ON K_*-ULTRAHOMOGENEOUS GRAPHS

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ABSTRACT. Let $\mathcal{C}$ be any class of finite graphs. A graph G is $\mathcal{C}$-ultrahomogeneous if every isomorphism between induced subgraphs belonging to $\mathcal{C}$ extends to an automorphism of G. We study finite graphs that are K_*-ultrahomogeneous, where K_* is the class of complete graphs. We also explicitly classify the finite graphs that are $\sqcup K_*$-ultrahomogeneous, where $\sqcup K_*$ is the class of disjoint unions of complete graphs.

1. INTRODUCTION

A mathematical structure is called ultrahomogeneous if every isomorphism between finite substructures can be extended to an automorphism. In the context of graph theory, a graph G is ultrahomogeneous if every isomorphism between finite induced subgraphs extends to an automorphism of G. Gardiner [5] gave an explicit classification of the finite ultrahomogeneous graphs using the previous work of Sheehan [13].

One can vary this basic definition in many interesting ways. For example, G is homogeneous if for every pair of finite isomorphic subgraphs H_1 and H_2, there exists an isomorphism $H_1 \to H_2$ that extends to an automorphism of G. The difference between homogeneity and ultrahomogeneity is whether or not the isomorphism $H_1 \to H_2$ is specified. Ronse [12] showed that a finite graph is homogeneous if and only if it is ultrahomogeneous. Two different variations are considered in [4] and [10].

Yet other variations arise by considering only certain types of subgraphs. A graph G is connected-ultrahomogeneous if every isomorphism between connected induced subgraphs extends to an automorphism of G. Gardiner [6] explicitly classified the connected-ultrahomogeneous graphs.

Definition 1.1. Let $\mathcal{C}$ be any class of finite graphs closed under isomorphism. A graph G is **$\mathcal{C}$-ultrahomogeneous** if every isomorphism between induced subgraphs belonging to $\mathcal{C}$ extends to an automorphism of G.

This definition recovers the original notion of ultrahomogeneity by taking $\mathcal{C}$ to be the class of all finite graphs. Similarly, we recover connected-ultrahomogeneity by taking $\mathcal{C}$ to be the class of all finite connected graphs.

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If H is a given fixed finite graph, we say that another graph G is H-ultrahomogeneous if it is $\{H\}$-ultrahomogeneous. This is equivalent to requiring that the automorphism group $\mathrm{Aut}(G)$ of the graph G acts transitively on the set of induced subgraphs that are isomorphic to H.

In this paper, we choose a few specific examples of reasonable classes $\mathcal{C}$ and study the finite $\mathcal{C}$-ultrahomogeneous graphs. Because complete subgraphs are a common subject of study for graph theorists (see, for example, [9] or [11]), we are interested in the class K_* of all finite complete graphs. Thus, we are considering graphs G such that any isomorphism between finite complete subgraphs of G extends to an automorphism. Note that the K_*-ultrahomogeneous graphs include all ultrahomogeneous and connected-ultrahomogeneous graphs, since these are stronger conditions than K_*-ultrahomogeneity.

Unfortunately, we are able to obtain only partial results about the K_*-ultrahomogeneous graphs. In order to prove something more substantial, we also consider the $\sqcup K_*$-ultrahomogeneous graphs, where $\sqcup K_*$ is the class of graphs that are disjoint unions of complete graphs. The class $\sqcup K_*$ consists of all graphs of the form

$$K_{r_1} \sqcup K_{r_2} \sqcup \cdots \sqcup K_{r_n}.$$

In Section 4, we classify the finite $\sqcup K_*$-ultrahomogeneous graphs explicitly. We find that the class of finite $\sqcup K_*$-ultrahomogeneous graphs is only slightly larger than the class of ultrahomogeneous graphs. This enlightens us about the nature of ultrahomogeneity for graphs. Considering all subgraphs turns out to be a highly redundant condition. The seemingly much weaker condition of $\sqcup K_*$-ultrahomogeneity turns out be nearly equivalent.

The classification of finite ultrahomogeneous graphs contains two sporadic graphs and two infinite families. In our classification, one of these sporadic graphs, $K_3 \times K_3$, becomes an infinite family. It would be interesting to find a homogeneity property that expands the other sporadic graph, C_5, into an infinite family.

In this paper, we consider ultrahomogeneity only for finite undirected graphs. Many of the questions that we answer can be asked for infinite graphs, but we focus exclusively on finite graphs. Therefore, from now on, whenever we use the word "graph", we always mean a finite graph.

Ultrahomogeneity makes sense in many other combinatorial contexts, such as finite geometries [2] and infinite graph theory [8]. Ultrahomogeneity tends to be a highly redundant condition in these contexts also. For example, [2] shows that ultrahomogeneity with respect to subsets of at most 6 points implies ultrahomogeneity in general for a certain kind of finite geometry. Also, [1, Thm. 3.2] implies that ultrahomogeneity for finite graphs is implied by ultrahomogeneity with respect to subgraphs on at most 5 vertices. We strongly suspect that it is possible to establish more redundancy results of this type.

1.1. **Notation.** For any graph G and any positive integer t, let tG be the graph that consists of t disjoint copies of G.

Recall that if G and H are graphs, then the Cartesian product $G \times H$ is graph whose vertex set is the product of the vertex sets of G and H such that two vertices (v_1, w_1) and (v_2, w_2) are adjacent if and only if $v_1 = v_2$ and w_1 and w_2 are adjacent in H, or if $w_1 = w_2$ and v_1 and v_2 are adjacent in G. Let G^t denote the t-fold Cartesian product of G with itself.

If t and n are positive integers, let $K_{t;n}$ be the complete regular multipartite graph containing t partite sets, each of which has n elements. Thus, $K_{t;n}$ is the complete regular multipartite graph with order nt and degree $n(t-1)$. For example, $K_{1;n}$ is the graph nK_1, and $K_{t;1}$ is the graph K_t.

2. $\mathcal{C}$-Ultrahomogeneous graphs

We present in this section some general results about $\mathcal{C}$-ultrahomogeneity that will be useful later.

Definition 2.1. If $\mathcal{C}$ is any class of graphs, then $\overline{\mathcal{C}}$ is the class of graphs whose complements belong to $\mathcal{C}$.

For example, the class $\overline{K_*}$ contains all graphs that consist entirely of disjoint vertices. Also, the class $\overline{\sqcup K_*}$ consists of all multipartite complete graphs.

The following theorem will allow us to check whether a graph is $\mathcal{C}$-ultrahomogeneous simply by examining its complement, which in some cases is an easier task.

Theorem 2.2. *A graph G is $\mathcal{C}$-ultrahomogeneous if and only if its complement $\overline{G}$ is $\overline{\mathcal{C}}$-ultrahomogeneous.*

Proof. We prove the backward direction. The proof in the other direction is identical because the complement of the complement of a graph is the original graph.

Suppose that $\overline{G}$ is $\overline{\mathcal{C}}$-ultrahomogeneous. We need to show that G is $\mathcal{C}$-ultrahomogeneous. Let $\phi : H_1 \to H_2$ be an isomorphism of subgraphs in G, where H_1 and H_2 belong to $\mathcal{C}$. Then ϕ canonically determines an isomorphism $\overline{\phi} : \overline{H_1} \to \overline{H_2}$ in $\overline{G}$, where $\overline{H_1}$ and $\overline{H_2}$ belong to $\overline{\mathcal{C}}$. Since $\overline{G}$ is $\overline{\mathcal{C}}$-ultrahomogeneous, there exists an automorphism $\overline{\psi}$ of $\overline{G}$ taking $\overline{H_1}$ to $\overline{H_2}$. Now $\overline{\psi}$ canonically determines an automorphism ψ of G taking H_1 to H_2, and ψ extends ϕ. □

3. K_*-ultrahomogeneous graphs

In this section, we give some examples of K_*-ultrahomogeneous graphs. Any ultrahomogeneous graph or connected-ultrahomogeneous graph is K_*-ultrahomogeneous. These are not the only examples. For example, any vertex-transitive, strongly edge-transitive, triangle-free graph is K_*-ultrahomogeneous.

Proposition 3.1. *Let G be a K_*-ultrahomogeneous graph. Then the t-fold Cartesian product G^t is K_*-ultrahomogeneous.*

Proof. Suppose that H is an induced subgraph of G^t such that H is isomorphic to K_n for some n. Then H is of the form $\{v_1\} \times \{v_2\} \times \cdots \times J \times \cdots \times \{v_{t-1}\}$, where J is isomorphic to K_n. By permuting factors, there exists an automorphism of G taking H to H', where H' is of the form $J \times \{v_1\} \times \{v_2\} \times \cdots \times \{v_{t-1}\}$. Since G is K_1-ultrahomogeneous (i.e., vertex-transitive), there exists an automorphism of G taking H' to H'', where H'' is of the form $J \times \{v\} \times \{v\} \times \cdots \times \{v\}$, for some fixed vertex v. Finally, since G is K_*-ultrahomogeneous, there exists an automorphism taking H'' to any other subgraph of the form $L \times \{v\} \times \{v\} \times \cdots \times \{v\}$, where L is isomorphic to K_n. □

Note that for an ultrahomogeneous graph G, the graph G^t is not necessarily ultrahomogeneous. In fact, $K_n \times K_n$ is K_*-ultrahomogeneous but not ultrahomogeneous for $n > 3$. This shows that there are K_*-ultrahomogeneous graphs which are not already ultrahomogeneous.

It may seem natural to guess that G is ultrahomogeneous if it is both K_*-ultrahomogeneous and $\overline{K}_*$-ultrahomogeneous, but this is not true. For example, $K_n \times K_n$ for $n > 3$ is K_*-ultrahomogeneous and $\overline{K}_*$-ultrahomogeneous but not ultrahomogeneous, as will be shown below in Lemma 3.3.

Lemma 3.2. *The graph $\overline{K_m \times K_n}$ is K_*-ultrahomogeneous.*

Proof. In the graph $\overline{K_m \times K_n}$, two vertices (v_1, w_1) and (v_2, w_2) are adjacent if and only if $v_1 \neq v_2$ and $w_1 \neq w_2$. Let H and H' be induced subgraphs isomorphic to K_t, and let $\{(v_1, w_1), (v_2, w_2,), \ldots, (v_t, w_t)\}$ and $\{(v'_1, w'_1), (v'_2, w'_2,), \ldots, (v'_t, w'_t)\}$ be the sets of vertices of H and H' respectively. Let ϕ be the isomorphism that takes (v_i, w_i) to (v'_i, w'_i). Note that the vertices $v_1, v_2, \ldots, v_t$ are distinct. Similarly, the vertices $w_1, w_2, \ldots, w_t$ are distinct; the vertices $v'_1, v'_2, \ldots, v'_t$ are distinct; and the vertices $w'_1, w'_2, \ldots, w'_t$ are distinct.

Choose any permutation of the vertices of K_m that takes each v_i to v'_i, and choose any permutation of the vertices of K_n that takes each w_i to w'_i. These permutations induce an automorphism of $\overline{K_m \times K_n}$, and this automorphism extends ϕ. □

Lemma 3.3. *For $n > 3$, the graph $K_n \times K_n$ is K_*-ultrahomogeneous and $\overline{K}_*$-ultrahomogeneous but not ultrahomogeneous.*

Proof. The graph $K_n \times K_n$ is K_*-ultrahomogeneous because of Proposition 3.1. Also, by Lemma 3.2, $\overline{K_n \times K_n}$ is K_*-ultrahomogeneous. Therefore, by Theorem 2.2, $K_n \times K_n$ is $\overline{K}_*$-ultrahomogeneous. However, $K_n \times K_n$ is not ultrahomogeneous. To see this, it is enough to show that $K_n \times K_n$ is not $2K_2$-ultrahomogeneous. Let H be the copy of $2K_2$ spanned by the vertices (v_1, w_1), (v_1, w_2), (v_2, w_3), (v_2, w_4), and let H' be the copy of $2K_2$ spanned by the vertices (v_1, w_1), (v_1, w_2), (v_2, w_3), and (v_3, w_3). It is straightforward to check that no automorphism of $K_n \times K_n$ takes H to H'. □

It is reasonable to ask whether all the graphs in the class K_* are relevant. Is there any subclass $\mathcal{C}$ of K_* such that the K_*-ultrahomogeneous graphs are precisely equal to the $\mathcal{C}$-ultrahomogeneous graphs? We do not know the answer, but the following example is a preliminary step.

Example 3.4. We construct a graph G that is K_1-ultrahomogeneous and K_2-ultrahomogeneous but *not* K_3-ultrahomogeneous. The idea to use Cayley graphs to find such an example was brought to our attention through the work of Doyle [3] who employed similar techniques to show that there are vertex-transitive, edge-transitive graphs that are not strongly edge-transitive.

Let A be the group $\mathbb{Z}_5 \times \mathbb{Z}_5$. Let $S = \{(x, 0), (0, x), (x, x) \mid x \in \mathbb{Z}_5, x \neq 0\}$ be the generating set. It can be shown that the resulting Cayley graph $G = \mathrm{Cay}(A; S)$ is K_1-ultrahomogeneous and K_2-ultrahomogeneous. To see that G is not K_3-ultrahomogeneous, just consider the induced subgraphs H_1 whose vertices are $(0,0)$, $(1,0)$, and $(1,1)$, and H_2 whose vertices are $(0,0)$, $(1,0)$, and $(2,0)$. Then any isomorphism between these two subgraphs does not extend to an automorphism.

The following proposition tells us that we only need to consider connected K_*-ultrahomogeneous graphs if we wish to understand all K_*-ultrahomogeneous graphs.

Proposition 3.5. *If G is K_*-ultrahomogeneous, then G is isomorphic to tH, where H is a connected, K_*-ultrahomogeneous graph.*

Proof. Suppose G is K_*-ultrahomogeneous. The case when G is connected is trivial, so suppose G is not connected. Then G has at least two components. Suppose for sake of contradiction that G has two components H_1 and H_2 that are not isomorphic. Let v_1 be a vertex of H_1 and v_2 a vertex of H_2. Since G is K_*-ultrahomogeneous, there exists an automorphism taking v_1 to v_2. This is a contradiction since H_1 is not isomorphic to H_2. □

4. $\sqcup K_*$-ULTRAHOMOGENEOUS GRAPHS

Let $\mathcal{M}$ be the class of all complete (not necessarily regular) multipartite graphs $K_{r_1, r_2, \ldots, r_n}$, with $n \geq 1$ and each $r_i \geq 1$. Note that $\mathcal{M}$ is the complement of the class $\sqcup K_*$. In order to understand the $\sqcup K_*$-ultrahomogeneous graphs, we will study the $\mathcal{M}$-ultrahomogeneous graphs and then apply Theorem 2.2.

Note in particular that K_n belongs to $\mathcal{M}$ (by taking $r_i = 1$) and that the disjoint union tK_1 of t vertices also belong to $\mathcal{M}$ (by taking $n = 1$ and

$r_1 = t$). Thus, every $\mathcal{M}$-ultrahomogeneous graph is K_*-ultrahomogeneous and also $\overline{K}_*$-ultrahomogeneous.

Definition 4.1. For any induced subgraph H of a graph G, let $N_H(G)$ be the induced subgraph of G consisting of the vertices β of G such that β is adjacent to every vertex of H but not in H. Also, let $\overline{N}_H(G)$ be the induced subgraph of G consisting of the vertices β of G such that β does not belong to either H or $N_H(G)$.

Note that when H consists just of a single vertex α, then $N_H(G)$ (which we also write as $N_\alpha(G)$) is the induced subgraph on the neighbors of α. Also $\overline{N}_\alpha(G)$ is the induced subgraph on the vertices that are not adjacent to α.

We will classify the $\mathcal{M}$-ultrahomogeneous graphs by induction on the number of vertices. The key inductive step is given by the following proposition.

Proposition 4.2. *If G is $\mathcal{M}$-ultrahomogeneous and H is an induced subgraph of G that belongs to $\mathcal{M}$, then $N_H(G)$ is also $\mathcal{M}$-ultrahomogeneous.*

Proof. Let $\phi : K \to K'$ be any isomorphism between induced subgraphs of $N_H(G)$ such that K and K' belong to $\mathcal{M}$. We need to show that ϕ extends to an automorphism of $N_H(G)$.

Let L be the induced subgraph of G consisting of the vertices of H together with the vertices of K. Define L' similarly, using the vertices of H and of K'. Then there is an isomorphism $\tilde{\phi} : L \to L'$; it is the identity on the vertices of H, and it is ϕ on the vertices of K.

Note that L and L' also belong to $\mathcal{M}$. Therefore, $\tilde{\phi}$ extends to an automorphism ψ of G since G is $\mathcal{M}$-ultrahomogeneous. Since ψ fixes H, it restricts to an automorphism of $N_H(G)$. □

We now give the explicit classification of $\mathcal{M}$-ultrahomogeneous graphs.

Theorem 4.3. *A graph is $\mathcal{M}$-ultrahomogeneous if and only if it is isomorphic to:*

(1) tK_n *for* $t \geq 1$ *and* $n \geq 1$;
(2) $K_{t;n}$ *for* $t \geq 2$ *and* $n \geq 2$;
(3) $K_n \times K_n$ *for* $n \geq 3$; *or*
(4) C_5.

We give the main steps in the proof here, but the technical details are recorded in lemmas later in this section.

Proof. By inspection, each listed graph is $\mathcal{M}$-ultrahomogeneous. For the other implication, suppose that G is $\mathcal{M}$-ultrahomogeneous. If G is disconnected, then G belongs to the above list by Lemma 4.5 below.

Now we may assume that G is connected. The proof is by induction on the number of vertices in G. If G contains one vertex, then G belongs to the list. Assume now that $n \geq 2$ and that any $\mathcal{M}$-ultrahomogeneous graph with strictly fewer than n vertices belongs to the list. Let G contain n vertices. We need to show that G belongs to the list. Choose any vertex α of G. By Proposition 4.2, $N_\alpha(G)$ is again $\mathcal{M}$-ultrahomogeneous, and it has strictly fewer vertices than G. By the induction assumption, $N_\alpha(G)$ belongs to the list in the statement of the theorem. Lemmas 4.8, 4.9, and 4.10 below indicate that $N_\alpha(G)$ must be isomorphic to

(1) $2K_n$ with $n \geq 2$;
(2) tK_1 with $t \geq 1$; or
(3) $K_{t;n}$ with $t \geq 2$ and $n \geq 1$.

The situation when $N_\alpha(G) = K_n$ is included in Case (3). Lemmas 4.11, 4.12, and 4.13 provide an explicit description of G in cases (1), (2), and (3) respectively. □

The main point of Theorem 4.3 is to provide a classification of the $\sqcup K_*$-ultrahomogeneous graphs. This is stated in the following corollary.

Corollary 4.4. *A graph is $\sqcup K_*$-ultrahomogeneous if and only if it is isomorphic to:*

(1) tK_n *for* $t \geq 1$ *and* $n \geq 1$;
(2) $K_{t;n}$ *for* $t \geq 2$ *and* $n \geq 2$;
(3) $\overline{K_n \times K_n}$ *for* $n \geq 3$; *or*
(4) C_5.

Proof. By Theorem 2.2, we just need to find the complements of the graphs listed in Theorem 4.3. □

The only difference between our classification and the classification of ultrahomogeneous graphs [5] is that the graph $\overline{K_n \times K_n}$ is not ultrahomogeneous when $n > 3$.

The rest of this section is dedicated to proving the technical lemmas necessary for the proof of Theorem 4.3.

Lemma 4.5. *If G is disconnected and $\mathcal{M}$-ultrahomogeneous, then G is isomorphic to tK_n for some $t \geq 2$ and $n \geq 1$.*

Proof. Since G is $2K_1$-ultrahomogeneous, every pair of non-adjacent vertices belong to distinct components of G. This implies that G is a disjoint union of complete graphs. But G is also vertex-transitive (i.e., K_1-ultrahomogeneous), so each component has the same order. □

Lemma 4.6. *If G is connected and $\mathcal{M}$-ultrahomogeneous, then G has diameter at most 2.*

Proof. The graph G is vertex-transitive (i.e., K_1-ultrahomogeneous), so the graph $N_\alpha(G)$ is independent (up to isomorphism) of a choice of vertex α in G. If $\overline{N}_\alpha G$ is empty, then G has diameter 1. Otherwise, let β be any vertex of $\overline{N}_\alpha(G)$ (i.e., β is not adjacent to α) such that the distance from α to β is

2. Since G is connected and since G is $2K_1$-ultrahomogeneous, the distance between any two non-adjacent vertices must equal 2. □

In other words, in a connected $\mathcal{M}$-ultrahomogeneous graph, any two vertices are either adjacent or have at least one common neighbor.

Remark 4.7. Many of the proofs that follow use similar techniques, which we introduce here. Some of these ideas were inspired by the methods of [12].

In Lemma 4.6, we proved that two non-adjacent vertices in a connected $\mathcal{M}$-ultrahomogeneous graph have at least one common neighbor. In fact, every pair of non-adjacent vertices has the same number of common neighbors. Let m be this number; note that it is at least one.

Let r be the number of vertices in $\overline{N}_\alpha(G)$ (i.e., the number of vertices not adjacent to α). The total number of paths of length 2 from α to $\overline{N}_\alpha(G)$ is equal to rm.

Now let d be the number of vertices in $N_\alpha(G)$ (i.e., the degree of α) and let k be the degree of each vertex in $N_\alpha(G)$ (i.e., the number of copies of K_3 containing any given edge). Then each vertex in $N_\alpha(G)$ has exactly $d-k-1$ neighbors in $\overline{N}_\alpha(G)$, so the total number of paths of length 2 from α to $\overline{N}_\alpha(G)$ is equal to $d(d-k-1)$. Thus,

$$rm = d(d-k-1). \tag{4.1}$$

We will use this equation frequently in the following proofs.

Choose a vertex β in $\overline{N}_\alpha(G)$, and let s be the number of vertices in $\overline{N}_\alpha(G)$ that are adjacent to every vertex of H. The graph H does not depend (up to isomorphism) on the choice of β in $\overline{N}_\alpha(G)$ because G is $2K_1$-ultrahomogeneous; therefore s is independent of the choice of β. Suppose that H belongs to $\mathcal{M}$. If H' is isomorphic to H and also belongs to $N_\alpha(G)$, then there is an automorphism fixing α and taking H to H'. Therefore, in this case, if p is the number of subgraphs of $N_\alpha(G)$ that are isomorphic to H, then

$$r = sp. \tag{4.2}$$

The condition on H will always be satisfied in the situations where we use Equation (4.2) below.

Let v be the number of vertices in G, and let q_n be the number of copies of K_n in the subgraph $N_\alpha(G)$. Then α is contained in exactly q_n copies of K_{n+1}. Since G is K_1-ultrahomogeneous, every vertex is contained in exactly q_n copies of K_{n+1}. Therefore, $q_n v$ equals $n+1$ times the number of copies of K_{n+1} in G, so

$$n+1 \text{ divides } q_n v. \tag{4.3}$$

Lemma 4.8. *If G is connected and $\mathcal{M}$-ultrahomogeneous and α is any vertex of G, then $N_\alpha(G)$ is not isomorphic to C_5.*

Proof. For sake of contradiction, suppose that $N_\alpha(G)$ is isomorphic to C_5. Now G is regular of degree 5, and G has $6+r$ vertices. Equation 4.1 tells us that $rm = 10$, so r is a divisor of 10. Letting $n = 2$, Equation 4.3 tells us that 3 divides $5(6+r)$, so r is a multiple of 3. But there are no divisors of 10 that are also multiples of 3. □

Lemma 4.9. *If G is connected and $\mathcal{M}$-ultrahomogeneous and α is any vertex of G, then $N_\alpha(G)$ is not isomorphic to $K_n \times K_n$ with $n \geq 3$.*

Proof. For sake of contradiction, suppose that $N_\alpha(G)$ is isomorphic to $K_n \times K_n$ with $n \geq 3$. To fix notation, let $\{(\delta_i, \delta_j) | i, j = 1, 2, \ldots, n\}$ be the vertices of $N_\alpha(G)$, where (δ_i, δ_j) and (δ_k, δ_l) are adjacent if and only if $i = k$ or $j = l$.

Let β be any vertex of $\overline{N}_\alpha(G)$, and let γ be a vertex in $N_\alpha(G)$ that is also adjacent to β. We may assume that γ is the vertex (δ_1, δ_1). Consider $N_\gamma(G)$, which is also isomorphic to $K_n \times K_n$; we know that it contains α, β, (δ_i, δ_1), and (δ_1, δ_i) for $i \neq 1$. It follows that β is adjacent to at least two more vertices of $N_\alpha(G)$, one of the form (δ_1, δ_j) and one of the form (δ_k, δ_1). Moreover, β is not adjacent to any other vertices of the form (δ_1, δ_l) or (δ_l, δ_1). Similarly one can show that β is adjacent to exactly 0 or 2 vertices of each "row" or "column" of $N_\alpha(G)$.

Now $N_{\alpha,\beta}(G)$ must be isomorphic to a non-empty disjoint union of cycles of even length. But $N_{\alpha,\beta}(G)$ is $\mathcal{M}$-ultrahomogeneous by Proposition 4.2, so by Lemmas 4.5 and 4.6, $N_{\alpha,\beta}(G)$ is actually isomorphic to C_4 and $m = 4$.

There are $\binom{n}{2}^2$ copies of C_4 in $K_n \times K_n$, so Equation 4.2 tell us that

$$r = s\binom{n}{2}^2.$$

On the other hand, Equation 4.1 tells us that

$$4r = n^2(n-1)^2.$$

It follows that $s = 1$, which means that for every copy C of C_4 in $N_\alpha(G)$, there exists exactly one vertex β in $\overline{N}_\alpha(G)$ such that β is adjacent to each vertex of C.

Now we know that G contains $1+n^2+\binom{n}{2}^2$ vertices, and $N_\alpha(G)$ contains $2n$ copies of K_n. By Equation 4.3, $n+1$ divides $2n(1+n^2+\binom{n}{2}^2)$. Performing arithmetic modulo $n+1$,

$$0 \equiv 2n\left(1+n^2+\binom{n}{2}^2\right) \equiv -6.$$

Thus 0 is congruent to -6 modulo $n+1$, so $n+1$ must divide 6. Since $n \geq 3$, the only possibility is $n = 5$.

We are left only with the case $n = 5$. Since $n \geq 3$, the graph $N_\alpha(G)$ contains three vertices that are pairwise non-adjacent. Since G is $3K_1$-ultrahomogeneous, every set of three pairwise non-adjacent vertices has a common neighbor. If β_1 and β_2 are vertices of $\overline{N}_\alpha(G)$ with no common neighbor in $N_\alpha(G)$, then β_1 and β_2 must be adjacent; otherwise, α, β_1, and β_2 are three pairwise non-adjacent vertices with no common neighbor.

Let β be the vertex of $\overline{N}_\alpha(G)$ that is adjacent to every vertex of a given copy of C_4 in $N_\alpha(G)$. There are 51 copies of C_4 in $N_\alpha(G)$ that do not intersect this given copy of C_4, and there are 51 vertices of $\overline{N}_\alpha(G)$ that correspond to these 51 copies of C_4. None of these 51 vertices has a common neighbor with β in $N_\alpha(G)$, so these 51 vertices must all be adjacent to β because of the remark in the previous paragraph. This is impossible since G is 25-regular. □

Lemma 4.10. *Let G be connected and $\mathcal{M}$-ultrahomogeneous, and let α be any vertex of G. If $N_\alpha(G)$ is isomorphic to tK_n, then $n = 1$ or $t \leq 2$.*

Proof. Let $A_1, A_2, \ldots, A_t$ be the components of $N_\alpha(G)$, so each A_j is a complete graph. Let γ be any vertex of $N_\alpha(G)$; then $N_{\alpha,\gamma}(G)$ is isomorphic to K_{n-1}. Because G is K_2-ultrahomogeneous, every pair of adjacent vertices has exactly $n-1$ common neighbors.

If γ_1 and γ_2 both belong to A_j, then they are adjacent. Their common neighbors consist entirely of α together with the other vertices of A_j. In other words, if β is not a neighbor of α, then $N_\beta(G) \cap A_j$ contains at most one vertex. This implies that $1 \leq m \leq t$. Equation 4.1 gives us the formula

$$rm = n^2 t(t-1). \tag{4.4}$$

The number of copies of mK_1 in $N_\alpha(G)$ is $\binom{t}{m} n^m$, so Equation 4.2 gives the formula

$$r = \binom{t}{m} n^m s.$$

Combining the previous two equations, we get the formula

$$s n^{m-2} \binom{t-1}{m-1} = t - 1. \tag{4.5}$$

This means that $t-1$ cannot be strictly smaller than $\binom{t-1}{m-1}$, so there are only four possible values for m: 1, 2, $t-1$, and t. We treat each case separately. In each case, we assume that $t \geq 3$ and $n \geq 2$ and reach a contradiction.

First, note that when $t \geq 3$, the graph $N_\alpha(G)$ has three pairwise non-adjacent vertices. Since G is $3K_1$-ultrahomogeneous, every set of three non-adjacent vertices in G must have a common neighbor.

Case 1: $m = 1$. Let β be any vertex in $\overline{N}_\alpha(G)$, and let γ be its unique neighbor in $N_\alpha(G)$. If δ is another vertex in $\overline{N}_\alpha(G)$ that is not adjacent to γ, then β and δ must be adjacent; otherwise α, β, and δ would form a set of three pairwise non-adjacent vertices with no common neighbor. In other words, every vertex of $\overline{N}_\alpha(G)$ must be adjacent to β or γ (or both). From Equation 4.5, we know that $s = n(t-1)$. Thus, γ has $s = n(t-1)$ neighbors in $\overline{N}_\alpha(G)$, so β has at least $r - n(t-1)$ neighbors in $\overline{N}_\alpha(G)$. Now

$nt \geq n^2t(t-1) - n(t-1)$ since G is nt-regular and since $r = n^2t(t-1)$. This inequality can never hold when $n \geq 2$ and $t \geq 3$.

Case 2: $m = 2$. First of all, Equation 4.5 tells us that $s = 1$. Let β be any vertex in $\overline{N}_\alpha(G)$, and let γ_1 and γ_2 be its neighbors in $N_\alpha(G)$. Since γ_1 and γ_2 have exactly $n-1$ neighbors in $N_\alpha(G)$ and since G is tn regular, γ_1 and γ_2 have exactly $n(t-1)$ neighbors in $\overline{N}_\alpha(G)$. Thus, there are exactly $2n(t-1)-1$ vertices in $\overline{N}_\alpha(G)$ that are adjacent to γ_1 or γ_2 (or both). If δ is another vertex in $\overline{N}_\alpha(G)$ that is adjacent to neither γ_1 nor γ_2, then β and δ must be adjacent; otherwise α, β, and δ would form a set of three pairwise non-adjacent vertices with no common neighbor.

Therefore, β has at least $r - 2n(t-1) + 1$ neighbors in $\overline{N}_\alpha(G)$. Taking into account the two neighbors of β in $N_\alpha(G)$, $nt \geq \frac{1}{2}n^2t(t-1) - 2n(t-1) + 3$ since G is nt-regular and since $r = n^2t(t-1)/2$. This inequality can never hold when $n \geq 2$ and $t \geq 3$.

Case 3: $m = t-1$. Now Equation 4.5 becomes $sn^{t-3} = 1$, so $t \leq 3$. Thus $t = 3$ and $m = 2$. We have reduced this case to Case 2.

Case 4: $m = t$. Now Equation 4.5 becomes $sn^{t-2} = t-1$, so $t-1 \geq n^{t-2}$. Since $n \geq 2$ and $t \geq 3$, this can only happen when $n = 2$, $s = 1$, and $t = 3$. It follows that $m = 3$ and $r = 8$. Note that $N_\alpha(G)$ is isomorphic to $3K_2$. Therefore, every edge of G lies in exactly one copy of K_3.

Let $N_\alpha(G)$ consist of the three disjoint edges $\beta_1\beta_2$, $\gamma_1\gamma_2$, and $\delta_1\delta_2$. Since each of these edges lies in only one copy of K_3 and each vertex ϵ of $\overline{N}_\alpha(G)$ is adjacent to exactly three vertices in $N_\alpha(G)$, it follows that ϵ is adjacent to one vertex of the form β_i, one vertex of the form γ_j, and one vertex of the form δ_k. This accounts for all the edges between $N_\alpha(G)$ and $\overline{N}_\alpha(G)$ since there are 8 vertices in $\overline{N}_\alpha(G)$ and there are 8 possibilities for $(\beta_i, \gamma_j, \delta_k)$. Write ϵ_{ijk} for the vertex of $\overline{N}_\alpha(G)$ that is adjacent to β_i, γ_j, and δ_k.

Next we must account for the edges within $\overline{N}_\alpha(G)$. The edge $\beta_1\epsilon_{111}$ must belong to a copy of K_3, so ϵ_{111} is adjacent to a vertex of the form ϵ_{1jk}. Moreover, since the edge $\epsilon_{111}\epsilon_{1jk}$ belongs to only one copy of K_3, we must have that ϵ_{1jk} equals ϵ_{122}. By symmetric arguments, ϵ_{ijk} is adjacent to $\epsilon_{i'j'k'}$ if and only if i, j, and k differ from i', j', and k' in exactly two places.

We have completely described G. However, we can now see that G is not $\mathcal{M}$-ultrahomogeneous. In fact, it is not even vertex-transitive. For example, $\overline{N}_\alpha(G)$ is isomorphic to $2K_4$, but $\overline{N}_{\beta_1}(G)$ is a connected graph. □

Lemma 4.11. *Let G be an $\mathcal{M}$-ultrahomogeneous graph, and let α be any vertex of G. If $N_\alpha(G)$ is isomorphic to $2K_n$ with $n \geq 2$, then G is isomorphic to $K_{n+1} \times K_{n+1}$.*

Proof. As in the proof of Lemma 4.10, $1 \leq m \leq 2$.

Suppose that $m = 1$. Using Equation 4.4, $r = 2n^2$. Thus G has $1 + 2n + 2n^2$ vertices. Also, $N_\alpha(G)$ contains exactly 2 copies of K_n. Equation 4.3 tells us that $n+1$ divides $2+4n+4n^2$. Performing arithmetic modulo $n+1$,

$$0 \equiv 2 + 4n + 4n^2 \equiv 2 + 4(-1) + 4(-1)^2 \equiv 2.$$

This can never happen when $n \geq 2$.

We have contradicted the assumption that $m = 1$, so m must equal 2. Again using the Equation 4.4, $r = n^2$. Thus G has $1 + 2n + n^2$ vertices.

Let γ be any neighbor of α; then $N_{\alpha,\gamma}(G)$ is isomorphic to K_{n-1}. Since G is K_2-ultrahomogeneous, $N_{\gamma_1,\gamma_2}(G)$ is isomorphic to K_{n-1} for any pair of adjacent vertices γ_1 and γ_2.

Let A and B be the two copies of K_n in $N_\alpha(G)$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the vertices of A. Now each vertex α_i in A is adjacent to the copy of K_n consisting of α together with the other vertices of A. The vertex α_i must be adjacent to another copy B_i of K_n whose vertices are not in A, B, or equal to α. The subgraph $N_{\alpha_i,\alpha_j}(G)$ consists of the vertex α together with the other vertices of A. Therefore, none of the B_i intersect.

The vertex α together with the vertices of $A, B, B_1, \ldots, B_n$ give us $1 + 2n + n^2$ vertices, so we have accounted for all the vertices. We still must account for the edges between vertices belonging to $B, B_1, \ldots, B_n$. In fact, we must account for exactly n edges incident to each such vertex.

Let β be any vertex of B. If β had two neighbors γ_1 and γ_2 in B_i, then β would belong to $N_{\gamma_1,\gamma_2}(G)$. But this is not possible because $N_{\gamma_1,\gamma_2}(G)$ is the copy of K_{n-1} consisting of α_i and the other $n-2$ vertices of B_i. Therefore, β has exactly one neighbor in each B_i, and these neighbors together with β form a complete graph. From this fact it follows that G must be isomorphic to $K_{n+1} \times K_{n+1}$. □

Lemma 4.12. *Let G be connected and $\mathcal{M}$-ultrahomogeneous, and let α be any vertex of G. Suppose that $N_\alpha(G)$ is isomorphic to tK_1.*

(1) *If $t = 1$, then G is isomorphic to K_2.*
(2) *If $t = 2$, then G is isomorphic to C_4 or C_5.*
(3) *If $t \geq 3$, then G is isomorphic to $K_{2;t}$.*

Proof. When $t = 1$, the graph G is 1-regular, so it is isomorphic to K_2.

When $t = 2$, the graph G is connected and 2-regular. This means that G is isomorphic to a cycle. By Lemma 4.6, the only $\mathcal{M}$-ultrahomogeneous cycles are C_4 and C_5.

In the rest of the proof, we assume that $t \geq 3$. We follow the same outline as the proof of Lemma 4.10. In the case $m = 1$, we obtain the inequality $t \geq (t-1)^2$; this can never hold when $t \geq 3$.

In the case $m = t - 1$, Equation 4.4 tells us that $r = t$. Each vertex β in $\overline{N}_\alpha(G)$ has $t-1$ neighbors in $N_\alpha(G)$, so it must have exactly one neighbor γ in $\overline{N}_\alpha(G)$. Since β and γ both have $t-1$ neighbors amongst the t vertices of $N_\alpha(G)$, they must have a common neighbor δ. Then β, γ, and δ are the vertices of a copy of K_3. This is a contradiction since G contains no copies of K_3.

In the case $m = t$, the graph $\overline{N}_\alpha(G)$ consists entirely of vertices that are adjacent to every vertex of $N_\alpha(G)$. Equation 4.4 tells us that $r = t - 1$. Since G is t-regular, there are no more vertices and G is isomorphic to $K_{2;t}$.

Finally, in the case $m = 2$, we get $t \geq \frac{1}{2}t(t-1) - 2(t-1) + 3$, which implies that $t \leq 5$. We handle the cases $t = 3$, $t = 4$, and $t = 5$ separately.

Case 1: $t = 3$. Equation (4.4) implies that $r = 3$. Therefore, G is a 3-regular graph on 7 vertices. This is impossible.

Case 2: $t = 4$. Equation (4.4) implies that $r = 6$. Since $m = 2$, each of the 6 vertices of $\overline{N}_\alpha(G)$ is adjacent to one of the 6 pairs of vertices in $N_\alpha(G)$. Since G contains no copies of K_3, any two adjacent vertices of $\overline{N}_\alpha(G)$ cannot have a common neighbor in $N_\alpha(G)$. This means that every vertex of $\overline{N}_\alpha(G)$ has at most one neighbor in $\overline{N}_\alpha(G)$. This leads to a contradiction since G must be 4-regular.

Case 3: $t = 5$. Equation (4.4) implies that $r = 10$. Since $m = 2$, each of the 10 vertices of $\overline{N}_\alpha(G)$ is adjacent to one of the 10 pairs of vertices in $N_\alpha(G)$. Since G contains no copies of K_3, any two adjacent vertices of $\overline{N}_\alpha(G)$ cannot have a common neighbor in $N_\alpha(G)$. Since G is 5-regular, this implies that two vertices of $\overline{N}_\alpha(G)$ are adjacent if and only if they do not have any common neighbors in $N_\alpha(G)$.

We have completely described the graph G. It remains to observe that G is not $\mathcal{M}$-ultrahomogeneous. In fact, G is not $4K_1$-ultrahomogeneous. Because $N_\alpha(G)$ is isomorphic to $5K_1$, it follows that every set of 4 non-adjacent vertices can be extended (uniquely) to a set of 5 non-adjacent vertices. However, if we take 3 vertices of $N_\alpha(G)$ together with the unique non-adjacent vertex of $\overline{N}_\alpha(G)$, then we have a set of 4 vertices that cannot be extended. □

Lemma 4.13. *Let G be connected and $\mathcal{M}$-ultrahomogeneous, and let α be any vertex of G. If $N_\alpha(G)$ is isomorphic to the complete regular multipartite graph $K_{t;n}$ with $t \geq 2$ and $n \geq 1$, then G is isomorphic to $K_{t+1;n}$.*

Proof. Let $A_1, A_2, \ldots, A_t$ be the partite sets of $N_\alpha(G)$. Choose any vertex β of A_1. Since $N_\beta(G)$ is also isomorphic to $K_{t;n}$, there exist $n-1$ vertices $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ of G such that each α_j is adjacent to β and to every vertex of $A_2, \ldots, A_t$ but is not adjacent to α. Also, the vertices α_j and α_k are not adjacent for $j \neq k$.

It remains only to show that each α_j is adjacent to every vertex of A_1. To do this, choose a vertex γ in A_2 and consider $N_\gamma(G)$, which is again isomorphic to $K_{t;n}$. □

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