The generalized exponent sets of primitive, minimally strong digraphs (I)*

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Abstract

Let $D = (V, E)$ be a primitive digraph. The exponent of D at a vertex $u \in V$, denoted by $\exp_D(u)$, is defined to be the least integer k such that there is a walk of length k from u to v for each $v \in V$. Let $V = \{v_1, v_2, \cdots, v_n\}$. The vertices of V can be ordered so that $\exp_D(v_{i_1}) \leq \exp_D(v_{i_2}) \leq \cdots \leq \exp_D(v_{i_n})=\gamma(D)$. The number $\exp_D(v_{i_k})$ is called k-exponent of D, denoted by $\exp_D(k)$. We use $L(D)$ to denote the set of distinct lengths of the cycles of D. In this paper, we completely determinate 1-exponent sets of primitive, minimally strong digraphs of with n vertices and $L(D) = \{p, q\}\}$, where $3 \leq p < q$ and $p + q > n$.

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1 Introduction

We consider only the digraphs without multiple arcs. Let $D = (V, E)$ be a digraph with n vertices. A walk uWv of length p from u to v in D is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs (u, u_1),

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$(u_1, u_2), \ldots, (u_{p-1}, v)$, where the vertices and arcs need not to be distinct, and denoted by $uWv = (u, u_1, \ldots, u_{p-1}, v)$. The initial vertex of uWv is u, the terminal vertex is v, and $u_1, u_2, \ldots, u_{p-1}$ are the internal vertices of uWv. If $u = v$, then uWv is a circuit (or a closed walk). A path is a walk with distinct vertices. A cycle(an elementary circuit) is a circuit with distinct vertices except for $u = v$. For convenience, we treat a cycle as a path (a closed path) in this paper. An r-cycle is a cycle of length r. By $L(D)$ we denote the set of distinct lengths of the cycles of D. For the sake of simplicity, we use notation $[a, \ldots, b]$ to denote the set of all integers between a and b, namely $[a, \ldots, b] = \{m \mid m \in Z \text{ and } a \leq m \leq b\}$. We use notation $\lfloor a \rfloor$ and $\lceil a \rceil$, respectively, to denote the greatest integer which is not greater than a and the least integer which is not less than a.

The digraph D is called strongly connected(or strong) if for each ordered pair of distinct vertices u, v there is a walk from u to v. A strongly connected digraph D is called minimally strong(or ministrong) provided each digraph obtained from D by removing an arc is not strongly connected. A digraph D is primitive if there exists an integer $k > 0$ such that for each ordered pair of vertices $u, v \in V(D)$ (not necessarily distinct), there is a walk of length k from u to v in D, and the least such k is called the exponent of D, denoted by $\exp(D)$. It is well known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1.

In 1990, from the background of memoryless communication system, R. A. Brualdi and Bolian Liu [1] generalized the concept of the exponent for a primitive digraph and introduced the concept of k-exponent. Let $D = (V, E)$ be a primitive digraph with n vertices $v_1, v_2, \ldots, v_n$. For any $v_i, v_j \in V$, let $\exp_D(v_i, v_j) :=$ the smallest integer p such that there is a walk of length t from v_i to v_j for each integer $t \geq p$. Let the exponent of vertex v_i be defined by $\exp_D(v_i) := \max\{\exp_D(v_i, v_j) : v_j \in V\}$. Then $\exp_D(v_i)$ is the smallest integer p such that there is a walk of length p from v_i to each vertex of D. We arrange the vertices of D in such a way that $\exp_D(v_{i_1}) \leq \exp_D(v_{i_2}) \leq \cdots \leq \exp_D(v_{i_n})$, and we call the number $\exp_D(v_{i_k})$ the k-point exponent of D (the k-exponent for short), which is denoted by $\exp_D(k)$.

Let $PMSD_n$ be the set of all primitive, ministrong digraphs of order n. Bolian Liu[2] obtained the maximum value of the k-exponent for $PMSD_n$. Bo Zhou [5] characterized primitive, ministrong digraphs with n vertices whose k-exponent $(1 \leq k \leq n)$ achieve the maximum value. In 2002, Bo Zhou [5] pointed out that the complete determination of k-exponent set $(1 \leq k \leq n-1)$ of $PMSD_n$ is an interesting and difficult problem.

In this paper, we mainly study the 1-exponent of the primitive, ministrong digraphs with n vertices and $L(D) = \{p, q\}$, where $3 \leq p < q$ and $p + q > n$. In Section 2 we shall give a lower bound of 1-exponent(see

Theorem 2.1). In Section 3 we shall determinate completely 1-exponent set(see Theorems 3.5 and 3.6).

2 The lower bound of the 1-exponent

Let $D = (V, E)$ be a digraph. $D' = (V', E')$ is called a subdigraph of D if $V' \subseteq V$ and $E' \subseteq E$, and denoted by $D' \subseteq D$. We call D' a proper subdigraph of D (write $D' \subset D$) if $D' \subseteq D$ and $D' \neq D$. Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be two subdigraphs of D. We call the digraphs $D_1 \cap D_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $D_1 \cup D_2 = (V_1 \cup V_2, E_1 \cup E_2)$ the intersection and the union of D_1, D_2, respectively.

Let $D = (V, E)$ be a digraph. We use $R_t(u)$ to denote the set of vertices of D that can be reached by a walk with initial vertex u of length t (for $t = 0$, we define $R_t(u) = \{u\}$). Let uWv be a walk from vertex u to vertex v. We use $\eta(uWv)$ to denote the length of the walk uWv. Let $vW'\omega$ be a walk from vertex v to vertex ω. For convenience, we also use $uWvW'\omega$ to denote the walk $uWv + vW'\omega$ from u to ω.

Let D be a digraph, C a cycle of D with length at least 2. Let u and v be two vertices in $V(C)$. We define $uC^{(0)}u = u$, $uC^{(0)}v$ the path from u to v in C for $u \neq v$, and $uC^{(k)}v (k \geq 1)$ the walk $uC^{(0)}v + \underbrace{C + \cdots + C}_{k \text{ times}}$ from u to v.

Let $D = (V, E)$ be a primitive digraph with $L(D) = \{p, q\}$. For $u, v \in V(D)$, the distance $d(u, v)$ from u to v is defined to be the length of shortest walk from u to v in D, the relative distance $d_{L(D)}(u, v)$ from u to v is defined to be the length of the shortest walk from u to v that meets at least one p-cycle and one q-cycle. The Frobenius number $\phi(p, q)$ is defined to be the smallest integer m such that every integer with $t \geq m$ can be represented in the form $t = \mu_1 p + \mu_2 q$, where μ_1, μ_2 are nonnegative integers. It is well known that if p and q are coprime, then $\phi(p, q) = (p - 1)(q - 1)$.

Lemma 2.1 [4] *Let D be a primitive digraph with $L(D) = \{p, q\}$ $(p, q > 1)$. Write $\phi_{L(D)} = \phi(p, q)$. Then for any $u, v \in V(D)$, we have*

$\exp_D(u, v) \leq d_{L(D)}(u, v) + \phi_{L(D)}$,

$\exp_D(u) \leq \max\{d_{L(D)}(u, v) : v \in V\} + \phi_{L(D)}$.

Lemma 2.2 *Let D be a primitive digraph with $L(D) = \{p, q\} (p, q > 1)$. Let $u, v \in V(D)$ and let a be a positive integer. If the length of every walk from u to v of length at least a can be expressed as $\eta(uWv) = \mu_1 p + \mu_2 q$ $+a$, where μ_1, μ_2 are nonnegative integers, then* $\exp_D(u, v) \geq a + \phi_{L(D)}$.

Proof. If $\exp_D(u, v) < a + \phi_{L(D)}$, then there exists a walk from u to v of length $a + \phi_{L(D)} - 1$ by the definition of $\exp_D(u, v)$. Since $\phi_{L(D)} \geq 2$, it follows that $a + \phi_{L(D)} - 1 > a$, and so $a + \phi_{L(D)} - 1 = \mu_1 p + \mu_2 q + a$, where μ_1, μ_2 are nonnegative integers. Thus $\phi_{L(D)} - 1 = \mu_1 p + \mu_2 q$, which contradicts that $\phi_{L(D)}$ is Frobenius number of p, q. Therefore $\exp_D(u,v) \geq a + \phi_{L(D)}$. $\square$

By Lemmas 2.1 and 2.2, we have

Lemma 2.3 *Let D be a primitive digraph with $L(D) = \{p,q\}(p,q > 1)$. Let $u,v \in V(D)$, if the length of every walk from u to v of length at least $d_{L(D)}(u,v)$ can be expressed as $\eta(uWv) = \mu_1 p + \mu_2 q + d_{L(D)}(u,v)$, where μ_1, μ_2 are nonnegative integers, then $\exp_D(u,v) = d_{L(D)}(u,v) + \phi_{L(D)}$.*

Let D be a digraph, $3 \leq p < q$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ and C_p be respectively a cycle of length q and length p in D. We call C_p a consecutive p-cycle on C_q if $C_p \cap C_q = v_{t_1} C_q^{(0)} v_{t_2}$(where v_{t_1} and v_{t_2} are two vertices of C_q, not necessary distinct), v_{t_1} and v_{t_2} are respectively called the initial vertex and the terminal vertex of C_p on C_q. Let C_p and C_p' be two consecutive p-cycles on C_q. we call C_p' a greater consecutive p-cycle on C_q than C_p if $C_p \cap C_q \subset C_p' \cap C_q$. C_p is called a maximum consecutive p-cycle on C_q if there is no greater consecutive p-cycle on C_q than C_p in D. Let D be a digraph, C_q a q-cycle of D and σ a path of C_q, and let T be a set of some consecutive(maximum consecutive) p-cycle on C_q. We call T a consecutive (maximum consecutive)p-cycle cover of (C_q, σ) if $(\bigcup_{C_p \in T} C_p) \cap C_q = \sigma$.

Let T be a consecutive (maximum consecutive)p-cycles cover of (C_q, σ). We call T reducible if there exists some p-cycle $C_p \in T$ such that $T_1 = T \backslash \{C_p\}$ is still a consecutive (maximum consecutive) p-cycles cover of (C_q, σ) and call C_p a superfluous p-cycle in T. We call T irreducible if T is not reducible. We have

Lemma 2.4 *Let D be a digraph, $3 \leq p < q$, C_q a q-cycle of D and σ a path of C_q, and let T be an irreducible consecutive p-cycles cover of (C_q, σ). Then*

(i) *$C_p' \cap C_q \not\subseteq C_p'' \cap C_q$ for any distinct C_p', $C_p'' \in T$.*
(ii) *$V(C_p' \cap C_p'' \cap C_p''' \cap C_q) = \emptyset$ for any distinct C_p', C_p'', $C_p''' \in T$.*

Proof. (i) If there exist distinct C_p', $C_p'' \in T$ such that $C_p' \cap C_q \subseteq C_p'' \cap C_q$, then C_p' is a superfluous p-cycle in T. This contradicts that T is irreducible.

(ii) If there exist distinct C_p', C_p'', $C_p''' \in T$ such that $V(C_p' \cap C_p'' \cap C_p''' \cap C_q) \neq \emptyset$, then $(C_p' \cup C_p'' \cup C_p''') \cap C_q$ is a subpath of σ by C_p', C_p'', C_p''' being consecutive p-cycles on C_q. Let $(C_p' \cup C_p'' \cup C_p''') \cap C_q = u C_q^{(0)} v$. Then u is the initial vertex of one of the three p-cycles on C_q and v is the terminal vertex of one of the three p-cycles on C_q. Without loss of generality we assume that u is the initial vertex of C_p' on C_q and v is the terminal vertex of C''_p on C_q. Since $V(C'_p \cap C''_p \cap C_q) \neq \emptyset$ and C'_p, C''_p are consecutive p-cycles on C_q, then $(C'_p \cup C''_p) \cap C_q = uC_q^{(0)}v$, and thus C'''_p is a superfluous p-cycle in T. This contradicts that T is irreducible.

This completes the proof of Lemma 2.4. □

Let T be an irreducible consecutive (maximum consecutive) p-cycles cover of (C_q, σ). From (i) of Lemma 2.4, distinct cycles in T have distinct initial vertices and distinct terminal vertices on C_q. If $\sigma \neq C_q$, then $\sigma = v_a C_q^{(0)} v_b$, where v_a is the initial vertex of some $C_p \in T$ on C_q; if $\sigma = C_q$, then σ can be expressed as $\sigma = v_a C_q^{(1)} v_a$, where v_a is the initial vertex of any $C_p \in T$ on C_q. We arrange all p-cycles in T in the sequence: $C_p^1, C_p^2, \ldots, C_p^t$ such that along the path σ, we first meet the initial vertex v_a of C_p^1 on C_q, next we meet the initial vertex of C_p^2 on C_q, ..., finally we meet the initial vertex of C_p^t on C_q. The sequence $C_p^1, C_p^2, \ldots, C_p^t$ is called a irreducible consecutive (maximum consecutive) p-cycles chain of (C_q, σ).

Let $D = (V, E)$ be a digraph, C_p a p-cycle of D and C_q a q-cycle of D. We use $uP_{\bar{C}_q}v$ to denote any path from $u \in V$ to $v \in V$ whose internal vertices and arcs are not in C_q, use $uP_{C_p}v$ to denote the path in C_p from $u \in V(C_p)$ to $v \in V(C_p)$. Clearly,

$$uP_{C_p}v = \begin{cases} uC_p^{(0)}v, & \text{if} \quad u \neq v, \\ uC_p^{(1)}u \text{ (namly } C_p), & \text{if} \quad u = v. \end{cases}$$

We also denote $uP_{C_p}v$ by $uP_{C_p\bar{C}_q}v$ if all internal vertices and arcs of $uP_{C_p}v$ is not in C_q.

Lemma 2.5 *Let D be a digraph, $3 \leq p < q$, $C_q = (v_1, v_2, \ldots, v_q, v_1)$ a q-cycle of D and σ a path of C_q. Let $C_p^1, C_p^2, \ldots, C_p^t$ be an irreducible consecutive p-cycles chain of (C_q, σ), where $C_p^i = v_{x_i} C_q^{(0)} v_{y_i} P_{C_p^i \bar{C}_q} v_{x_i}$. We have the following results:*

(i) *If $\sigma \subseteq v_1 C_q^{(0)} v_q$, then $1 \leq x_1 < x_2 \leq y_1 < x_3 \leq y_2 < \cdots < x_{t-1} \leq y_{t-2} < x_t \leq y_{t-1} < y_t \leq q$.*

(ii) *If $\sigma = C_q$ and $x_1 = 1$, then $1 = x_1 \leq y_t < x_2 \leq y_1 < x_3 \leq y_2 < \cdots < x_{t-1} \leq y_{t-2} < x_t \leq y_{t-1} \leq q$.*

Proof. Let $T = \{C_p^1, C_p^2, \ldots, C_p^t\}$. Then T is an irreducible consecutive p-cycles cover of (C_q, σ). By (i) of Lemma 2.4, x_1, x_2, ..., x_t are distinct and y_1, y_2, ..., y_t are distinct.

(i) From $\sigma \subseteq v_1 C_q^{(0)} v_q$ and the definition of irreducible consecutive p-cycles chain, $x_1 < x_2 < \cdots < x_t$ and $x_i < y_i$ $(i = 1, 2, \ldots, t)$. We claim that $y_1 < y_2 < \cdots < y_t$. Otherwise, there must exist $k \in \{1, 2, \ldots, t-1\}$ such that $y_k \geq y_{k+1}$, it follows that $x_k < x_{k+1} < y_{k+1} \leq y_k$. And so $v_{x_{k+1}} C_q^{(0)} v_{y_{k+1}} \subseteq v_{x_k} C_q^{(0)} v_{y_k}$. Namely $C_p^{k+1} \cap C_q \subseteq C_p^k \cap C_q$. This contradicts that T is irreducible. It follows from $(\bigcup_{i=1}^{t} C_p^i) \cap C_q = \sigma$ that $\sigma = v_{x_1} C_q^{(0)} v_{y_t}$.

Now we prove that $x_{i+1} \le y_i$ $(i = 1, 2, \dots, t-1)$ and $y_i < x_{i+2}$ $(i = 1, 2, \dots, t-2)$. If there exists $j \in \{1, 2, \dots, t-1\}$ such that $x_{j+1} > y_j$, it follows from $x_i < y_i$ $(i = 1, 2, \dots, t)$, $x_1 < x_2 < \cdots < x_t \le q$ and $y_1 < y_2 < \cdots < y_{t-1} < y_t \le q$ that

$$v_{y_j} C_q^{(0)} v_{y_j+1} \not\subseteq \bigcup_{i=1}^{t} v_{x_i} C_q^{(0)} v_{y_i} = (\bigcup_{i=1}^{t} C_p^i) \cap C_q = v_{x_1} C_q^{(0)} v_{y_t},$$

which is impossible. Hence $x_{i+1} \le y_i$ $(i = 1, 2, \dots, t-1)$. If there exists $l \in \{1, 2, \dots, t-2\}$ such that $y_l \ge x_{l+2}$, it follows from $x_l < x_{l+1} < x_{l+2}$ and $y_l < y_{l+1}$ that $x_l < x_{l+1} < x_{l+2} \le y_l < y_{l+1}$. And so $v_{x_{l+2}} \in V(v_{x_i} C_q^{(0)} v_{y_i})$ $(i = l, l+1, l+2)$. Hence

$$v_{x_{l+2}} \in V(C_p^l \cap C_p^{l+1} \cap C_p^{l+2} \cap C_q).$$

This contradicts (ii) of Lemma 2.4. Hence $y_i < x_{i+2}$ $(i = 1, 2, \dots, t-2)$. Therefore $1 \le x_1 < x_2 \le y_1 < x_3 \le y_2 < \cdots < x_{t-1} \le y_{t-2} < x_t \le y_{t-1} < y_t \le q$.

(ii) Clearly $t \ge 2$. From $x_1 = 1$ and the definition of irreducible consecutive p-cycles chain, we have $x_1 < x_2 < \cdots < x_t$. First we prove that there exists an unique integer $j \in \{1, 2, \dots, t\}$ such that $x_j > y_j$. Clearly $x_1 < y_1$. If $x_i < y_i$ for each $i \in \{1, 2, \dots, t\}$, then $v_q C_q^{(0)} v_1 \not\subseteq v_{x_i} C_q^{(0)} v_{y_i}$ $(i = 1, 2, \dots, t)$, and so $v_q C_q^{(0)} v_1 \not\subseteq (\bigcup_{i=1}^{t} C_p^i) \cap C_q = C_q$, which is absurd. If there exist two distinct integers $i, i' \in \{1, 2, \dots, t\}$ such that $x_i > y_i$ and $x_{i'} > y_{i'}$, then $i, i' \in \{2, 3, \dots, t\}$ and $v_1 \in V(v_{x_i} C_q^{(0)} v_{y_i}) \cap V(v_{x_{i'}} C_q^{(0)} v_{y_{i'}})$. Note that $v_1 \in V(v_{x_1} C_q^{(0)} v_{y_1})$. Thus $v_1 \in V(C_p^1 \cap C_p^i \cap C_p^{i'} \cap C_q)$, which contradicts (ii) of Lemma 2.4.

We next prove that $j = t$. If $j \ne t$, then $j < t$ and $y_j < x_j < x_t < y_t$, and so $v_{x_t} C_q^{(0)} v_{y_t} \subseteq v_{x_j} C_q^{(0)} v_{y_j}$. Namely $C_p^t \cap C_q \subseteq C_p^j \cap C_q$. This contradicts that T is irreducible.

We next prove that $y_t < x_2$. Clearly it holds when $t = 2$. Hence it suffices to prove that $y_t < x_2$ for $t \ge 3$. If $y_t \ge x_2$ and $x_2 > y_1$, then $x_1 < y_1 < x_2 \le y_t < x_t$, and so $v_{x_1} C_q^{(0)} v_{y_1} \subseteq v_{x_t} C_q^{(0)} v_{y_t}$. Namely $C_p^1 \cap C_q \subseteq C_p^t \cap C_q$, which contradicts that T is irreducible. If $y_t \ge x_2$ and $x_2 \le y_1$, it follows from $x_t > y_t$, $x_1 < x_2$ and $y_1 < y_t$ that $x_1 < x_2 \le y_1 < y_t < x_t$. Thus for each $i \in \{1, 2, t\}$, $v_{x_2} \in V(v_{x_i} C_q^{(0)} v_{y_i})$, and so $v_{x_2} \in V(C_p^1 \cap C_p^2 \cap C_p^t \cap C_q)$, which contradicts (ii) of Lemma 2.4.

Now we prove that $y_1 < y_2 < \cdots < y_{t-1}$. If there exists $k \in \{1, 2, \ldots, t-2\}$ such that $y_k \geq y_{k+1}$, then $x_k < x_{k+1} < y_{k+1} \leq y_k$, and so $v_{x_{k+1}} C_q^{(0)} v_{y_{k+1}} \subseteq v_{x_k} C_q^{(0)} v_{y_k}$. Namely $C_p^{k+1} \cap C_q \subseteq C_p^k \cap C_q$. This contradicts that T is irreducible.

Finally we prove that $x_{i+1} \leq y_i$ $(i = 1, 2, \ldots, t-1)$ and $y_i < x_{i+2}$ $(i = 1, 2, \ldots, t-2)$. If there exists $i_1 \in \{1, 2, \ldots, t-1\}$ such that $x_{i_1+1} > y_{i_1}$, observe that $y_t < y_1$ by $v_{x_1} C_q^{(0)} v_{y_1} \not\subseteq v_{x_t} C_q^{(0)} v_{y_t}$, it follows that

$$y_{i_1} < y_{i_1} + 1 \leq x_{i_1+1} \leq x_i < y_i \quad \text{for } i_1 + 1 \leq i \leq t-1,$$

and

$$x_1 \leq y_t < y_1 \leq y_{i_1} < y_{i_1} + 1 \leq x_{i_1+1} \leq x_t.$$

Thus

$$v_{y_{i_1}} C_q^{(0)} v_{y_{i_1}+1} \not\subseteq v_{x_i} C_q^{(0)} v_{y_i} \quad \text{for } i_1 + 1 \leq i \leq t.$$

Moreover, since

$$x_i < y_i \leq y_{i_1} < y_{i_1} + 1 \quad \text{for } 1 \leq i \leq i_1,$$

then

$$v_{y_{i_1}} C_q^{(0)} v_{y_{i_1}+1} \not\subseteq v_{x_i} C_q^{(0)} v_{y_i} \quad \text{for } 1 \leq i \leq i_1,$$

and so

$$v_{y_{i_1}} C_q^{(0)} v_{y_{i_1}+1} \not\subseteq \bigcup_{i=1}^{t} v_{x_i} C_q^{(0)} v_{y_i} = (\bigcup_{i=1}^{t} C_p^i) \cap C_q = C_q,$$

which is absurd. Hence $x_{i+1} \leq y_i$ $(i = 1, 2, \ldots, t-1)$. If there exists $l \in \{1, 2, \ldots, t-2\}$ such that $y_l \geq x_{l+2}$, it follows from $x_l < x_{l+1} < x_{l+2}$ and $y_l < y_{l+1}$ that $x_l < x_{l+1} < x_{l+2} \leq y_l < y_{l+1}$, and so

$$v_{x_{l+2}} \in V(v_{x_i} C_q^{(0)} v_{y_i})(i = l, l+1, l+2).$$

Hence

$$v_{x_{l+2}} \in V(C_p^l \cap C_p^{l+1} \cap C_p^{l+2} \cap C_q).$$

This contradicts (ii) of Lemma 2.4. Hence $y_i < x_{i+2}$ $(i = 1, 2, \ldots, t-2)$. Consequently $1 = x_1 \leq y_t < x_2 \leq y_1 < x_3 \leq y_2 < \cdots < x_{t-1} \leq y_{t-2} < x_t \leq y_{t-1} \leq q$.

We have completed the proof of Lemma 2.5. □

Lemma 2.6 *Let D be a primitive digraph with n vertices and $L(D) = \{p, q\}$ with $3 \leq p < q$ and $p + q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be a q-cycle in D and C_p a p-cycle in D. Then there exists a consecutive p-cycle C'_p on C_q such that $C_p \cap C_q \subseteq C'_p \cap C_q$.*

Proof. Clearly the result holds if C_p is a consecutive p-cycle on C_q. Now we suppose that C_p is not a consecutive p-cycle on C_q. Since $p+q>n$, we have $V(C_p \cap C_q) \neq \emptyset$, and so we can express C_p as

$$C_p = v_{t_1} C_q^{(0)} v_{l_1} P_{C_p \bar{C}_q} v_{t_2} C_q^{(0)} v_{l_2} P_{C_p \bar{C}_q} v_{t_3} C_q^{(0)} v_{l_3} \cdots v_{t_s} C_q^{(0)} v_{l_s} P_{C_p \bar{C}_q} v_{t_1},$$

where $s \geq 2$ and $v_{t_i}, v_{l_i} \in V(C_q)$ $(i=1,2,\ldots,s)$. Without loss of generality we assume that $t_1 = 1$. We first prove that any internal vertex of the paths $v_{l_i} C_q^{(0)} v_{t_{i+1}}$ $(i=1,2,\ldots,s-1)$, $v_{l_s} C_q^{(0)} v_{t_1}$ is not in $V(C_p)$. If $v \in V(C_p)$ is a internal vertex of $v_{l_1} C_q^{(0)} v_{t_2}$, then $C_p = v_{l_1} P_{C_p \bar{C}_q} v_{t_2} P_{C_p} v P_{C_p} v_{l_1}$. Note that $v_{l_1} P_{C_p \bar{C}_q} v_{t_2} C_q^{(0)} v_{l_1}$ is a cycle. If $\eta(v_{l_1} P_{C_p \bar{C}_q} v_{t_2} C_q^{(0)} v_{l_1}) = p$, then $\eta(v_{t_2} C_q^{(0)} v_{l_1}) = \eta(v_{t_2} P_{C_p} v P_{C_p} v_{l_1})$, and so

$$\begin{aligned}\eta(v_{l_1} C_q^{(0)} v P_{C_p} v_{l_1}) &= \eta(v_{l_1} C_q^{(0)} v) + \eta(v P_{C_p} v_{l_1}) \\ &< \eta(v_{l_1} C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v P_{C_p} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v_{t_2}) + \eta(v_{t_2} C_q^{(0)} v_{l_1}) = q,\end{aligned}$$

$$\begin{aligned}\eta(v C_q^{(0)} v_{t_2} P_{C_p} v) &= \eta(v C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v) \\ &< \eta(v_{l_1} C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v P_{C_p} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v_{t_2}) + \eta(v_{t_2} C_q^{(0)} v_{l_1}) = q.\end{aligned}$$

Since $v_{l_1} C_q^{(0)} v P_{C_p} v_{l_1}$, $v C_q^{(0)} v_{t_2} P_{C_p} v$ are two circuits and $L(D) = \{p,q\}$, then $\eta(v_{l_1} C_q^{(0)} v P_{C_p} v_{l_1}) = k_1 p$ (k_1 is a positive integer) and $\eta(v C_q^{(0)} v_{t_2} P_{C_p} v) = k_2 p$ (k_2 is a positive integer). It follows that

$$\begin{aligned}q &= \eta(v_{l_1} C_q^{(0)} v C_q^{(0)} v_{t_2} C_q^{(0)} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v) + \eta(v C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v P_{C_p} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v) + \eta(v P_{C_p} v_{l_1}) + \eta(v C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v) \\ &= \eta(v_{l_1} C_q^{(0)} v P_{C_p} v_{l_1}) + \eta(v C_q^{(0)} v_{t_2} P_{C_p} v) = (k_1 + k_2) p\,.\end{aligned}$$

This contradicts that $(p,q)=1$. If $\eta(v_{l_1} P_{C_p \bar{C}_q} v_{t_2} C_q^{(0)} v_{l_1}) = q$, then $\eta(v_{l_1} P_{C_p \bar{C}_q}$ $= \eta(v_{l_1} C_q^{(0)} v_{t_2})$. Note that $v_{l_1} C_q^{(0)} v P_{C_p} v_{l_1}$, $v C_q^{(0)} v_{t_2} P_{C_p} v$ are circuits. Hence

$$\begin{aligned}p &= \eta(v_{l_1} P_{C_p \bar{C}_q} v_{t_2} P_{C_p} v_{l_1}) = \eta(v_{l_1} P_{C_p \bar{C}_q} v_{t_2}) + \eta(v_{t_2} P_{C_p} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v P_{C_p} v_{l_1}) \\ &= \eta(v_{l_1} C_q^{(0)} v) + \eta(v P_{C_p} v_{l_1}) + \eta(v C_q^{(0)} v_{t_2}) + \eta(v_{t_2} P_{C_p} v) \\ &= \eta(v_{l_1} C_q^{(0)} v P_{C_p} v_{l_1}) + \eta(v C_q^{(0)} v_{t_2} P_{C_p} v) \geq p + p = 2p\,,\end{aligned}$$

which is absurd. Therefore, any internal vertex of the path $v_{l_1}C_q^{(0)}v_{t_2}$ is not in $V(C_p)$. Similarly, any internal vertex of the paths $v_{l_i}C_q^{(0)}v_{t_{i+1}}$ $(i = 2, \ldots, s-1)$, $v_{l_s}C_q^{(0)}v_{t_1}$ is not in $V(C_p)$. It follows that $1 = t_1 \leq l_1 < t_2 \leq l_2 < \cdots < t_s \leq l_s$. Note that each of $v_{t_1}C_q^{(0)}v_{l_i}P_{C_p\bar{C}_q}v_{t_{i+1}}C_q^{(0)}v_{t_1}$ $(i = 1, 2, \ldots, s-1)$ and $v_{t_1}C_q^{(0)}v_{l_s}P_{C_p\bar{C}_q}v_{t_1}$ is a cycle of D. We claim that there exists a p-cycle in these cycles. Otherwise, if all of these cycles are q-cycle, then

$$\eta(v_{l_i}P_{C_p\bar{C}_q}v_{t_{i+1}}) = \eta(v_{l_i}C_q^{(0)}v_{t_{i+1}})(i = 1, 2, \ldots, s-1)$$

and

$$\eta(v_{l_s}P_{C_p\bar{C}_q}v_{t_1}) = \eta(v_{l_s}C_q^{(0)}v_{t_1}),$$

and so

$$\eta(C_p) = \eta(v_{t_1}C_q^{(0)}v_{l_1}C_q^{(0)}v_{t_2}C_q^{(0)}v_{l_2}\cdots v_{t_s}C_q^{(0)}v_{l_s}C_q^{(0)}v_{t_1}) = q,$$

a contradiction. Without loss of generality we assume that

$$\eta(v_{t_1}C_q^{(0)}v_{l_1}P_{C_p\bar{C}_q}v_{t_2}C_q^{(0)}v_{t_1}) = p\,.$$

Take $C_p' = v_{t_1}C_q^{(0)}v_{l_1}P_{C_p\bar{C}_q}v_{t_2}C_q^{(0)}v_{t_1}$. Then C_p' is a consecutive p-cycle on C_q and $C_p \cap C_q \subseteq v_{t_2}C_q^{(0)}v_{l_1} = C_p' \cap C_q$. The proof of Lemma 2.6 is complete. □

Lemma 2.7 *Let D be a primitive digraph on n vertices and $L(D) = \{p, q\}$ with $3 \leq p < q$ and $p + q > n$, and let C_q be a q-cycle of D and $C_p^i = v_{x_i}C_q^{(0)}v_{y_i}P_{C_p^i\bar{C}_q}v_{x_i}$ $(i = 1, 2)$ two distinct maximum consecutive p-cycles on C_q and $C_p^1 \cap C_q \neq C_p^2 \cap C_q$. Then there exists no common internal vertex in paths $v_{y_1}P_{C_p^1\bar{C}_q}v_{x_1}$ and $v_{y_2}P_{C_p^2\bar{C}_q}v_{x_2}$.*

Proof. We assume that there exists vertex $u \in V(D)$ such that u is a common internal vertex of $v_{y_1}P_{C_p^1\bar{C}_q}v_{x_1}$ and $v_{y_2}P_{C_p^2\bar{C}_q}v_{x_2}$, then $v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}$ is a walk of D, and any internal vertex of $v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}$ is not in $V(C_q)$. It follows that $v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}$ is a path (otherwise, there is a common internal vertex v in $v_{y_1}P_{C_p^1\bar{C}_q}u$ and $uP_{C_p^2\bar{C}_q}v_{x_2}$. Thus there exists the circuit $vP_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v$ which must contain a cycle with no vertex in $V(C_q)$, which contradicts that $p + q > n$). Hence $v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}C_q^{(0)}v_{y_1}$ is a cycle. Similarly $v_{y_2}P_{C_p^2\bar{C}_q}uP_{C_p^1\bar{C}_q}v_{x_1}C_q^{(0)}v_{y_2}$ is also a cycle. Let

$$\eta(v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}C_q^{(0)}v_{y_1}) = a$$

and

$$\eta(v_{y_2}P_{C_p^2\bar{C}_q}uP_{C_p^1\bar{C}_q}v_{x_1}C_q^{(0)}v_{y_2}) = b.$$

If $V(C_p^1 \cap C_p^2 \cap C_q) = \emptyset$, then

$$\eta(v_{x_2}C_q^{(0)}v_{y_1}) = \eta(v_{x_2}C_q^{(0)}v_{y_2}) + \eta(v_{y_2}C_q^{(0)}v_{y_1}),$$
$$\eta(v_{x_1}C_q^{(0)}v_{y_2}) = \eta(v_{x_1}C_q^{(0)}v_{y_1}) + \eta(v_{y_1}C_q^{(0)}v_{y_2}),$$

and so

$$\begin{aligned}
a+b &= \eta(v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}C_q^{(0)}v_{y_1}) + \eta(v_{y_2}P_{C_p^2\bar{C}_q}uP_{C_p^1\bar{C}_q}v_{x_1}C_q^{(0)}v_{y_2})\\
&= \eta(v_{y_1}P_{C_p^1\bar{C}_q}u) + \eta(uP_{C_p^2\bar{C}_q}v_{x_2}) + \eta(v_{x_2}C_q^{(0)}v_{y_1})\\
&\quad +\eta(v_{y_2}P_{C_p^2\bar{C}_q}u) + \eta(uP_{C_p^1\bar{C}_q}v_{x_1}) + \eta(v_{x_1}C_q^{(0)}v_{y_2})\\
&= (\eta(v_{y_1}P_{C_p^1\bar{C}_q}u) + \eta(uP_{C_p^1\bar{C}_q}v_{x_1}) + \eta(v_{x_1}C_q^{(0)}v_{y_1}))\\
&\quad +(\eta(v_{x_2}C_q^{(0)}v_{y_2}) + \eta(v_{y_2}P_{C_p^2\bar{C}_q}u) + \eta(uP_{C_p^2\bar{C}_q}v_{x_2}))\\
&\quad +(\eta(v_{y_2}C_q^{(0)}v_{y_1}) + \eta(v_{y_1}C_q^{(0)}v_{y_2}))\\
&= \eta(C_p^1) + \eta(C_p^2) + \eta(C_q) = 2p+q > p+q\ .
\end{aligned}$$

Since $a, b \in L(D) = \{p, q\}$ and $p < q$, then $a = b = q$, and so $2q = 2p+q$. It follows that $q = 2p$, which contradicts that $(p, q) = 1$. If $V(C_p^1 \cap C_p^2 \cap C_q) \neq \emptyset$, $v_{x_1} = v_{y_2}$ and $v_{x_2} = v_{y_1}$, then

$$\begin{aligned}
a+b &= \eta(v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{y_1}) + \eta(v_{y_2}P_{C_p^2\bar{C}_q}uP_{C_p^1\bar{C}_q}v_{y_2})\\
&= \eta(v_{y_1}P_{C_p^1\bar{C}_q}u) + \eta(uP_{C_p^2\bar{C}_q}v_{y_1}) + \eta(v_{y_2}P_{C_p^2\bar{C}_q}u) + \eta(uP_{C_p^1\bar{C}_q}v_{y_2})\\
&= \eta(v_{y_1}P_{C_p^1\bar{C}_q}u) + \eta(uP_{C_p^1\bar{C}_q}v_{y_2}) + \eta(v_{y_2}P_{C_p^2\bar{C}_q}u) + \eta(uP_{C_p^2\bar{C}_q}v_{y_1})\\
&= \eta(C_p^1) + \eta(C_p^2) - (\eta(v_{y_1}C_q^{(0)}v_{y_2}) + \eta(v_{y_2}C_q^{(0)}v_{y_1})) = 2p - q,
\end{aligned}$$

which contradicts that $a+b \geq 2p$. If $V(C_p^1 \cap C_p^2 \cap C_q) \neq \emptyset$, and $v_{x_1} \neq v_{y_2}$ or $v_{x_2} \neq v_{y_1}$, similarly we have $a+b = \eta(C_p^1) + \eta(C_p^2) = 2p$. Hence $a = b = p$ since $a, b \geq p$. Since

$$(v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}C_q^{(0)}v_{y_1}) \cap C_q = v_{x_2}C_q^{(0)}v_{y_1},$$

$$(v_{y_2}P_{C_p^2\bar{C}_q}uP_{C_p^1\bar{C}_q}v_{x_1}C_q^{(0)}v_{y_2}) \cap C_q = v_{x_1}C_q^{(0)}v_{y_2},$$

and we can check from $C_p^1 \cap C_q \neq C_p^2 \cap C_q$ that either

$$C_p^1 \cap C_q \subset v_{x_2}C_q^{(0)}v_{y_1} \text{ and } C_p^2 \cap C_q \subset v_{x_2}C_q^{(0)}v_{y_1}$$

or

$$C_p^1 \cap C_q \subset v_{x_1}C_q^{(0)}v_{y_2} \text{ and } C_p^2 \cap C_q \subset v_{x_1}C_q^{(0)}v_{y_2}\ ,$$

it follows that

$$\text{either } v_{y_1}P_{C_p^1\bar{C}_q}uP_{C_p^2\bar{C}_q}v_{x_2}C_q^{(0)}v_{y_1} \text{ or } v_{y_2}P_{C_p^2\bar{C}_q}uP_{C_p^1\bar{C}_q}v_{x_1}C_q^{(0)}v_{y_2}$$

is a greater consecutive p-cycle on C_q than C_p^1 and C_p^2. This contradicts that C_p^1, C_p^2 are maximum consecutive p-cycles on C_q. The proof of Lemma 2.7 is complete. □

Lemma 2.8 *Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \le p < q$ and $p + q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be a q-cycle of D. Then in C_q there must exist two distinct arcs which are not in any p-cycles.*

Proof. Define that $v_{q+1} = v_1$. If there is precisely one arc (v_a, v_{a+1}) of C_q such that (v_a, v_{a+1}) is not an arc of any p-cycle of D, then there exists a walk from v_a to v_{a+1} along some p-cycles which does not pass through arc (v_a, v_{a+1}). This contradicts that D is minimally strong digraph.

Suppose that for each $a \in \{1, 2, \ldots, q\}$, in D there is a p-cycle containing the arc (v_a, v_{a+1}). By Lemma 2.6, in D there exists a consecutive p-cycle on C_q which contains the arc (v_a, v_{a+1}). We take a maximum consecutive p-cycle $C_p^{\bar{a}}$ on C_q containing the arc (v_a, v_{a+1}). Let $T = \{C_p^{\bar{a}} : a \in [1, \ldots, q]\}$. Then T is a maximum consecutive p-cycles cover of (C_q, C_q), and we obtain an irreducible maximum consecutive p-cycles cover T_1 of (C_q, C_q) by removing the superfluous p-cycles from T. Furthermore, we can obtain an irreducible maximum consecutive p-cycles chain $C_p^1, C_p^2, \ldots, C_p^t$ of (C_q, C_q) by properly arranging the order of the p-cycles of T_1. Let $C_p^i = v_{x_i}C_q^{(0)}v_{y_i}P_{C_p^i\bar{C}_q}v_{x_i}$ and without loss of generality we assume that $x_1 = 1$. By Lemma 2.5, $1 = x_1 \le y_t < x_2 \le y_1 < x_3 \le y_2 < \cdots < x_{t-1} \le y_{t-2} < x_t \le y_{t-1} \le q$. By Lemma 2.7, for any distinct $i, j \in \{1, 2, \ldots, t\}$, there exists no common internal vertex in the paths $v_{y_i}P_{C_p^i\bar{C}_q}v_{x_i}$ and $v_{y_j}P_{C_p^j\bar{C}_q}v_{x_j}$, and so

$$\begin{aligned} & v_{y_1}P_{C_p^1}v_{y_t}P_{C_p^t}v_{y_{t-1}}P_{C_p^{t-1}}v_{y_{t-2}}\cdots v_{y_2}P_{C_p^2}v_{y_1} \\ = {} & v_{y_1}P_{C_p^1\bar{C}_q}v_{x_1}C_q^{(0)}v_{y_t}P_{C_p^t\bar{C}_q}v_{x_t}C_q^{(0)}v_{y_{t-1}}P_{C_p^{t-1}\bar{C}_q} \\ & v_{x_{t-1}}C_q^{(0)}v_{y_{t-2}}\cdots v_{y_2}P_{C_p^2\bar{C}_q}v_{x_2}C_q^{(0)}v_{y_1} \end{aligned}$$

is a cycle (denoted by C_r, where r is its length). It is easy to see that $\eta(C_p^1) + \eta(C_p^2) + \cdots + \eta(C_p^t) = \eta(C_q) + \eta(C_r)$, namely $tp = q + r$. Since $L(D) = \{p, q\}$, then $r \in \{p, q\}$. If r=p, then $q = (t-1)p$, which contradicts that $(p, q) = 1$ and $p \ge 3$. If $r = q$, then $2q = tp$, and so $p \mid 2q$, which contradicts $(p, q) = 1$ and $p \ge 3$. The proof of Lemma 2.8 is complete. □

Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \le p < q$ and $p + q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be a q-cycle of D. By Lemma 2.8, in C_q there exist two arcs not being in any p-cycle. Without loss of generality we assume that the two arcs are (v_q, v_1) and (v_a, v_{a+1}) $(1 \le a \le q-1)$. We have

Lemma 2.9 *Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \le p < q$ and $p + q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be a q-cycle of D and two arcs (v_q, v_1), (v_a, v_{a+1}) $(1 \le a \le q-1)$ not in any p-cycle. Let $v_i, v_j \in V(C_q)$ and $v_i W v_j$ be any walk from v_i to v_j in D. Then*

$$\eta(v_i W v_j) = \begin{cases} \mu_1 p + \mu_2 q + \eta(v_i C_q^{(0)} v_j), & \text{if} \quad i < j, \\ \mu_1 p + \mu_2 q - \eta(v_i C_q^{(0)} v_j), & \text{if} \quad i > j, \\ \mu_1 p + \mu_2 q, & \text{if} \quad i = j, \end{cases}$$

where μ_1, μ_2 are nonnegative integers.

Proof. $v_i W v_j$ can be expressed as $v_i W v_j =$

$$v_{i_1} C_q^{(k_1)} v_{j_1} P_{\bar{C}_q} v_{i_2} C_q^{(k_2)} v_{j_2} P_{\bar{C}_q} v_{i_3} C_q^{(k_3)} v_{j_3} \cdots v_{i_{t-1}} C_q^{(k_{t-1})} v_{j_{t-1}} P_{\bar{C}_q} v_{i_t} C_q^{(k_t)} v_{j_t},$$

where k_l $(1 \le l \le t)$ are nonnegative integers, v_{i_l}, v_{j_l} $(1 \le l \le t)$ are the vertices of C_q, $i_1 = i, j_t = j$. We first consider $\eta(v_x C_q^{(k)} v_y)$ and $\eta(v_x P_{\bar{C}_q} v_y)$ for any $x, y \in \{1, 2, \ldots, q\}$. Clearly

$$\begin{aligned} \eta(v_x C_q^{(k)} v_y) &= kq + \eta(v_x C_q^{(0)} v_y) \\ &= \begin{cases} kq + \eta(v_x C_q^{(0)} v_y), & \text{if} \quad x < y, \\ (k+1)q - \eta(v_y C_q^{(0)} v_x), & \text{if} \quad x > y, \\ kq, & \text{if} \quad x = y. \end{cases} \end{aligned}$$

For $\eta(v_x P_{\bar{C}_q} v_y)$, we consider the following two cases.

Case 1: $1 \le x \le a$. If $x < y$, then $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ contains the arc (v_q, v_1), and so $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ is a q-cycle. Hence $\eta(v_x P_{\bar{C}_q} v_y) = \eta(v_x C_q^{(0)} v_y)$. If $x > y$, then $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ is a p-cycle or a q-cycle, and so

$$\eta(v_x P_{\bar{C}_q} v_y) = p(\text{or} \;\; q) - \eta(v_y C_q^{(0)} v_x).$$

If $x = y$, then $v_x P_{\bar{C}_q} v_y$ is a p-cycle or a q-cycle, and so $\eta(v_x P_{\bar{C}_q} v_y) = p(\text{or} \;\; q)$.

Case 2: $a + 1 \le x \le q$. If $x < y$, then $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ contains the arc (v_q, v_1), and so $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ is a q-cycle. Hence $\eta(v_x P_{\bar{C}_q} v_y) = \eta(v_x C_q^{(0)} v_y)$. If $x > y$ and $y \ge a + 1$, then $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ is a p-cycle or a q-cycle, and so

$$\eta(v_x P_{\bar{C}_q} v_y) = p(\text{or} \;\; q) - \eta(v_y C_q^{(0)} v_x).$$

If $x > y$ and $y \le a$, then $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ contains the arc (v_a, v_{a+1}), and so $v_x P_{\bar{C}_q} v_y C_q^{(0)} v_x$ is a q-cycle. Hence

$$\eta(v_x P_{\bar{C}_q} v_y) = q - \eta(v_y C_q^{(0)} v_x).$$

If $x = y$, then $v_x P_{\bar{C}_q} v_y$ is a p-cycle or a q-cycle, and so $\eta(v_x P_{\bar{C}_q} v_y) = p(\text{or } q)$.

By the above discussions, we have

$$\eta(v_x P_{\bar{C}_q} v_y) = \begin{cases} \eta(v_x C_q^{(0)} v_y), & \text{if } x < y, \\ p(\text{or } q) - \eta(v_y C_q^{(0)} v_x), & \text{if } x > y, \\ p(\text{or } q), & \text{if } x = y. \end{cases}$$

It follows that

$$\begin{aligned} \eta(v_i W v_j) &= \sum_{l=1}^{t} \eta(v_{i_l} C_q^{(k_l)} v_{j_l}) + \sum_{l=1}^{t-1} \eta(v_{j_l} P_{\bar{C}_q} v_{i_{l+1}}) \\ &= \begin{cases} \mu_1 p + \mu_2 q + \eta(v_i C_q^{(0)} v_j), & \text{if } i < j, \\ \mu_1 p + \mu_2 q - \eta(v_j C_q^{(0)} v_i), & \text{if } i > j, \\ \mu_1 p + \mu_2 q, & \text{if } i = j, \end{cases} \end{aligned}$$

where μ_1, μ_2 are nonnegative integers. The proof of Lemma 2.9 is complete. □

Let D be a digraph and $u, v \in V(D)$, and let uPv be a path from u to v and u', v' two vertices in uPv. We use $u'Pv'$ to denote the path from u' to v' in uPv.

Lemma 2.10 *Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \le p < q$ and $p + q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ is a q-cycle in D. Let (v_q, v_1), (v_a, v_{a+1}) $(1 \le a \le q-1)$ are two arcs not being in any p-cycle. We have*

(i) *If there exists the path $v_i P_{\bar{C}_q} v_j$ $(1 \le i < j \le a)$ in D, then $\eta(v_i P_{\bar{C}_q} v_j) = \eta(v_i C_q^{(0)} v_j)$.*

(ii) *If $v_i P_{\bar{C}_q} v_j$ $(1 \le i < j \le a)$ is a path of some p-cycle C_p in D, then any internal vertex of $v_i C_q^{(0)} v_j$ is not in $V(C_p)$.*

(iii) *For any $v_i \in \{v_1, v_2, \ldots, v_a\}$ and any $v_j \in \{v_{a+1}, v_{a+2}, \ldots, v_q\}$, there exists no p-cycle containing both v_i and v_j in D.*

(iv) *Let C_p be a p-cycle with $V(C_p) \cap \{v_1, v_2, \ldots, v_a\} \ne \emptyset$, and let i, j be respectively the least and the greatest subscript of the vertices in $V(C_p) \cap \{v_1, v_2, .., v_a\}$. Then C_p can be expressed as*

$$\begin{aligned} C_p &= v_{i_1} C_q^{(0)} v_{j_1} P_{C_p \bar{C}_q} v_{i_2} C_q^{(0)} v_{j_2} P_{C_p \bar{C}_q} v_{i_3} C_q^{(0)} v_{j_3} \cdots \\ &\quad v_{i_{t-1}} C_q^{(0)} v_{j_{t-1}} P_{C_p \bar{C}_q} v_{i_t} C_q^{(0)} v_{j_t} P_{C_p \bar{C}_q} v_{i_1}, \end{aligned}$$

where $1 \le i = i_1 \le j_1 < i_2 \le j_2 < i_3 \le j_3 < \cdots < i_{t-1} \le j_{t-1} < i_t \le j_t = j \le a$.

Proof. (i) Clearly $v_i P_{\bar{C}_q} v_j C_q^{(0)} v_i$ is a cycle containing the arc (v_a, v_{a+1}). Hence $v_i P_{\bar{C}_q} v_j C_q^{(0)} v_i$ is a q-cycle, and so $\eta(v_i P_{\bar{C}_q} v_j) = \eta(v_i C_q^{(0)} v_j)$

(ii) If there exists vertex v_t with $i+1 \le t \le j-1$ such that $v_t \in V(C_p)$, then C_p can be expressed as $C_p = v_i P_{\bar{C}_q} v_j P_{C_p} v_t P_{C_p} v_i$, and so $\eta(v_i P_{\bar{C}_q} v_j) = \eta(v_i C_q^{(0)} v_j)$ by (i). Hence $\eta(v_i C_q^{(0)} v_j P_{C_p} v_t P_{C_p} v_i) = p$. On the other hand, since $v_t C_q^{(0)} v_j P_{C_p} v_t$ is a circuit,

$$\begin{aligned} \eta(v_i C_q^{(0)} v_j P_{C_p} v_t P_{C_p} v_i) &= \eta(v_i C_q^{(0)} v_t C_q^{(0)} v_j P_{C_p} v_t P_{C_p} v_i) \\ &> \eta(v_t C_q^{(0)} v_j P_{C_p} v_t) \ge p, \end{aligned}$$

which is absurd.

(iii) If there exist $v_i \in \{v_1, v_2, \ldots, v_a\}$, $v_j \in \{v_{a+1}, v_{a+2}, \ldots, v_q\}$ and some p-cycle C_p such that both v_i and v_j are in $V(C_p)$, then there exists a path $v_{i_1} P_{\bar{C}_q} v_{j_1}$ with $v_{i_1} \in \{v_1, v_2, \ldots, v_a\}$ and $v_{j_1} \in \{v_{a+1}, v_{a+2}, \ldots, v_q\}$ such that $v_{i_1} P_{\bar{C}_q} v_{j_1} \subset C_p$. Clearly $v_{i_1} P_{\bar{C}_q} v_{j_1} C_q^{(0)} v_{i_1}$ is a cycle containing the arc (v_q, v_1). Hence $v_{i_1} P_{\bar{C}_q} v_{j_1} C_q^{(0)} v_{i_1}$ is a q-cycle since $L(D) = \{p, q\}$, and so $\eta(v_{i_1} P_{\bar{C}_q} v_{j_1}) = \eta(v_{i_1} C_q^{(0)} v_{j_1})$. It follows that $v_{i_1} C_q^{(0)} v_{j_1} P_{C_p} v_{i_1}$ is a p-cycle. However $v_{i_1} C_q^{(0)} v_{j_1}$ contains the arc (v_a, v_{a+1}), a contradiction.

(iv) Let i_1 and j_t be respectively the least and the greatest subscript of the vertices in $V(C_p) \cap \{v_1, v_2, \ldots, v_a\}$. Then there must exist integer $j_1 \in [i_1, \ldots, a]$ such that $v_{i_1} C_q^{(0)} v_{j_1}$ $(\subset C_p \cap C_q)$ is the longest path with the initial vertex v_{i_1}. If $j_1 = j_t$, then by (iii), C_p can be expressed as

$$C_p = v_{i_1} C_q^{(0)} v_{j_1} P_{C_p \bar{C}_q} v_{i_1}.$$

If $j_1 < j_t$, let v_{i_2} be the vertex in C_q that the path in C_p beginning at vertex v_{j_1} first meet, and let $v_{i_2} C_q^{(0)} v_{j_2}$ $(\subset C_p \cap C_q)$ be the longest path beginning at vertex v_{i_2}. Then by (iii) and C_p being a cycle, we have $i_1 \le j_1 < i_2 \le j_2 \le j_t \le a$. If $j_2 = j_t$, then by (ii), (iii) and C_p being a cycle, the vertex in C_q that the path in C_p beginning at vertex v_{j_2} first meet must be v_{i_1}. Hence C_p can be expressed as

$$C_p = v_{i_1} C_q^{(0)} v_{j_1} P_{C_p \bar{C}_q} v_{i_2} C_q^{(0)} P_{C_p \bar{C}_q} v_{i_1}.$$

If $j_2 < j_t$, continue the above process, finally we obtain that

$$\begin{aligned} C_p &= v_{i_1} C_q^{(0)} v_{j_1} P_{C_p \bar{C}_q} v_{i_2} C_q^{(0)} v_{j_2} P_{C_p \bar{C}_q} v_{i_3} C_q^{(0)} v_{j_3} \cdots \\ &\qquad v_{i_{t-1}} C_q^{(0)} v_{j_{t-1}} P_{C_p \bar{C}_q} v_{i_t} C_q^{(0)} v_{j_t} P_{C_p \bar{C}_q} v_{i_1}, \end{aligned}$$

where $1 \le i = i_1 \le j_1 < i_2 \le j_2 < i_3 \le j_3 < \cdots < i_{t-1} \le j_{t-1} < i_t \le j_t = j \le a$. The proof of Lemma 2.10 is complete. □

Lemma 2.11 *Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \le p < q$ and $p + q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be a q-cycle in D, both (v_q, v_1) and (v_a, v_{a+1}) be not arcs of any p-cycle in D. Let C_p, C'_p be p-cycles containing at least two vertices of $\{v_1, v_2, \ldots, v_a\}$, and let i and j (i' and j') be respectively the least and the greatest subscript of the vertices in $V(C_p) \cap \{v_1, v_2, \ldots, v_a\}$ ($V(C'_p) \cap \{v_1, v_2, \ldots, v_a\}$). We have*

(i) *If $[i, j] \cap [i', j'] = \emptyset$, then no common internal vertex of $v_j P_{C_p} v_i$ and $v_{j'} P_{C'_p} v_{i'}$ exists.*

(ii) *If $[i, j] \cap [i', j'] \ne \emptyset$ and there exists a common internal vertex in $v_j P_{C_p} v_i$, $v_{j'} P_{C'_p} v_{i'}$, then $\max\{j, j'\} - \min\{i, i'\} \le p - 1$.*

Proof. By (iv) of Lemma 2.10, we have

$$v_j P_{C_p} v_i = v_j P_{C_p \bar{C}_q} v_i \text{ and } v_{j'} P_{C'_p} v_{i'} = v_{j'} P_{C'_p \bar{C}_q} v_{i'}.$$

By (i) and (iv) of Lemma 2.10, we have

$$\eta(v_i P_{C_p} v_j) = \eta(v_i C_q^{(0)} v_j) \text{ and } \eta(v_{i'} P_{C'_p} v_{j'}) = \eta(v_{i'} C_q^{(0)} v_{j'}).$$

Hence both $v_i C_q^{(0)} v_j P_{C_p} v_i$ and $v_{i'} C_q^{(0)} v_{j'} P_{C'_p} v_{i'}$ are p-cycles.

(i) If $[i, j] \cap [i', j'] = \emptyset$, then either $i' > j$ or $i > j'$ hold. Without loss of generality we assume $i' > j$. If there exists a common internal vertex u in $v_j P_{C_p} v_i$ and $v_{j'} P_{C'_p} v_{i'}$, since each internal vertex of $v_j P_{C_p} u P_{C'_p} v_{i'}$ not in $V(C_q)$, it follows that internal vertices of $v_j P_{C_p} u P_{C'_p} v_{i'}$ are distinct (otherwise, $v_j P_{C_p} u P_{C'_p} v_{i'}$ contains a cycle whose all vertices are not in $V(C_q)$, which contradicts that $L(D) = \{p, q\}$ and $p + q > n$). By (i) of Lemma 2.10, $\eta(v_j P_{C_p} u P_{C'_p} v_{i'}) = \eta(v_j C_q^{(0)} v_{i'})$. Hence

$$\begin{aligned} 2p &= \eta(v_{j'} P_{C'_p} u P_{C_p} v_i P_{C_p} v_j P_{C_p} u P_{C'_p} v_{i'} P_{C'_p} v_{j'}) \\ &= \eta(v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_j C_q^{(0)} v_{i'} C_q^{(0)} v_{j'}) \\ &= \eta(v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_{j'}). \end{aligned}$$

Since internal vertices of $v_{j'} P_{C'_p} u P_{C_p} v_i$ are distinct and are not in $V(C_q)$ (just as internal vertices of $v_j P_{C_p} u P_{C'_p} v_{i'}$ are distinct and are not in $V(C_q)$), hence $v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_{j'}$ is a cycle. It follows from $L(D) = \{p, q\}$ and $\eta(v_{j'} P_{C'_p} u P_{C_p} v_i\ C_q^{(0)} v_{j'}) = 2p$ that

$$\eta(v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_{j'}) = q,$$

and so $q = 2p$, which contradicts that $(p, q) = 1$. Therefore, no common internal vertex of $v_j P_{C_p} v_i$, $v_{j'} P_{C'_p} v_{i'}$ exists.

(ii) If $[i, j] \cap [i', j'] \ne \emptyset$, then either $i \le i' \le j$ or $i' \le i \le j'$. Without loss of generality we assume that $i \le i' \le j$. If $j' \le j$, then

$\max\{j,j'\} - \min\{i,i'\} = j - i \leq p-1$. If $j' > j$ and u is a common internal vertex of $v_j P_{C_p} v_i$, $v_{j'} P_{C'_p} v_{i'}$, then $v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_{j'}$ is a cycle (the method of the proof is the same as in (i)). Note that any arc of the cycle $v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_{j'}$ belong to either the p-cycle $v_i C_q^{(0)} v_j P_{C_p} v_i$ or the p-cycle $v_{i'} C_q^{(0)} v_{j'} P_{C'_p} v_{i'}$. By Lemma 2.8, the cycle $v_{j'} P_{C'_p} u P_{C_p} v_i C_q^{(0)} v_{j'}$ is not q-cycle, and so it must be a p-cycle. Therefore $\max\{j,j'\} - \min\{i,i'\} = j' - i \leq p-1$.

We have completed the proof of Lemma 2.11. □

Lemma 2.12 *Let $D \in PMSD_n$ and $L(D) = \{p,q\}$ with $3 \leq p < q$ and $p+q > n$, and let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be a q-cycle of D and σ a path in C_q. If there is a consecutive p-cycles cover of (C_q,σ) in D, then $\eta(\sigma) \leq \min\{q-2, (n-q)(p-2)\}$.*

Proof. By Lemma 2.8, in C_q there exist two arcs not being in any p-cycle. Without loss of generality we assume that (v_q, v_1), (v_a, v_{a+1}) are the two arcs, and $\sigma \subseteq v_1 C_q^{(0)} v_a$. Let T be a consecutive p-cycles cover of (C_q,σ). We remove those superfluous p-cycles in T to obtain an irreducible consecutive p-cycles cover T_1 ($\subseteq T$) of (C_q,σ). Afterwards, we get an irreducible consecutive p-cycles chain $C_p^1, C_p^2, \ldots, C_p^t$ of (C_q,σ) by properly arranging the order of the p-cycles of T_1. Let $C_p^l = v_{i_l} C_q^{(0)} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ $(l = 1,2,\ldots,t)$. By Lemma 2.5,

$$i_1 < i_2 \leq j_1 < i_3 \leq j_2 < \cdots < i_{t-1} \leq j_{t-2} < i_t \leq j_{t-1} < j_t.$$

We first prove that $\bigcup_{l=1}^{t} C_p^l$ contains at least t vertices not in $V(C_q)$. It suffices to prove that $\bigcup_{l=1}^{t} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ contains at least t vertices not in $V(C_q)$. Clearly, $v_{j_1} P_{C_p^1 \bar{C}_q} v_{i_1}$ contains at least a vertex not in $V(C_q)$ by D ministrong. Suppose that for $k \in \{1,2,\ldots,t-1\}$, $\bigcup_{l=1}^{k} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ contains at least k vertices not in $V(C_q)$. We prove that $\bigcup_{l=1}^{k+1} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ contains at least $k+1$ vertices not in C_q. For each $l \in \{1,2,\ldots,k-1\}$, since $[i_l, j_l] \cap [i_{k+1}, j_{k+1}] = \emptyset$, by (i) of Lemma 2.11, there is no common internal vertex in $v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ and $v_{j_{k+1}} P_{C_p^{k+1} \bar{C}_q} v_{i_{k+1}}$. If there is no common internal vertex in $v_{j_k} P_{C_p^k \bar{C}_q} v_{i_k}$ and $v_{j_{k+1}} P_{C_p^{k+1} \bar{C}_q} v_{i_{k+1}}$, then $\bigcup_{l=1}^{k+1} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ contains at least $k+1$ vertices not in $V(C_q)$ by D ministrong. If there exists some common internal vertices in $v_{j_k} P_{C_p^k \bar{C}_q} v_{i_k}$ and

$v_{j_{k+1}} P_{C_p^{k+1}\bar{C}_q} v_{i_{k+1}}$, then $v_{j_{k+1}} P_{C_p^{k+1}\bar{C}_q} v_{i_{k+1}}$ contains at least an internal vertex which is different from those common internal vertices by D ministrong, and so $\bigcup_{l=1}^{k+1} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ contains at least $k+1$ vertices not in $V(C_q)$. By induction, $\bigcup_{l=1}^{t} v_{j_l} P_{C_p^l \bar{C}_q} v_{i_l}$ contains at least t vertices not in $V(C_q)$.

Now we prove $\eta(\sigma) \le \min\{q-2, (n-q)(p-2)\}$. Note that in D there are precisely $n-q$ vertices not in the q-cycle C_q. By the above arguments, any irreducible consecutive p-cycles chain contains at most $n-q$ p-cycles. Hence $t \le n-q$. Since $\eta(v_{i_l} C_q^{(0)} v_{j_l}) \le p-2$ $(l = 1, 2, \ldots, t)$, then

$$\begin{aligned}\eta(\sigma) &= \eta(v_{i_1} C_q^{(0)} v_{j_t}) \le \sum_{l=1}^{t} \eta(v_{i_l} C_q^{(0)} v_{j_l}) \\ &\le \sum_{l=1}^{t} (p-2) = t(p-2) \le (n-q)(p-2).\end{aligned}$$

We can check from Lemma 2.8 that $\eta(\sigma) \le q-2$, and so

$$\eta(\sigma) \le \min\{q-2, (n-q)(p-2)\}.$$

The proof of Lemma 2.12 is complete. □

Theorem 2.1 *Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \le p < q$ and $p + q > n$. Then*

$$\exp_D(1) \ge \max\{(p-1)(q-1)+1, p(q-1)-(n-q)(p-2)\}.$$

Proof. Let $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be any q-cycle of D. By Lemma 2.8, in C_q there exist at least two arcs not belonging to any p-cycle. Without loss of generality we assume that the arc (v_q, v_1) does not belong to any p-cycle. Then C_q can be expressed as

$$C_q = v_1 C_q^{(0)} v_{l_1} C_q^{(0)} v_{l_1+1} C_q^{(0)} v_{l_2} C_q^{(0)} v_{l_2+1} \cdots v_{l_{k-1}} C_q^{(0)} v_{l_{k-1}+1} C_q^{(0)} v_{l_k} C_q^{(0)} v_{l_k+1}$$

(where $k \ge 2$, $l_k = q$, $v_{q+1} = v_1$) such that each arc in $v_1 C_q^{(0)} v_{l_1}$, $v_{l_i+1} C_q^{(0)} v_{l_{i+1}}$ $(i = 1, 2, \ldots k-1)$ is in some p-cycle, and for each $i \in \{1, 2, \ldots, k\}$, $v_{l_i} C_q^{(0)} v_{l_i+1} = (v_{l_i}, v_{l_i+1})$ is not in any p-cycle. We first prove that

$$\eta(v_1 C_q^{(0)} v_{l_1}) \le \min\{q-2, (n-q)(p-2)\}.$$

If $l_1 = 1$, then $\eta(v_1 C_q^{(0)} v_{l_1}) = 0 \le \min\{q-2, (n-q)(p-2)\}$. If $l_1 > 1$, for each $i \in [1, \ldots l_1 - 1]$, let (v_i, v_{i+1}) be an arc of the p-cycle C_p^i (perhaps $C_p^i = C_p^j$ for $i \ne j$), and let h_i and j_i be respectively the least and the greatest subscript of the vertices in $V(C_p^i) \cap \{v_1, v_2, \ldots, v_{l_1}\}$. By (i),(iv) of Lemma 2.10, C_p^i can be expressed as $C_p^i = v_{h_i} P_{C_p^i} v_{j_i} P_{C_p^i \bar{C}_q} v_{h_i}$ and $\eta(v_{h_i} P_{C_p^i} v_{j_i}) = \eta(v_{h_i} C_q^{(0)} v_{j_i})$, where $1 \le h_i < j_i \le l_1$. Hence $\bar{C}_p^i = v_{h_i} C_q^{(0)} v_{j_i} P_{C_p^i \bar{C}_q} v_{h_i}$ is a p-cycle (a consecutive p-cycle) containing the arc (v_i, v_{i+1}), and thus $T = \{\bar{C}_p^i : i = 1, 2, \ldots, l_1 - 1\}$ is a consecutive p-cycles cover of $(C_q, v_1 C_q^{(0)} v_{l_1})$. By Lemma 2.12,

$$\eta(v_1 C_q^{(0)} v_{l_1}) \le \min\{q-2, (n-q)(p-2)\}.$$

Similarly

$$\eta(v_{l_i+1} C_q^{(0)} v_{l_{i+1}}) \le \min\{q-2, (n-q)(p-2)\} (i = 1, 2, \ldots, k-1).$$

Next we prove that for any $v \in V(C_q)$,

$$\exp_D(v) \ge \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}.$$

Let $v_1 W v_q$ be any walk from v_1 to v_q in D. By Lemma 2.9, $\eta(v_1 W v_q) = \mu_1 p + \mu_2 q + \eta(v_1 C_q^{(0)} v_q)$, where μ_1, μ_2 are nonnegative integers. By Lemma 2.2, $\exp_D(v_1, v_q) \ge \eta(v_1 C_q^{(0)} v_q) + \phi_{L(D)} = \phi_{L(D)} + q - 1$. Hence

$$\begin{aligned} \exp_D(v_1) &= \max\{\exp_D(v_1, \omega) : \omega \in V(D)\} \\ &\ge \exp_D(v_1, v_q) \ge \phi_{L(D)} + q - 1. \end{aligned}$$

Let i be any integer in $[1, \ldots, l_1]$. Clearly for any positive integer x, $R_x(v_i) \subseteq R_{x+(i-1)}(v_1)$. Hence

$$V = R_{\exp_D(v_i)}(v_i) \subseteq R_{\exp_D(v_i)+(i-1)}(v_1),$$

it follows that $\exp_D(v_1) \le \exp_D(v_i) + (i-1)$. Thus

$$\exp_D(v_i) \ge \exp_D(v_1) - (i-1) \ge \phi_{L(D)} + q - 1 - (i-1).$$

By Lemma 2.12, $i - 1 \le l_1 - 1 \le \min\{q-2, (n-q)(p-2)\}$. Hence

$$\exp_D(v_i) \ge \phi_{L(D)} + q - 1 - \min\{q-2, (n-q)(p-2)\},$$

and so, for each $v \in V(v_1 C_q^{(0)} v_{l_1})$,

$$\exp_D(v) \ge \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}.$$

By similar argument, for each $v \in V(v_{l_i+1} C_q^{(0)} v_{l_{i+1}})$ $(i = 1, 2, \ldots, k-1)$,

$$\exp_D(v) \ge \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}.$$

Therefore for any $v \in V(C_q)$,

$$\exp_D(v) \geq \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}.$$

Now we show that for any vertex v not in any q-cycle,

$$\exp_D(v) \geq \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}.$$

Let $X = \{x : x \in [1, \ldots, q]$ and there is a path from v to v_x whose internal vertices are not in $V(C_q)\}$ and $Y = \{y : y \in [1, \ldots, q]$ and there is a path from v_y to v whose internal vertices are not in $V(C_q)\}$. Let $b = \max\{x : x \in X\}$ and $c = \min\{y : y \in Y\}$. Then no common internal vertex exists in $vP_{\bar{C}_q}v_b$, $v_cP_{\bar{C}_q}v$. Otherwise, if u is a common internal vertex in $vP_{\bar{C}_q}v_b$, $v_cP_{\bar{C}_q}v$, then $vP_{\bar{C}_q}uP_{\bar{C}_q}v$ contains a cycle whose vertices are not in $V(C_q)$, which contradicts that $L(D) = \{p, q\}$ and $p + q > n$. Hence $vP_{\bar{C}_q}v_bC_q^{(0)}v_cP_{\bar{C}_q}v$ is a cycle (is a p-cycle since it contains vertex v). Similarly, for any $x \in X$ and $y \in Y$, $vP_{\bar{C}_q}v_xC_q^{(0)}v_cP_{\bar{C}_q}v$, $vP_{\bar{C}_q}v_bC_q^{(0)}v_yP_{\bar{C}_q}v$ are p-cycles. Note that $v_bC_q^{(0)}v_c$ is a path of the p-cycle $vP_{\bar{C}_q}v_bC_q^{(0)}v_cP_{\bar{C}_q}v$. Hence $v_bC_q^{(0)}v_c$ does not contain the arc (v_{l_i}, v_{l_i+1}) $(i = 1, 2, \ldots, k)$. It follows that $v_bC_q^{(0)}v_c$ is contained in one of the paths $v_1C_q^{(0)}v_{l_1}$, $v_{l_1+1}C_q^{(0)}v_{l_2}$, $v_{l_2+1}C_q^{(0)}v_{l_3}, \ldots, v_{l_{k-1}+1}C_q^{(0)}v_{l_k}$. Without loss of generality we assume that $v_bC_q^{(0)}v_c \subseteq v_1C_q^{(0)}v_{l_1}$. Then $1 \leq b \leq c \leq l_1$, and so $v_xC_q^{(0)}v_c \subseteq v_1C_q^{(0)}v_{l_1}$, $v_bC_q^{(0)}v_y \subseteq v_1C_q^{(0)}v_{l_1}$. Namely, $1 \leq x \leq c \leq l_1$ and $1 \leq b \leq y \leq l_1$. From the definition of b, c, we have $b \geq x$ and $c \leq y$. Therefore $1 \leq x \leq b \leq c \leq y \leq l_1$. Let vWv_q be any walk from v to v_q. Then vWv_q can be expressed as $vWv_q = vP_{\bar{C}_q}v_xWv_q$, where x is a vertex in X, v_xWv_q is a subwalk of vWv_q which is obtained by removing the path $vP_{\bar{C}_q}v_x$ from vWv_q. By Lemma 2.9, $\eta(v_xWv_q) = \mu_1p + \mu_2q + \eta(v_xC_q^{(0)}v_q)$, where μ_1, μ_2 are nonnegative integers. Hence

$$\begin{aligned}
&\eta(vWv_q) = \eta(vP_{\bar{C}_q}v_x) + \mu_1p + \mu_2q + \eta(v_xC_q^{(0)}v_q) \\
&= \eta(vP_{\bar{C}_q}v_x) + \mu_1p + \mu_2q + \eta(v_xC_q^{(0)}v_c) + \eta(v_cC_q^{(0)}v_q) \\
&= \mu_1p + \mu_2q + \eta(vP_{\bar{C}_q}v_xC_q^{(0)}v_cP_{\bar{C}_q}v) - \eta(v_cP_{\bar{C}_q}v) + \eta(v_cC_q^{(0)}v_q) \\
&= \mu_1p + \mu_2q + p - \eta(v_cP_{\bar{C}_q}v) + (q-c).
\end{aligned}$$

Note that for any path $v_cP_{\bar{C}_q}v$ from v_c to v whose internal vertices are not in $V(C_q)$, $\eta(vP_{\bar{C}_q}v_x\ C_q^{(0)}v_cP_{\bar{C}_q}v) = p$. Hence $\eta(v_cP_{\bar{C}_q}v)$ is a constant, and so $p - \eta(v_cP_{\bar{C}_q}v) + q - c$ is a constant. By Lemma 2.2,

$$\exp_D(v, v_q) \geq \phi_{L(D)} + p - \eta(v_cP_{\bar{C}_q}v) + (q-c).$$

By Lemma 2.12, $l_1 - 1 \leq \min\{q-2, (n-q)(p-2)\}$. Hence

$$\begin{aligned} q - c \geq q - l_1 &= q - 1 - (l_1 - 1) \\ &\geq q - 1 - \min\{q-2, (n-q)(p-2)\} \\ &= \max\{1, q-1-(n-q)(p-2)\}. \end{aligned}$$

Note that $\eta(v_c P_{\bar{C}_q} v) \leq p - 1$. It follows that

$$\begin{aligned} \exp_D(v, v_q) &\geq \phi_{L(D)} + p - (p-1) + \max\{1, (q-1) - (n-q)(p-2)\} \\ &> \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}, \end{aligned}$$

and so

$$\begin{aligned} \exp_D(v) &= \max\{\exp_D(v, \omega) : \omega \in V(D)\} \geq \exp_D(v, v_q) \\ &> \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}. \end{aligned}$$

To sum up, for any $v \in V(D)$,

$$\exp_D(v) \geq \max\{\phi_{L(D)} + 1, \phi_{L(D)} + q - 1 - (n-q)(p-2)\}.$$

Therefore

$$\exp_D(1) \geq \max\{(p-1)(q-1) + 1, p(q-1) - (n-q)(p-2)\}.$$

The proof of Theorem 2.1 is complete. □

3 the 1-exponent set

Lemma 3.1 [3] *Let D be a primitive digraph on n vertices and $L(D) = \{p, q\}$ with $p < q$ and $p + q > n$. Then*

$$(p-1)(q-1) \leq \exp_D(1) \leq (p-1)(q-1) + n - p.$$

By Lemma 3.1 and Theorem 2.1, we have

Theorem 3.1 *Let $D \in PMSD_n$ and $L(D) = \{p, q\}$ with $3 \leq p < q$ and $p + q > n$.*
(i) *If $q + \lceil (q-2)/(p-2) \rceil \leq n$, then*
$(p-1)(q-1) + 1 \leq \exp_D(1) \leq (p-1)(q-1) + n - p.$
(ii) *If $q + \lceil (q-2)/(p-2) \rceil > n$, then*
$p(q-1) - (n-q)(p-2) \leq \exp_D(1) \leq (p-1)(q-1) + n - p.$

Proof. (i) If $q + \lceil (q-2)/(p-2) \rceil \leq n$, then $\frac{q-2}{p-2} \leq \lceil \frac{q-2}{p-2} \rceil \leq n - q$, and so $q - 2 \leq (n-q)(p-2)$. By Lemma 3.1 and Theorem 2.1,

$$(p-1)(q-1) + 1 \leq \exp_D(1) \leq (p-1)(q-1) + n - p.$$

(ii) If $q + \lceil (q-2)/(p-2) \rceil > n$, then $\frac{q-2}{p-2} > n - q$ since $n - q$ is an integer, and so $q - 2 > (n-q)(p-2)$. By Lemma 3.1 and Theorem 2.1,

$$p(q-1) - (n-q)(p-2) \leq \exp_D(1) \leq (p-1)(q-1) + n - p.$$

The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Let p, q be two integers with $3 \leq p < q \leq n-1$, $(p,q) = 1$ and $p + q > n$. Then for each $m \in [(p-1)(q-1) + q - p + 1, \ldots, (p-1)(q-1) + n - p]$, there exists a primitive, minimally strong digraph D with n vertices and $L(D) = \{p, q\}$ such that $\exp_D(1) = m$.*

Proof. For each $m \in [(p-1)(q-1)+q-p+1, \ldots, (p-1)(q-1)+n-p]$, there exists an unique integer $a \in [q-p+1, \ldots, n-p]$ such that $m = (p-1)(q-1) + a$. Let $D = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and

$$\begin{aligned} E = \{(v_i, v_{i+1}) : i = 1, 2, \ldots, q, \ldots, a+p-1\} \cup \{(v_q, v_1), (v_{a+p}, v_{a+1})\} \\ \cup\{(v_q, v_i), (v_i, v_2) : i = a+p+1, a+p+2, \ldots, n \text{ if } a+p < n\}, \end{aligned}$$

where $q - p + 1 \leq a \leq n - p$. Then $D \in PMSD_n$ and $L(D) = \{p, q\}$. Note that $d_{L(D)}(v_1, v_a) = a - 1 + q$ and the length of any walk $v_1 W v_a$ from v_1 to v_a of length at least $d_{L(D)}(v_1, v_a)$ can be represented in the form $z_1 p + z_2 q + d_{L(D)}(v_1, v_a)$. By Lemma 2.3,

$$\exp_D(v_1, v_a) = d_{L(D)}(v_1, v_a) + \phi_{L(D)} = q + a - 1 + (p-1)(q-1).$$

Hence

$$\begin{aligned} \exp_D(v_1) &= \max\{\exp_D(v_1, v) : v \in V(D)\} \\ &= \exp_D(v_1, v_a) = (p-1)(q-1) + q + a - 1. \end{aligned}$$

Let x be any positive integer, we have

$R_{x+1}(v_i) = R_x(v_{i+1})$ for each $i \in [1, \ldots, q-1] \cup [q+1, \ldots, a+p-1]$;
$R_{x+1}(v_j) = R_x(v_2)$ for each $j \in \{1\} \cup [a+p+1, \ldots, n]$;
$R_{x+1}(v_{a+p}) = R_x(v_{a+1})$. Hence

$\exp_D(v_{i+1}) = \exp_D(v_i) - 1$ for $i \in [1, \ldots, q-1] \cup [q+1, \ldots, a+p-1]$,
$\exp_D(v_2) = \exp_D(v_j) - 1$ for $j \in \{1\} \cup [a+p+1, \ldots, n]$,
$\exp_D(v_{a+1}) = \exp_D(v_{a+p}) - 1$.

Therefore

$$\begin{aligned} \exp_D(1) &= \exp_D(v_q) = \exp_D(v_1) - (q-1) \\ &= (p-1)(q-1) + a = m. \end{aligned}$$

The proof of Theorem 3.2 is complete. □

Lemma 3.2 *Let $D = C_q \cup C_p^1 \cup C_p^2 \cup \cdots \cup C_p^k$ be a primitive digraph with $3 \le p < q$, where $C_q = (v_1, v_2, \dots v_q, v_1)$ is a q-cycle, $C_p^1, C_p^2, \dots C_p^k$ an irreducible consecutive p-cycles chain of $(C_q, v_1 C_q^{(0)} v_t)$ $(1 < t \le q)$, and for any distinct $i, j \in \{1, 2, \dots, k\}$, $V(C_p^i) \cap V(C_p^j) \subset V(C_q)$. Then $\exp_D(v_i) = \exp_D(v_{i+1}) + 1$ for each $i \in [1, \dots, t-1]$.*

Proof. Let $C_p^i = v_{s_i} C_q^{(0)} v_{l_i} P_{C_p^i \bar{C}_q} v_{s_i}$ $(i = 1, 2, \dots, k)$. By Lemma 2.5, $1 = s_1 < s_2 \le l_1 < s_3 \le l_2 < \cdots < s_{k-1} \le l_{k-2} < s_k \le l_{k-1} < l_k = t$. Clearly $L(D) = \{p, q\}$. It is easy to see that for any positive integer x and each $i \in [1, \dots, l_1 - 1] \cup \bigcup_{j=1}^{k-1} [l_j + 1, \dots, l_{j+1} - 1]$, $R_{x+1}(v_i) = R_x(v_{i+1})$. Hence for each $i \in [1, \dots, l_1 - 1] \cup \bigcup_{j=1}^{k-1} [l_j + 1, \dots, l_{j+1} - 1]$, $\exp_D(v_i) = \exp_D(v_{i+1}) + 1$. Thus it suffices to prove that

$$\exp_D(v_{l_i}) = \exp_D(v_{l_i+1}) + 1 \ \ (i = 1, 2, \dots, k-1).$$

We first prove that

$$R_{ip-(l_i-l_1)}(v_{l_i}) = R_{ip-(l_i-l_1)-1}(v_{l_i+1}) \ \ (i = 1, 2, \dots, k-1). \tag{3.1}$$

Let u_i be the terminal vertex of the arc in C_p^i with the initial vertex v_{l_i} $(i = 1, 2, \dots, k)$. It is easy to see that

$$\begin{aligned} R_p(v_{l_1}) &= R_{p-1}(v_{l_1+1}) \cup R_{p-1}(u_1) \\ &= R_{p-1}(v_{l_1+1}) \cup \{v_{l_1}\} \\ &= R_{p-1}(v_{l_1+1}) \ \text{ since } \ v_{l_1} \in R_{p-1}(v_{l_1+1}). \end{aligned}$$

So (3.1) holds for $i = 1$. Since

$$\begin{aligned} R_{2p-(l_2-l_1)}(v_{l_2}) &= R_{2p-(l_2-l_1)-1}(v_{l_2+1}) \cup R_{2p-(l_2-l_1)-1}(u_2) \\ &= R_{2p-(l_2-l_1)-1}(v_{l_2+1}) \cup R_p(v_{l_1}), \end{aligned}$$

$$\begin{aligned} R_p(v_{l_1}) &= R_{p-1}(v_{l_1+1}) \cup R_{p-1}(u_1) \\ &= R_{p-(l_2-l_1)}(R_{l_2-l_1-1}(v_{l_1+1})) \cup \{v_{l_1}\} \\ &= R_{p-(l_2-l_1)}(v_{l_2}) \cup \{v_{l_1}\} = R_{p-(l_2-l_1)}(v_{l_2}), \end{aligned} \tag{3.2}$$

and

$$R_{2p-(l_2-l_1)-1}(v_{l_2+1}) = R_{p-(l_2-l_1)}(R_{p-1}(v_{l_2+1})) \supseteq R_{p-(l_2-l_1)}(v_{l_2}),$$

then $R_{2p-(l_2-l_1)}(v_{l_2}) = R_{2p-(l_2-l_1)-1}(v_{l_2+1})$. Namely (3.1) holds for $i = 2$.

Suppose that $i \in [3, \ldots, k-1]$. Since

$$R_{ip-(l_i-l_1)}(v_{l_i}) = R_{ip-(l_i-l_1)-1}(v_{l_i+1}) \cup R_{ip-(l_i-l_1)-1}(u_i) \tag{3.3}$$

and

$$\begin{aligned} R_{ip-(l_i-l_1)-1}(u_i) &= R_{(i-1)p-(l_{i-1}-l_1)}(R_{p-(l_i-l_{i-1})-1}(u_i)) \\ &= R_{(i-1)p-(l_{i-1}-l_1)}(v_{l_{i-1}}), \qquad (3.4) \end{aligned}$$

then

$$R_{ip-(l_i-l_1)}(v_{l_i}) = R_{ip-(l_i-l_1)-1}(v_{l_i+1}) \cup R_{(i-1)p-(l_{i-1}-l_1)}(v_{l_{i-1}}).$$

Since

$$\begin{aligned} & R_{(i-1)p-(l_{i-1}-l_1)}(v_{l_{i-1}}) \\ =\ & R_{(i-1)p-(l_{i-1}-l_1)-1}(v_{l_{i-1}+1}) \cup R_{(i-1)p-(l_{i-1}-l_1)-1}(u_{i-1}) \\ =\ & R_{(i-1)p-(l_i-l_1)}(R_{l_i-l_{i-1}-1}(v_{l_{i-1}+1})) \\ & \qquad \cup R_{(i-2)p-(l_{i-2}-l_1)}(R_{p-(l_{i-1}-l_{i-2}-1)}(u_{i-1})) \\ =\ & R_{(i-1)p-(l_i-l_1)}(v_{l_i}) \cup R_{(i-2)p-(l_{i-2}-l_1)}(v_{l_{i-2}}), \end{aligned}$$

and similarly

$$R_{(i-2)p-(l_{i-2}-l_1)}(v_{l_{i-2}}) = R_{(i-2)p-(l_{i-1}-l_1)}(v_{l_{i-1}}) \cup R_{(i-3)p-(l_{i-3}-l_1)}(v_{l_{i-3}}),$$

....

$$R_{2p-(l_2-l_1)}(v_{l_2}) = R_{2p-(l_3-l_1)}(v_{l_3}) \cup R_p(v_{l_1}).$$

It follows from (3.2) that

$$\begin{aligned} R_{(i-1)p-(l_{i-1}-l_1)}(v_{l_{i-1}}) &= (\bigcup_{j=3}^{i} R_{(j-1)p-(l_j-l_1)}(v_{l_j})) \cup R_p(v_{l_1}) \\ &= (\bigcup_{j=2}^{i} R_{(j-1)p-(l_j-l_1)}(v_{l_j})) \cup \{v_{l_1}\}. \end{aligned}$$

Since for each $j \in \{3, 4, \ldots, i\}$,

$$\begin{aligned} R_{(j-1)p-(l_j-l_1)}(v_{l_j}) &= R_{(j-1)p-(l_j-l_1)-1}(R_1(v_{l_j})) \\ &\supseteq R_{(j-1)p-(l_j-l_1)-1}(u_j) \\ &= R_{(j-2)p-(l_{j-1}-l_1)}(R_{p-(l_j-l_{j-1})-1}(u_j)) \\ &= R_{(j-2)p-(l_{j-1}-l_1)}(v_{l_{j-1}}), \end{aligned}$$

and $R_{p-(l_2-l_1)}(v_{l_2}) \supseteq \{v_{l_1}\}$. Hence

$$R_{(i-1)p-(l_{i-1}-l_1)}(v_{l_{i-1}}) = R_{(i-1)p-(l_i-l_1)}(v_{l_i}).$$

From (3.3) and (3.4), we have

$$R_{ip-(l_i-l_1)}(v_{l_i}) = R_{ip-(l_i-l_1)-1}(v_{l_i+1}) \cup R_{(i-1)p-(l_i-l_1)}(v_{l_i}).$$

Since

$$\begin{aligned} R_{ip-(l_i-l_1)-1}(v_{l_i+1}) &= R_{(i-1)p-(l_i-l_1)}(R_{p-1}(v_{l_i+1})) \\ &\supseteq R_{(i-1)p-(l_i-l_1)}(v_{l_i}) \text{ for each } i \in [3,\dots,k-1], \end{aligned}$$

then (3.1) holds for each $i \in [3,\dots,k-1]$.

Summing, (4.1) holds for each $i \in [1,\dots,k-1]$.

Now we prove that

$$\exp_D(v_{l_i}) = \exp_D(v_{l_i+1}) + 1 \text{ for } i = 1,2,\dots,k-1.$$

By (3.1), we have

$$R_x(v_{l_i}) = R_{x-1}(v_{l_i+1}) \text{ for } i \in \{1,2,\dots,k-1\} \text{ and } x \geq ip-(l_i-l_1).$$

Let $v_{l_i}Wv_q$ be any walk from v_{l_i} to v_q in D. We can check that

$$\eta(v_{l_i}Wv_q) = \mu_1 p + \mu_2 q + q - l_i,$$

where μ_1, μ_2 are nonnegative integers. By Lemma 2.2,

$$\begin{aligned} \exp_D(v_{l_i}, v_q) &\geq \phi_{L(D)} + q - l_i \\ &= (p-1)(q-1) + q - l_i = pq - p - l_i + 1. \end{aligned}$$

Hence

$$\begin{aligned} \exp_D(v_{l_i}) &= \max\{\exp_D(v_{l_i}, v) : v \in V(D)\} \\ &\geq \exp_D(v_{l_i}, v_q) \geq pq - p - l_i + 1. \end{aligned}$$

Since $q \geq t = l_k \geq k+1$, then for each $i \in \{1,2,\dots,k-1\}$, $(pq-p-l_i+1) - [ip-(l_i-l_1)] = (q-i-1)p - l_1 + 1 \geq (q-k)p - l_1 + 1 \geq p - (l_1-1) > 0$, and so

$$pq - p - l_i + 1 > ip - (l_i - l_1) \ (i = 1,2,\dots,k-1).$$

It follows that $\exp_D(v_{l_i}) > ip - (l_i - l_1)$ $(i = 1,2,\dots,k-1)$. Hence

$$V = R_{\exp_D(v_{l_i})}(v_{l_i}) = R_{\exp_D(v_{l_i})-1}(v_{l_i+1}) \ (i = 1,2,\dots,k-1),$$

and so

$$\exp_D(v_{l_i+1}) \le \exp_D(v_{l_i}) - 1 \ \ (i = 1, 2, \ldots, k-1).$$

Namely

$$\exp_D(v_{l_i}) \ge \exp_D(v_{l_i+1}) + 1 \ \ (i = 1, 2, \ldots, k-1).$$

On the other hand, since

$$\begin{aligned} R_{\exp_D(v_{l_i+1})+1}(v_{l_i}) &= R_{\exp_D(v_{l_i+1})}(R_1(v_{l_i})) \\ &\supseteq R_{\exp_D(v_{l_i+1})}(v_{l_i+1}) \\ &= V \quad \text{for each } i \in \{1, 2, \ldots, k-1\}, \end{aligned}$$

then

$$\exp_D(v_{l_i}) \le \exp_D(v_{l_i+1}) + 1 \ \ (i = 1, 2, \ldots, k-1),$$

and so

$$\exp_D(v_{l_i}) = \exp_D(v_{l_i+1}) + 1 \ \ (i = 1, 2, \ldots, k-1).$$

Consequently

$$\exp_D(v_i) = \exp_D(v_{i+1}) + 1 \quad (i = 1, 2, \ldots, t-1).$$

The proof of Lemma 3.2 is complete. □

Theorem 3.3 *Let p, q be two integers with $3 \le p < q \le n-1$, $(p, q) = 1$, $p+q > n$, and $q + \lceil \frac{q-2}{p-2} \rceil \le n$. Then for each $m \in [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+q-p+1]$, there exists a primitive, ministrong digraph D with n vertices and $L(D) = \{p, q\}$ such that $\exp_D(1) = m$.*

Proof. For each $m \in [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+q-p+1]$, there exists an unique integer $a \in [1, \ldots, q-p+1]$ such that $m = (p-1)(q-1)+a$, and for such integer a, there exist an unique integer $k (= \lceil \frac{q-a-1}{p-2} \rceil)$ such that $1 + (k-1)(p-2) < q - a \le 1 + k(p-2)$. Clearly

$$q + k = q + \lceil \frac{q-a-1}{p-2} \rceil \le q + \lceil \frac{q-2}{p-2} \rceil \le n.$$

Let $D = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = E_1 \cup E_2 \cup E_3$, where

$$\begin{aligned} E_1 &= \{(v_i, v_{i+1}) : i = 1, 2, \ldots, q-1\} \cup \{(v_q, v_1)\}, \\ E_2 &= \{(v_{1+i(p-2)}, v_{q+i}), (v_{q+i}, v_{1+(i-1)(p-2)}) : i = 1, 2, \ldots, k-1\} \\ &\quad \cup \{(v_{q-a}, v_{q+k}), (v_{q+k}, v_{q-a-p+2})\} \\ E_3 &= \{(v_{p-1}, v_i), (v_i, v_1) : i \in [q+k+1, \ldots, n]\}. \end{aligned}$$

Let $D' = (V', E')$ with $V' = \{v_1, v_2, \ldots, v_{q+k}\}$ and $E' = E_1 \cup E_2$. It is not difficult to see that $D(D')$ is strongly connected and $L(D) = L(D') =$

$\{p, q\}$. Hence $D(D')$ is primitive for p and q being coprime. We can check that each digraph obtained from $D(D')$ by removal of an arc is not strongly connected. Hence D, D' are primitive, minimally strong digraphs with $L(D) = L(D') = \{p, q\}$. Clearly we have

$$\begin{aligned} \exp_D(v_i) &= \exp_D(v_{q+1}) \quad (i = q+k+1, q+k+2, \ldots, n), \\ \exp_D(v_i) &= \exp_{D'}(v_i) \quad (i = 1, 2, \ldots, q+k). \end{aligned}$$

It follows that $\exp_D(1) = \exp_{D'}(1)$. Let

$$\begin{aligned} C_q &= (v_1, v_2, \ldots, v_q, v_1), \\ C_p^i &= v_{1+(i-1)(p-2)} C_q^{(0)} v_{1+i(p-2)} \cup (v_{1+i(p-2)}, v_{q+i}, v_{1+(i-1)(p-2)}) \\ & \qquad\qquad (i = 1, 2, \ldots, k-1), \\ C_p^k &= v_{q-a-p+2} C_q^{(0)} v_{q-a} \cup (v_{q-a}, v_{q+k}, v_{q-a-p+2}). \end{aligned}$$

Then $D' = C_q \cup C_p^1 \cup C_p^2 \cup \cdots \cup C_p^k$, and $C_p^1, C_p^2, \ldots, C_p^k$ is an irreducible consecutive p-cycles chain of $(C_q, v_1 C_q^{(0)} v_{q-a})$, and for any distinct $i, j \in \{1, 2, \ldots, k\}$, $V(C_p^i) \cap V(C_p^j) \subset V(C_q)$. We can check that

$$\begin{aligned} &\exp_{D'}(v_q) = \exp_{D'}(v_{q+1}), \\ &\exp_{D'}(v_{q+i}) = \exp_{D'}(v_{(i-1)(p-2)}) \quad (i = 2, 3, \ldots, k-1), \\ &\exp_{D'}(v_{q+k}) = \exp_{D'}(v_{q-a-p+1}), \\ &\exp_{D'}(v_i) = \exp_{D'}(v_{i+1}) + 1 \quad (i = q-a+1, q-a+2, \ldots, q-1). \end{aligned}$$

It follows from Lemma 3.2 that

$$\exp_{D'}(1) = \exp_{D'}(v_{q-a}) = \exp_{D'}(v_1) - (q-a-1).$$

Let $v_1 W v_q$ be any walk in D' from v_1 to v_q. Then $\eta(v_1 W v_q)$ can be expressed as

$$\eta(v_1 W v_q) = \mu_1 p + \mu_2 q + d_{L(D')}(v_1, v_q),$$

where μ_1, μ_2 are nonnegative integers. By Lemma 2.3,

$$\exp_{D'}(v_1, v_q) = \phi_{L(D')} + d_{L(D')}(v_1, v_q) = (p-1)(q-1) + q - 1,$$

then

$$\begin{aligned} \exp_{D'}(v_1) &= \max\{\exp_{D'}(v_1, v) : v \in V(D)\} \\ &= \exp_{D'}(v_1, v_q) = (p-1)(q-1) + q - 1, \end{aligned}$$

and so $\exp_{D'}(1) = (p-1)(q-1) + a = m$. Therefore $\exp_D(1) = m$. The proof of Theorem 3.3 is complete. □

Theorem 3.4 *Let p, q be two integers with $3 \leq p < q \leq n-1$, $(p,q) = 1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil > n$. Then for each $m \in [p(q-1)-(n-q)(p-2), \ldots, (p-1)(q-1)+q-p+1]$, there exists a primitive, minimally strong digraph D with n vertices and $L(D) = \{p, q\}$ such that $\exp_D(1) = m$.*

Proof. For each $m \in [p(q-1)-(n-q)(p-2), \ldots, (p-1)(q-1)+q-p+1]$, there exists an unique integer $a \in [q-1-(n-q)(p-2), \ldots, q-p+1]$ such that $m = (p-1)(q-1)+a$, and for such integer a, there exists an unique integer $k (= \lceil \frac{q-a-1}{p-2} \rceil)$ such that $1+(k-1)(p-2) < q-a \leq 1+k(p-2)$. Clearly

$$q + k = q + \lceil \frac{q-a-1}{p-2} \rceil \leq q + \lceil \frac{(n-q)(p-2)}{p-2} \rceil = n.$$

Let $D = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and

$$\begin{aligned} E &= \{(v_i, v_{i+1}) : i = 1, 2, \ldots, q-1\} \cup \{(v_q, v_1)\} \\ &\cup \{(v_{1+i(p-2)}, v_{q+i}), (v_{q+i}, v_{1+(i-1)(p-2)}) : i = 1, 2, \ldots, k-1\} \\ &\cup \{(v_{q-a}, v_{q+k}), (v_{q+k}, v_{q-a-p+2})\} \\ &\cup \{(v_{p-1}, v_i), (v_i, v_1) : i \in [q+k+1, \ldots, n]\}. \end{aligned}$$

From the proof of Theorem 3.3, D is a primitive, minimally strong digraph with n vertices and $L(D) = \{p, q\}$. By the same argument as in the proof of Theorem 3.3, we obtain that

$$\begin{aligned} \exp_D(1) &= \exp_D(v_{q-a}) = \exp_D(v_1) - (q-a-1) \\ &= (p-1)(q-1) + q - 1 - (q-a-1) \\ &= (p-1)(q-1) + a = m. \end{aligned}$$

The proof of Theorem 3.4 is complete. □

By Theorems 3.1, 3.2 and 3.3, we have

Theorem 3.5 *Let S be the set of 1-exponent of all primitive, minimally strong digraphs with n vertices and $L(D) = \{p, q\}$, where $3 \leq p < q$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil \leq n$. Then $S = [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p]$.*

By Theorems 3.1, 3.2 and 3.4, we have

Theorem 3.6 *Let S be the set of 1-exponent of all primitive, minimally strong digraphs with n vertices and $L(D) = \{p, q\}$, where $3 \leq p < q$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil > n$. Then $S = [p(q-1)-(n-q)(p-2), \ldots, (p-1)(q-1)+n-p]$.*

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