Monochromatic Fibonacci Numbers of Graphs

by

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Abstract

We call the graph G an edge m-coloured if its edges are coloured with m colours. A path (or a cycle) is called monochromatic if all its edges are coloured alike. A subset $S \subseteq V(G)$ is independent by monochromatic paths if for every pair of different vertices from S there is no monochromatic path between them. In [5] it was defined the Fibonacci number of a graph to be the number of all independent sets of G; recall that S is independent if no two of its vertices are adjacent. In this paper we define the concept of a monochromatic Fibonacci number of a graph which gives the total number of monochromatic independent sets of G. Moreover we give the number of all independent by monochromatic paths sets of generalized lexicographic product of graphs using the concept of a monochromatic Fibonacci polynomial of a graph. These results generalize the Fibonacci number of a graph and the Fibonacci polynomial of a graph.

Keywords: independent set by monochromatic paths, Fibonacci number of a graph, Fibonacci polynomial of a graph, counting
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1 Introduction

For concepts not defined here, see [1] and [2]. We consider only finite, undirected, simple graphs. By P_n and C_n for $n \geq 2$ we mean graphs with the vertex sets $V(P_n) = V(C_n) = \{v_1, ..., v_n\}$ and the edge set $E(P_n) = \{\{v_i, v_{i+1}\}; i = 1, ..., n-1\}$ and $E(C_n) = E(P_n) \cup \{v_n, v_1\}$, respectively. Moreover P_1 is a graph with one vertex. Let $X \subset V(G) \cup E(G)$. The notation $G \setminus X$ means the graph obtained from G by deleting the set X. A subset $S \subseteq V(G)$ is independent of G if no two of its vertices are adjacent. Moreover we assume that the subset containing exactly one vertex and the empty set also are independent. In [5] a graph representation of the Fibonacci numbers F_n and the Lucas numbers L_n was presented. It was defined the Fibonacci number of a graph G to be the number of all independent sets S in G and following the Fibonacci number of G was denoted by $F(G)$. It is interesting to know that $F(P_n) = F_{n+1}$, where F_n is the n-th Fibonacci number defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. Moreover for $n \geq 2$ holds $F(C_n) = L_n$, where L_n is the n-th Lucas number defined by $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$. The number $F(C_n)$ for $n \geq 3$ has also another recurrence form $F(C_n) = F(P_{n-1}) + F(P_{n-3})$ with $F(P_n) = F_{n+1}$, $n = 0, 1, 2$. Fibonacci numbers of graphs were investigated for example in [3], [4], [5], [7], [8]. In [4] it was introduced more general concept, namely generalized Fibonacci number of a graph which give the total number of r-independent sets (i.e. generalization of an independent set in distance sense) of a graph G.

In this paper we generalize the concept of Fibonacci numbers of a graph G with respect to sets independent by monochromatic paths. The definition of an independent by monochromatic paths set was introduced in [6]. Let G be an edge coloured graph. A subset $S \subset V(G)$ is independent by monochromatic paths of G if for every pair of different vertices $v_1, v_2 \in S$ there is no monochromatic path between them. Moreover the empty set and the subset containing exactly one vertex also are independent by monochromatic paths of G. Throughout this paper we will write an imp-set of G instead of an independent by monochromatic paths set of G. By $MPF(G)$ we will denote the number of all imp-sets in the graph G and we will call it the monochromatic Fibonacci number of the graph G. The notation $MPF_k(G)$ means the total number of imp-sets with k elements, $k \geq 0$ in the graph G. It is obvious that $MPF(G) = \sum_{k \geq 0} MPF_k(G)$. Moreover for graph G_n on n vertices, $n = 0, 1$ we put that $MPF(G_n) = n + 1$ and also $MPF_0(G_n) = 1$, $MPF_1(G_1) = 1$.

2 The total number of imp-sets of P_n and C_n

Let P_n be an edge coloured graph and $\eta = (H_i)_{i \in \{1,...,t\}}$ be a sequence of monochromatic subpaths of P_n of length n_i, $n_i \geq 1$, $i = 1, ..., t$ such that $H_i \cap H_j \neq \emptyset$ if and only if $j = i + 1$. Then for the graph P_n we put notation $P_n^{n_1,...,n_t}$. Evidently n_1 is the length of the monochromatic path containing the initial vertex of P_n and n_t is the length of the monochromatic path containing the end vertex of P_n. Let $P_n^{n_1,...,n_t}$ be a graph with the vertex set numbered in the natural fashion. Then by a graph $P_{n-p}^{n_1,...,n_j}$, $j \leq t$ we mean a graph isomorphic to $P_n^{n_1,...,n_t} \setminus \{v_n, v_{n-1}, ..., v_{n-p+1}\}$. If $n_i = 1$ for $i = 1, ..., t$ then we will write P_n instead of $P_n^{1,...,1}$.

Theorem 1 *Let $P_n^{n_1,...,n_t}$ be an edge coloured graph on n vertices, $n \geq 2$ and let $k \geq 0$, $t \geq 1$ be integers. Then $MPF_0(P_n^{n_1,...,n_t}) = 1$, $MPF_1(P_n^{n_1,...,n_t}) = n$, for $k \geq 2$ holds $MPF_k(P_n^{n_1}) = 0$ and for $t \geq 2$ we have the formula $MPF_k(P_n^{n_1,...,n_t}) = MPF_k(P_{n-n_t}^{n_1,...,n_{t-1}}) + n_t MPF_{k-1}(P_{n-n_t-1}^{n_1,...,n_{t-1}-1})$.*

P R O O F: The statements $MPF_0(P_n^{n_1,...,n_t}) = 1$ and $MPF_1(P_n^{n_1,...,n_t}) = n$ are obvious. Let $k \geq 2$. If $t = 1$, then all edges of the graph $P_n^{n_1}$ are coloured alike. This implies that every imp-set has at most one vertex, hence for $k > 1$, $MPF_k(P_n^{n_1}) = 0$. Assume now that $t \geq 2$ and $k \geq 2$. Let S be an arbitrary k-element imp-set of a graph $P_n^{n_1,\dots,n_t}$ with the vertex set $V(P_n^{n_1,\dots,n_t})$ numbered in the natural fashion. Evidently n_t is the length of the monochromatic path containing the end vertex of P_n. Because only one vertex from monochromatic path can belong to S, hence we distinguish two possible cases:

Case 1. For every $i = 0, 1, \dots, n_t - 1$ holds $v_{n-i} \notin S$.

Let $\mathcal{S}_1$ be a family of all k-element sets S such that $v_{n-i} \notin S$, for every $i = 0, 1, \dots, n_t - 1$. Hence the definition of the graph $P_n^{n_1,\dots,n_t}$ implies that $S = S'$, where S' is an arbitrary k-element imp-set of the graph $P_n^{n_1,\dots,n_t} \setminus \{v_{n-i}; i = 0, 1, \dots, n_t - 1\}$ isomorphic to $P_{n-n_t}^{n_1,\dots,n_{t-1}}$. In other words $|\mathcal{S}_1| = MPF_k(P_{n-n_t}^{n_1,\dots,n_{t-1}})$.

Case 2. There exists $0 \leq i \leq n_t - 1$ such that $v_{n-i} \in S$.

Let $\mathcal{S}_2$ be a family of all k-element imp-sets S such that there exists $0 \leq i \leq n_t - 1$ where $v_{n-i} \in S$. Because $v_{n-i} \in S$, so all remaining vertices belonging to the monochromatic path H_t do not belong to S. In the other words $v_{n-j} \notin S$ for every $j \in \{0, 1, \dots, n_t\} \setminus \{i\}$. This implies that $S = S^* \cup \{v_{n-i}\}$, where S^* is an arbitrary $(k-1)$-element imp-set of the graph $P_n^{n_1,\dots,n_t} \setminus \{v_{n-s}; s = 0, 1, \dots, n_t\}$ isomorphic to $P_{n-(n_t+1)}^{n_1,\dots,n_{t-1}-1}$. Evidently we have $MPF_{k-1}(P_{n-n_t-1}^{n_1,\dots,n_{t-1}-1})$ sets S. Moreover $0 \leq i \leq n_t - 1$, so we can choose a vertex v_{n-i} belonging to S on n_t ways. Whence by fundamental combinatorial statements $|\mathcal{S}_2| = n_t MPF_{k-1}(P_{n-n_t-1}^{n_1,\dots,n_{t-1}-1})$.

In consequence if $t \geq 2$ and $k \geq 2$ then for the numbers $MPF_k(P_n^{n_1,\dots,n_t})$ we have the recurrence formula $MPF_k(P_n^{n_1,\dots,n_t}) = MPF_k(P_{n-n_t}^{n_1,\dots,n_{t-1}}) + n_t MPF_{k-1}(P_{n-n_t-1}^{n_1,\dots,n_{t-1}-1})$. This completes the proof. □

Theorem 2 *Let $P_n^{n_1,\dots,n_t}$ be an edge coloured graph on n, $n \geq 2$ vertices and let $t \geq 1$ be integer. Then $MPF(P_n^{n_1}) = n + 1$ and for $t \geq 2$ the numbers $MPF(P_n^{n_1,\dots,n_t})$ satisfy the following recurrence $MPF(P_n^{n_1,\dots,n_t}) = MPF(P_{n-n_t}^{n_1,\dots,n_{t-1}}) + n_t MPF(P_{n-n_t-1}^{n_1,\dots,n_{t-1}-1})$.*

P R O O F: If $t = 1$, then $P_n^{n_1}$ is monochromatic, so k can be 0 or 1. Hence $MPF(P_n^{n_1}) = MPF_0(P_n^{n_1}) + MPF_1(P_n^{n_1}) = n+1$, by Theorem 1. If $t \geq 2$, then using the Theorem 1 we have that

$$MPF(P_n^{n_1,\dots,n_t}) = \sum_{k\geq 0} MPF_k(P_n^{n_1,\dots,n_t}) = MPF_0(P_n^{n_1,\dots,n_t}) +$$

$$MPF_1(P_n^{n_1,\dots,n_t}) + \sum_{k\geq 2} \left(MPF_k(P_{n-n_t}^{n_1,\dots,n_{t-1}}) + n_t MPF_{k-1}(P_{n-n_t-1}^{n_1,\dots,n_{t-1}-1}) \right)$$

$$= 1 + n + \sum_{k\geq 2} MPF_k(P_{n-n_t}^{n_1,\dots,n_{t-1}}) + n_t \sum_{k\geq 2} MPF_{k-1}(P_{n-n_t-1}^{n_1,\dots,n_{t-1}-1})$$

$= 1+(n-n_t)+n_t+\sum_{k\geq 2} MPF_k(P_{n-n_t}^{n_1,\ldots,n_{t-1}})+n_t \sum_{r=k-1\geq 1} MPF_r(P_{n-n_t-1}^{n_1,\ldots,n_{t-1}-1})$

$= 1+n-n_t+\sum_{k\geq 2} MPF_k(P_{n-n_t}^{n_1,\ldots,n_{t-1}})+n_t \left(1+\sum_{r\geq 1} MPF_r(P_{n-n_t-1}^{n_1,\ldots,n_{t-1}-1})\right)$

$= \sum_{k\geq 0} MPF_k(P_{n-n_t}^{n_1,\ldots,n_{t-1}}) + n_t \sum_{r\geq 0} MPF_r(P_{n-n_t-1}^{n_1,\ldots,n_{t-1}-1})$

$= MPF(P_{n-n_t}^{n_1,\ldots,n_{t-1}}) + n_t MPF(P_{n-n_t-1}^{n_1,\ldots,n_{t-1}-1})$ as required. Thus the theorem is proved. □

Corollary 1 *For an arbitrary proper edge colouring of the graph P_n, $n \geq 1$ holds $MPF(P_n) = F_{n+1}$.*

P R O O F: Let ζ be a proper edge colouring of the graph P_n. If $n = 1$, then from the definition of the number $MPF(P_n)$ we obtain that $MPF(P_1) = F(P_1) = F_2$. If $n = 2$, then $t = 1$ so $MPF(P_2) = F(P_2) = F_3$ and result is obvious. Let $n \geq 3$. Then $t \geq 2$ and for every two adjacent edges $e_1, e_2 \in E(P_n^{n_1,\ldots,n_t})$ holds $\zeta(e_1) \neq \zeta(e_2)$. The proper edge colouring of the graph P_n implies that all monochromatic paths of P_n have the length equal to 1. Hence the Theorem 2 gives that $MPF(P_n) = MPF(P_{n-2})+MPF(P_{n-1})$, for $n \geq 3$. Moreover by initial condition we obtain that for an arbitrary proper edge colouring of P_n holds $MPF(P_n) = F(P_n) = F_{n+1}$. □

Let C_n be an edge coloured graph and $\mu = (H_i)_{i\in\{1,\ldots,t\}}$ be a sequence of monochromatic paths of C_n of length n_i, $n_i \geq 1$ $i = 1, \ldots, t$, such that $H_i \cap H_j \neq \emptyset$ if and only if $j = i+1$ or $i = 1$ and $j = t$. Then for the graph C_n we put notation $C_n^{n_1,\ldots,n_t}$.

Theorem 3 *Let $C_n^{n_1,\ldots,n_t}$ be an edge coloured graph on n vertices, $n\geq 3$ and let $t\geq 1$, $k\geq 0$ be integers. Then $MPF_0(C_n^{n_1,\ldots,n_t})=1$, $MPF_1(C_n^{n_1,\ldots,n_t})= n$, for $k\geq 2$ holds $MPF_k(C_n^{n_1}) = 0$ and for $t \geq 2$ we have the formula $MPF_k(C_n^{n_1,\ldots,n_t}) =$
$MPF_k(P_{n-1}^{n_1-1,n_2,\ldots,n_{t-1},n_t-1}) + MPF_{k-1}(P_{n-n_1-n_t-1}^{n_2-1,n_3,\ldots,n_{t-2},n_{t-1}-1})$.*

P R O O F: The initial conditions are obvious. Assume that $k \geq 2$ and $t \geq 2$. Let S be an arbitrary k-element imp-set of $C_n^{n_1,\ldots,n_t}$ with the vertex set $V(C_n^{n_1,\ldots,n_t})$ numbered in the natural fashion and assume without loos of generalization that $H_1 \cap H_t = \{v_1\}$. To calculate the number $MPF_k(C_n^{n_1,\ldots,n_t})$ we consider two possible cases:

Case 1. Let $v_1 \notin S$.

Let $\mathcal{S}_1$ be a family of all k-element imp-sets S of $C_n^{n_1,\ldots,n_t}$ such that $v_1 \notin S$. Evidently $S = S'$ where S' is an arbitrary k-element imp-set of the graph $C_n^{n_1,\ldots,n_t} \setminus \{v_1\}$ isomorphic to $P_{n-1}^{n_1-1,n_2,\ldots,n_{t-1},n_t-1}$. In the other words $|\mathcal{S}_1| = MPF_k(P_{n-1}^{n_1-1,n_2,\ldots,n_{t-1},n_t-1})$.

Case 2. Let $v_1 \in S$.

Assume that $\mathcal{S}_2$ is the family of all sets S such that $v_1 \in S$. Because only one vertex from monochromatic path can belong to the set S hence the assumption of the set S gives that $v_i \notin S$ for every $i = n - n_t + 1, ..., n, 2, ..., n_1 + 1$. This implies that $S = S^* \cup \{v_1\}$, where S^* is an arbitrary $(k-1)$-element imp-set of the graph $C_n^{n_1,...,n_t} \setminus \{v_s; s = n - n_t + 1, .., n_1 + 1\}$ isomorphic to $P_{n-n_1-n_t-1}^{n_2-1,n_3,...,n_{t-2},n_{t-1}-1}$. Hence it is clear that $|\mathcal{S}_2| = MPF_{k-1}(P_{n-n_1-n_t-1}^{n_2-1,n_3,...,n_{t-2},n_{t-1}-1})$. All this together completes the proof. □

Using the same method as in the Theorem 2 and in the Corollary 1 we can prove:

Theorem 4 *Let $C_n^{n_1,...,n_t}$ be an edge coloured graph on n vertices, $n \geq 3$ and let $t \geq 1$ be integer. Then $MPF(C_n^{n_1}) = n+1$ and for $t \geq 2$ the numbers $MPF(C_n^{n_1,...,n_t})$ satisfy the following recurrence $MPF(C_n^{n_1,...,n_t}) = MPF(P_{n-1}^{n_1-1,n_2,...,n_{t-1},n_t-1}) + MPF(P_{n-n_1-n_t-1}^{n_2-1,n_3,...,n_{t-2},n_{t-1}-1})$.*

Corollary 2 *For an arbitrary proper edge colouring of the graph C_n, $n \geq 3$ holds $MPF(C_n) = L_n$.*

3 Fibonacci numbers in edge coloured graphs

Let G be an edge coloured simple graph. By $\mathcal{Q} = \{Q_1, ..., Q_t\}$, $t \geq 1$ we denote the family of all maximal (with respect to set inclusion) monochromatic subgraphs of G. The vertex $x \in V(G)$ is an inner vertex of a monochromatic subgraph if there exists $1 \leq i \leq t$ such that x and all vertices adjacent to x belong to $V(Q_i)$. The vertex $x \in V(G)$ is an end vertex of a monochromatic subgraph if either $deg_G x = 1$ or $x \in V(Q_i)$ and there exists a vertex $y \notin V(Q_i)$ such that $\{x, y\} \in E(G)$. Evidently for every $x \in V(G)$, either x is an inner vertex of a monochromatic subgraph or x is an end vertex of a monochromatic subgraph. By $V_i(G)$ (respectively: $V_e(G)$) we denote the subset of $V(G)$ containing all inner vertices of a monochromatic subgraph (respectively: all end vertices of monochromatic subgraphs). Then $V(G) = V_i(G) \cup V_e(G)$ is a partition of $V(G)$ into two disjoint subsets. We define uncolored simple graph $G(\mathcal{Q})$ as follows: $V(G(\mathcal{Q})) = V(G)$ and $E(G(\mathcal{Q})) = \{\{v_p, v_q\}; v_p, v_q \in V(Q_i), i = 1, ..., t\}$ with replacing multiple edges by one edge.

Theorem 5 *For an arbitrary edge coloured graph G holds $MPF(G) = F(G(\mathcal{Q}))$.*

P R O O F: Let $S \subseteq V(G)$ be an imp-set of G. We shall show that S is an independent set of $G(\mathcal{Q})$. Assume on the contrary that S is not an independent set of $G(\mathcal{Q})$. This means that there exist $v_p, v_q \in S$ and $\{v_p, v_q\} \in E(G(\mathcal{Q}))$. The definition of the graph $G(\mathcal{Q})$ implies that $v_p, v_q \in V(Q_i)$ where Q_i is a monochromatic subgraph of G. Consequently there is a monochromatic path between v_p, v_q in G, a contradiction with the assumption.

Assume now that S^* is an independent set of $G(\mathcal{Q})$ but S^* is not an imp-set of G. Hence there exist $v_p, v_q \in S^*$ and a monochromatic path between them in G. This implies that $v_p, v_q \in V(Q_j)$ where Q_j is a monochromatic subgraph of G. By the definition of $G(\mathcal{Q})$ holds $\{v_p, v_q\} \in E(G(\mathcal{Q}))$, a contradiction with the independence of S^*.

Thus the theorem is proved. □

Theorem 6 *Let G be an edge coloured graph. If $\mathcal{S}$ is a family of all imp-sets S of G such that $S \subseteq V_e(G)$, then $|\mathcal{S}| = F(G(\mathcal{Q}) \setminus V_i(G))$.*

P R O O F: Let $\mathcal{S}$ be as in the assumption of the Theorem and $S \in \mathcal{S}$. We shall prove that S is an independent set of $G(\mathcal{Q}) \setminus V_i(G)$. Assume on the contrary that S is not independent of $G(\mathcal{Q}) \setminus V_i(G)$. This means that there exist $v_p, v_q \in S$ such that $\{v_p, v_q\} \in E(G(\mathcal{Q}) \setminus V_i(G))$. The definition of the graph $G(\mathcal{Q})$ implies that there is a monochromatic subgraph Q_i such that $v_p, v_q \in V(Q_i)$. Hence there exists a monochromatic path between v_p, v_q in G, a contradiction with the assumption.

Assume now that S^* is an independent set of $G(\mathcal{Q}) \setminus V_i(G)$ and S^* is not an imp-set of G. So there are $v_p, v_q \in S^* \in \mathcal{S}$ and a monochromatic path between them in G. Hence there is a monochromatic subgraph Q_i such $v_p, v_q \in V(Q_i)$ and by the definition of $G(\mathcal{Q})$ we obtain that $\{v_p, v_q\} \in E(G(\mathcal{Q}) \setminus V_i(G))$, a contradiction. This completes the proof. □

Corollary 3 *If $P_n^{n_1,\ldots,n_t}$ is an edge coloured graph, then*

$$|\mathcal{S}| = F_{n+1-\sum_{i=1}^{t}(n_i-1)}.$$

4 Monochromatic polynomial of graphs

In [3] it was introduced the concept of the Fibonacci polynomial of a graph which gives the total number of independent sets of the composition of two graphs. They define the Fibonacci polynomial $F_G(x)$ of the graph G by $F_G(x) = F(G[K_x])$ for integer x, where K_x is a complete graph on x vertices, $x \geq 1$. It has been proved:

Theorem 7 [3] *Let $n \geq 1$, $p \geq 0$, $x \geq 1$ be integers. Then for an arbitrary graph G on n vertices $F_G(x) = \sum_{k \geq 0} F_k(G)x^k$, where $F_k(G)$ is the number of all k-element independent sets of G.*

For general results concerning the total number of r-independent sets (i.e. independent sets generalized in distance sense) in generalized lexicographic product of graphs, see [8].

In this section we define the monochromatic Fibonacci polynomial of a graph G which gives the total number of imp-sets in edge coloured generalized lexicographic product of graphs. This concept generalize the Fibonacci polynomial of a graph introduced in [3]. The definition of the generalized lexicographic product of graphs is applied for edge coloured graphs in the following way.

Let G be an edge coloured graph on $V(G) = \{v_1, ..., v_n\}, n \geq 2$, and $\alpha = (R_i)_{i \in \{1,...,n\}}$ be a sequence of vertex disjoint edge coloured graphs on $V(R_i) = V = \{y_1, ..., y_x\}, x \geq 1$. By generalized lexicographic product of G and $\alpha = (R_i)_{i \in \{1,...,n\}}$ we mean a graph $G[\alpha]$ such that $V(G[\alpha]) = V(G) \times V$ *and* $E(G[\alpha]) = \{\{(v_i, y_p), (v_j, y_q)\}$ *coloured* $i; (v_i = v_j$ *and* $\{y_p, y_q\} \in E(R_i)$ *coloured* $i)$ *or* $\{v_i, v_j\} \in E(G)$ *coloured* $i\}$. By R_i^c, $i = 1, ..., n$ we will denote the copy of the graph R_i in $G[\alpha]$. If $R_i = R$ for $i = 1, ..., n$, then $G[\alpha] = G[R]$, where $G[R]$ is a composition of two graphs.

We define the monochromatic Fibonacci polynomial $MPF_G(x)$ of the edge coloured graph G on n vertices, $n \geq 2$, by $MPF_G(x) = MPF(G[\alpha])$, where $\alpha = (R_i)_{i \in \{1,...,n\}}$ is an arbitrary sequence of vertex disjoint edge coloured graphs on $|V(R_i)| = |V| = x$.

Theorem 8 *Let $x \geq 1$, $n \geq 2$ be integers. Then for an arbitrary edge coloured graph G on n vertices and for an arbitrary sequence α of vertex disjoint edge coloured graphs R_i, $i = 1, ..., n$, on x vertices $MPF_G(x) = \sum_{k \geq 0} MPF_k(G)x^k$.*

P R O O F: Let G be an edge coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge coloured graphs $R_1, ..., R_n$ on x, $x \geq 1$ vertices. To prove this Theorem it suffices to calculate the number $MPF(G[\alpha])$. From the definition of the graph $G[\alpha]$ we deduce that to obtain a k-element, $k \geq 1$, imp-set of $G[\alpha]$ first we have to choose a k-element imp-set of G. Of course we can do it on $MPF_k(G)$ ways. Let $S = \{v_i; i \in \mathcal{I}\}$ be an imp-set of G where $\mathcal{I}$ be a subset of a set of indexes of vertices belonging to $V(G)$ and $|\mathcal{I}| = k$, $k \geq 1$. Next in each R_i^c, $i \in \mathcal{I}$ we have to choose an imp-set. Evidently by the definition of $G[\alpha]$ for every two vertices from $V(R_i^c)$, $i \in \mathcal{I}$ there exists a monochromatic path between them in $G[\alpha]$, so exactly one vertex from each $V(R_i^c)$, $i \in \mathcal{I}$ can be chosen to belong to imp-set of $G[\alpha]$. Because every vertex from R_i^c can be chosen on x ways, so by fundamental combinatorial statements we have $MPF_k(G)x^k$ imp-sets having exactly k-element in $G[\alpha]$. Moreover the empty set also is imp-set of $G[\alpha]$, whence $MPF_G(x) = 1 + \sum_{k\geq 1} MPF_k(G)x^k = \sum_{k\geq 0} MPF_k(G)x^k$. Thus the theorem is proved. □

Using results from Theorem 1, Theorem 3 and applying Theorem 8 we can determine $MPF_{P_n}(x)$ and $MPF_{C_n}(x)$, respectively.

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