal chord in $dom(U_n^+)$.*

Proof: By Theorem 4.3 and Proposition 2.2. □

Theorem 4.6 *For odd $n \geq 5$, $i \in V(U_n)$, any set of internal chords can be created in $dom(U_n^+)$ by adding to U_n the arcs $\{(i, i+d) \mid d = 4, 6, ..., n-3\}$ and $\bigcup_L \{(i+2m+1, i+2)\}$, where $L \subseteq \left\{m \in \mathbb{Z} \mid 1 \leq m \leq \frac{n-3}{2}\right\}$, such that none of these sets of arcs will create any extraneous chords in $dom(U_n^+)$.*

Proof: For odd $n \geq 5$, $i \in V(U_n)$, by Corollary 4.5, exactly $\frac{n-3}{2}$ arcs can be added to U_n to create any specific internal chord in $dom(U_n^+)$. In particular, add the arcs $\{(i, i+d) \mid d = 4, 6, ..., n-3\}$ and the arc $(i+2m_1+1, i+2)$ to U_n to create U_n^+, for any $1 \leq m_1 \leq \frac{n-3}{2}$ and $i \in V(U_n)$. Then, for $i \in V(U_n)$ and $m_1 \in \left\{m \in \mathbb{Z} \mid 1 \leq m \leq \frac{n-3}{2}\right\}$, $dom(U_n^+) = C_n \cup \{i, i+2m_1+1\}$. By Theorem 4.3, no additional chords are produced in $dom(U_n^+)$. This remains true for all $L \subseteq \left\{m \in \mathbb{Z} \mid 1 \leq m \leq \frac{n-3}{2}\right\}$ since each of the added arcs in U_n^+ are all of the form $(i+2m+1, i+2)$, i.e., $i+2m+1$ always beats the vertex $i+2$. □

5 Cycles in $dom\,(U_n^+)$

Recall that there is no tournament T_n that generates an even cycle as a subgraph in $dom\,(T_n)$ [2]. The following lemma shows that there exist extended tournaments, namely U_n^+, for which even cycles of length λ, $4 \leq \lambda \leq (n-1)$, exist as subgraphs of $dom\,(U_n^+)$ for every $n \geq 5$.

Lemma 5.1 *For odd $n \geq 5$, by properly adding arcs to U_n to create U_n^+, there exists an even cycle of length λ for all $\lambda \in \{4, 6, ..., n-1\}$ in $dom\,(U_n^+)$.*

Proof: For each odd $n \geq 5$, consider U_n and $i \in V\,(U_n)$. By Theorem 4.3, there exists a distinct set of m arcs that can be added to U_n to produce only the m-tie internal chord $\{i, i+2m+1\}$ for $1 \leq m \leq \frac{n-3}{2}$ in $dom\,(U_n^+)$. For each odd k, there exists k edges plus the chord $\{i, i+k\}$, creating an even cycle of length $k+1$. Therefore, for each specific odd $n \geq 5$, an even cycle in $dom\,(U_n^+)$ of length λ is created where $\lambda \in \{4, 6, ..., n-1\}$. □

From Lemma 5.1, it follows that at least $\frac{n-3}{2}$ different even cycles can be generated in $dom\,(U_n^+)$, for each odd $n \geq 5$. For example, see Figure 3 where $n = 9$. In this figure, the 1-tie chord creates C_4, the 2-tie chord creates C_6, and the 3-tie chord creates C_8 as generated subgraphs in $dom\,(U_9^+)$. Here, there exists $\frac{9-3}{2} = 3$ different lengths of even cycles and $4 \leq \lambda \leq 8$ for even λ.

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