On 2-Factors with Quadrilaterals Containing Specified Vertices in a Bipartite Graph*

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Abstract

In this paper we consider the problem as follows: Given a bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2| = n$ and a positive integer k, what degree condition is sufficient to ensure that for any k distinct vertices $v_1, v_2, \cdots, v_k$ of G, G contains k independent quadrilaterals $Q_1, Q_2, \cdots, Q_k$ such that $v_i \in V(Q_i)$ for every $i \in \{1, 2, \cdots, k\}$, or G has a 2-factor with k independent cycles of specified lengths with respect to $\{v_1, v_2, \cdots, v_k\}$? We will prove that if $d(x) + d(y) \geq \lceil (4n + k)/3 \rceil$ for each pair of nonadjacent vertices $x \in V_1$ and $y \in V_2$, then, for any k distinct vertices $v_1, v_2, \cdots, v_k$ of G, G contains k independent quadrilaterals $Q_1, Q_2, \cdots, Q_k$ such that $v_i \in V(Q_i)$ for each $i \in \{1, \cdots, k\}$. Moreover, G has a 2-factor with k cycles with respect to $\{v_1, v_2, \cdots, v_k\}$ such that $k - 1$ of them are quadrilaterals. We also discuss the degree conditions in the above results.

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1 Introduction

We only consider finite graphs without loops and multiple edges. Let $G = (V, E)$ be a graph. The order of G is $|G| = |V|$ and its size is $e(G) = |E(G)|$. For two subgraphs G_1 and G_2, the set of edges incident to one vertex in G_1 and one in G_2 will be written as $E(G_1, G_2)$, and $e(G_1, G_2) = |E(G_1, G_2)|$. Let H be a subgraph of G and $x \in V(G)$ a vertex, $N(x, H)$ is the set of neighbors of x contained in H. We let $d(x, H) = |N(x, H)|$. Thus, $d(x, H)$

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is the degree of x in H, $d(x, G)$ is the degree of x in G, and we write $d(x)$ to replace $d(x, G)$. The minimum degree of G will be denoted by $\delta(G)$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. Let v be a vertex of G, a v-subgraph of G is a subgraph H of G such that $v \in V(H)$. A hamiltonian cycle of G is a cycle of G which contains every vertex of G. A 2-factor of G is a 2-regular spanning subgraph of G. Clearly, each component of a 2-factor of G is a cycle. Let $H_1, H_2, \cdots, H_k$ be subgraphs of G. We say that $H_1, H_2, \cdots, H_k$ are independent, if $V(H_i) \cap V(H_j) = \emptyset$ for any $\{i, j\} \subseteq \{1, 2, \cdots, k\}$ and $i \neq j$. Let $v_1, v_2, \cdots, v_k$ be k distinct vertices, and let $C_1, C_2, \cdots, C_k$ be k independent cycles that contain $v_1, v_2, \cdots, v_k$, respectively, in G. We say that G has a 2-factor with k cycles $C_1, C_2, \cdots, C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$, if $V(G) = V(C_1 \cup C_2 \cup \cdots \cup C_k)$. Let C be a cycle, use $l(C)$ to denote the length of C. That is, $l(C)$ is the number of vertices of C. A cycle of length 4 is called a quadrilateral. For a bipartite graph $G = (V_1, V_2; E)$, if $|V_1| = |V_2|$, then G is called balanced. We define $\sigma_{1,1}(G) = \min\{d(x) + d(y) : x \in V_1, y \in V_2, xy \notin E(G)\}$. Unexplained terminology and notation can be found in [1].

The problem on graph partition into cycles is one of the most interesting problems. Corrádi and Hajnal [3] proved that if G is a graph of order $n \geq 3k$ with the minimum degree at least $2k$, then G contains k independent cycles. When $n = 3k$, G contains k independent triangles. El-Zahar [4] conjectured that if a graph G of order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 3 (1 \leq i \leq k)$ has minimum degree at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k independent cycles of lengths $n_1, n_2, \cdots, n_k$, respectively. He proved it for $k = 2$. For a bipartite graph, Wang proved the following result.

Theorem A [6]. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2$ and $\delta(G) \geq \lceil n/2 \rceil + 1$. If $k \geq 0$ and $t \geq 3$ are two integers such that $n = 2k + t$, then G contains k independent quadrilaterals and a cycle of order $2t$ such that the cycle is independent of all the k quadrilaterals.*

Clearly, for a bipartite graph, quadrilateral is the smallest cycle. Recently, Wang considered the independent small cycles containing specified edges in a bipartite graph, proved the following theorem.

Theorem B [7]. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 3k$, where $k \geq 2$ is an integer. Suppose that $d(x) + d(y) \geq n + k$ for each pair of nonadjacent vertices x and y of G with $x \in V_1$ and $y \in V_2$. Then, for any k independent edges $e_1, \cdots, e_k$ of G, G has k vertex-disjoint cycles $C_1, \cdots, C_k$ of length at most 6 such that $e_i \in E(C_i)$ for each $i \in \{1, \cdots, k\}$.*

In this paper, we consider the independent quadrilaterals containing specified vertices in a bipartite graph, give the following results.

Theorem 1. *Let $k \geq 1$ be an integer. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2k + 1$. Suppose that $\sigma_{1,1}(G) \geq \lceil (4n + k)/3 \rceil$. Then for any k distinct vertices $v_1, v_2, \cdots, v_k$ of G, G contains k independent quadrilaterals $Q_1, Q_2, \cdots, Q_k$ such that $v_i \in V(Q_i)$ for every $i \in \{1, 2, \cdots, k\}$.*

Theorem 2. *Let $k \geq 1$ be an integer and $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2k + 1$. If $\sigma_{1,1}(G) \geq \lceil (4n + k)/3 \rceil$, then for any k distinct vertices $v_1, v_2, \cdots, v_k$ of G, G has a 2-factor with k cycles $C_1, C_2, \cdots, C_k$ such that $k - 1$ of them are quadrilaterals and $v_i \in V(C_i)$ for every $i \in \{1, 2, \cdots, k\}$.*

Remark. The degree condition of G in Theorem 1 is sharp when $2n = 4k+2$. Generally, we can give an example to show that $\sigma_{1,1}(G) \geq n+k$ is necessary.

Example 1. Let G be a bipartite graph with the bipartition $(A \cup B \cup \{y_1, y_3, y_5\}, X \cup Y \cup \{y_2, y_4, y_6\})$ with $|A| = |B| = |X| = |Y| = k - 1$ such that the subgraph $G[A \cup B, X \cup Y]$ is a complete bipartite graph, and $C = y_1y_2y_3y_4y_5y_6y_1$ is a cycle of length 6 satisfying $y_1y_4 \notin E$, $y_3y_6 \notin E$ and $y_2y_5 \notin E$. Let $H = G[A \cup B, X \cup Y]$. We construct G as follows. $N(y_1, H) = X \cup Y$, $N(y_2, H) = B$, $N(y_3, H) = X$, $N(y_4, H) = A$, $N(y_5, H) = X \cup Y$, $N(y_6, H) = A \cup B$. Clearly, G is of order $2k + 1$ and $\sigma_2(G) \geq 3k + 1 = (4n+k-1)/3$. Set $X = \{v_1, v_2, \cdots, v_{k-1}\}$ and $y_3 = v_k$. It is easy to see that each quadrilateral containing v_k is joint some v_i for $i \in \{1, 2, \cdots, k - 1\}$. Therefore, there are no k independent quadrilaterals in G satisfying the requirement.

Example 2. Let G be a bipartite graph with the bipartition $(A \cup B \cup \{u_1, u_2\}, X \cup Y \cup \{u_3\})$ with $|A| = |X| = k - 1$, $|B| + 1 = |Y| = n - k$. We obtain the graph G as follows: From the complete bipartite graph $(A \cup B \cup \{u_1, u_2\}, X \cup Y \cup \{u_3\})$, we delete the edge u_1u_3 and the edges between u_2 and Y. Then $\sigma_{1,1}(G) \geq n + k - 1$. Let $X = \{v_1, v_2, \cdots, v_{k-1}\}$ and $u_2 = v_k$. Then each quadrilateral containing v_k is joint some v_i for $i \in \{1, 2, \cdots, k - 1\}$. Consequently, there are no k required quadrilaterals in G.

We would like to mention a result in [2]. In [2], Chen etc. proved that, if k is a positive integer and G is a balanced bipartite graph of order $2n$ with $n \geq 9k$ and $\delta(G) \geq (n + 2)/2$, then, for every perfect matching M, G has a 2-factor with exactly k components including every edge of M.

2 Lemmas

Let $G = (V_1, V_2; E)$ be a balanced bipartite graph with $|V_1| = |V_2| = n \geq 2$. If P is a v-path of G, we define

$$\lambda(v, P) = \min\{|V(P_1)|, |V(P_2)|\}.$$

Where P_1 and P_2 are the two components of $P - v$. In the following, we give a number of lemmas. Lemma 2.4 and Lemma 2.6 are easy observations

Lemma 2.1 **[5]**. *Let $P = x_1y_1 \cdots x_ky_k$ be a path of G with $k \geq 2$. If $d(x_1, P) + d(y_k, P) \geq k+1$, then G has a cycle C such that $V(C) = V(P)$.*

Lemma 2.2 **[5]**. *Let x and y be any pair of nonadjacent vertices with $x \in V_1$ and $y \in V_2$. If $d(x) + d(y) \geq n+1$, then G is hamiltonian.*

Lemma 2.3. *Let $P = x_1x_2 \cdots x_{2r+d}$ be a path in G, where $d = 0$ or 1. Let $y \in V(G) - V(P)$ such that $\{x_{2r+d}, y\} \not\subseteq V_i$ for every $i \in \{1, 2\}$. Then the following two statements hold.*

(a) If $d(y, P) + d(x_{2r+d}, P) \geq r+2$, then $G[V(P) \cup \{y\}]$ has a hamiltonian path P' with two endvertices x_1 and y.

(b) If $d(y, P) + d(x_{2r+d}, P) \geq r+1$, then $G[V(P) \cup \{y\}]$ has a hamiltonian path.

Proof. Clearly, if $yx_{2r+d} \in E$, then $P' = x_1x_2 \cdots x_{2r+d}y$ is the required hamiltonian path. In the following, we may assume that $yx_{2r+d} \notin E$. Let $S = \{x_{i+1} | x_ix_{2r+d} \in E\}$. Clearly, $d(x_{2r+d}, P) = |S|$. First we suppose that $d(y, P) + d(x_{2r+d}, P) \geq r+2$. Then

$$|N(y, P) \cap S| = |N(y, P)| + |S| - |N(y, P) \cup S| \geq r + 2 - (r+1) = 1.$$

This implies that there exists $x_{i+1} \in N(y, P) \cap S$. It follows that $P' = x_1x_2 \cdots x_ix_{2r+d}\, x_{2r+d-1} \cdots x_{i+1}y$ is the required hamiltonian path of $G[V(P) \cup \{y\}]$. Second, we assume that $d(y, P) + d(x_{2r+d}, P) \geq r+1$. If $d = 1$ and $yx_1 \in E$, then $P + y$ is a hamiltonian path of $G[V(P) \cup \{y\}]$. Hence if $d = 1$ then $yx_1 \notin E$. Consequently, we have $|N(y, P) \cup S| \leq r$, and then $|N(y, P) \cap S| \geq 1$. By the same argument as above, we see that $G[V(P) \cup \{y\}]$ has a hamiltonian path. □

Lemma 2.4. *Let Q be a v-quadrilateral . Let $x \in V_1$ and $y \in V_2$ be two vertices not on Q. Suppose that $d(x, Q) + d(y, Q) = 4$. Then there exist two vertices $u_1 \in V_1$ and $u_2 \in V_2$ of Q such that $Q - u_1 + x$ contains a v-quadrilateral and $u_1y \in E$, and $Q - u_2 + y$ contains a v-quadrilateral and $u_2x \in E$.*

Lemma 2.5. *Let Q be a v-quadrilateral and P a u-path of length 5 such that they are independent. Suppose that $\lambda(u, P) \neq 0$ and 1. If $e(Q, P) \geq 10$,* *then $G[V(Q \cup P)]$ contains two independent quadrilaterals Q_1 and Q_2 such that $u \in V(Q_1)$ and $v \in V(Q_2)$.*

Proof. Let $Q = a_1a_2a_3a_4a_1$ and $P = x_1 \cdots x_6$ with $\{a_1, x_1\} \subseteq V_1$. As $\lambda(u, P) \neq 0$ and 1, $u = x_3$ or x_4. If $u = x_3$ and u, v are in the same bipartition of G, without loss of generality, we may assume that $v = a_1$. If $u = x_3$ and u, v are in the different bipartition of G, we may assume that $v = a_2$. Similarly, if $u = x_4$, we may assume that $v = a_2$ or $v = a_1$ according to u, v being in the same bipartition or different bipartition of G. By symmetry, we only prove the two cases $v = a_1, u = x_3$ and $v = a_1, u = x_4$.

Case 1. $v = a_1$ and $u = x_3$.
Let $P_1 = x_1x_2x_3x_4$. It is not difficult to see that if $e(Q, P_1) = 8$, then we have two independent quadrilaterals Q_1 and Q_2 such that $u \in V(Q_1)$ and $v \in V(Q_2)$. So in the following we assume that $e(Q, P_1) \leq 7$. On the other hand, $e(Q, P) \geq 10$. Therefore, $6 \leq e(Q, P_1) \leq 7$. We distinguish the following three cases.

Case 1.1. $e(Q, P_1) = 7$ and $d(x_4, Q) = 2$.
Since $e(Q, P_1) = 7$, there exists $i \in \{2, 4\}$ such that $\{a_ix_1, a_ix_3\} \subseteq E$. Then we have two independent quadrilaterals $Q_1 = x_1x_2x_3a_ix_1$ and $Q_2 = a_1a_ja_3x_4a_1$ for $\{i, j\} = \{2, 4\}$.

Case 1.2. $e(Q, P_1) = 7$ and $d(x_4, Q) = 1$.
Note that $d(x_1, Q) = d(x_2, Q) = d(x_3, Q) = 2$. If $a_3x_4 \in E$, then $G[V(Q \cup P_1)]$ contains two independent quadrilaterals $Q_1 = a_3a_4x_3x_4a_3$ and $Q_2 = a_1a_2x_1x_2a_1$ such that $u \in V(Q_1)$ and $v \in V(Q_2)$. Otherwise $a_1x_4 \in E$. As $e(Q, P) \geq 10$, we have $e(x_5x_6, Q) \geq 3$ and then $d(x_5, Q) \geq 1$. Suppose that $a_ix_5 \in E$. Then $G[V(Q \cup P)]$ contains two required quadrilaterals $Q_1 = x_3x_4x_5a_ix_3$ and $Q_2 = a_1a_jx_1x_2a_1$ for $\{i, j\} = \{2, 4\}$.

Case 1.3. $e(Q, P_1) = 6$.
In this case, we see that $d(x_5, Q) = d(x_6, Q) = 2$ because $e(Q, P) \geq 10$. If there exists $i \in \{2, 4\}$ such that $a_ix_3 \in E$, say $i = 2$, then we have two independent quadrilaterals $Q_1 = x_3x_4x_5a_2x_3$ and $Q_2 = a_1a_4a_3x_6a_1$. Otherwise $d(x_3, Q) = 0$. As $e(Q, P) \geq 10$, we see that $d(x_4, Q) = d(x_1, Q) = d(x_2, Q) = 2$. Then $G[V(Q \cup P)]$ contains two independent quadrilaterals $Q_1 = a_3x_2x_3x_4a_3$ and $Q_2 = a_1a_2x_1a_4a_1$ such that $u \in V(Q_1)$ and $v \in V(Q_2)$.

Case 2. $v = a_1$ and $u = x_4$.
Let $P_1 = x_3x_4x_5x_6$. Similar as the discussion in Case 1, if $e(Q, P_1) = 8$, then there exist two independent quadrilaterals Q_1 and Q_2 such that $u \in V(Q_1)$ and $v \in V(Q_2)$. So we may assume that $(Q, P) \leq 7$. Since $e(Q, P) \geq 10$, we have $e(Q, P_1) = 7$ or $e(Q, P_1) = 6$. We consider the following two cases.

Case 2.1. $e(Q, P_1) = 7$.

In this case, $d(x_6, Q) \geq 1$. Suppose that $d(x_6, Q) = 2$. Since $e(Q, P_1) = 7$, there exists $i \in \{2, 4\}$ such that $\{a_i x_3, a_i x_5\} \subseteq E$, say $i = 2$. Consequently, $G[V(Q \cup P_1)]$ contains two required quadrilaterals $Q_1 = x_3 x_4 x_5 a_2 x_3$ and $Q_2 = a_1 a_4 a_3 x_6 a_1$. Otherwise $d(x_6, Q) = 1$. It follows that $d(x_3, Q) = d(x_4, Q) = d(x_5, Q) = 2$. If $a_3 x_6 \in E$, then we obtain two independent quadrilaterals $Q_1 = a_3 x_4 x_5 x_6 a_3$ and $Q_2 = x_3 a_2 a_1 a_4 x_3$ such that $u \in V(Q_1)$ and $v \in V(Q_2)$. If $a_1 x_6 \in E$. Then $G[V(Q \cup P_1)]$ contains two required independent quadrilaterals $Q_1 = a_3 a_4 x_3 x_4 a_3$ and $Q_2 = a_1 a_2 x_5 x_6 a_1$.

Case 2.2 $e(Q, P_1) = 6$.

In this case, $d(x_1, Q) = d(x_2, Q) = 2$. If $\{a_i x_3, a_i x_5\} \subseteq E$ holds for some $i \in \{2, 4\}$, say $i = 2$, then we obtain two independent quadrilaterals $Q_1 = a_2 x_3 x_4 x_5 a_2$ and $Q_2 = x_2 a_1 a_4 a_3 x_2$. Otherwise, $e(\{x_3, x_5\}, Q) \leq 2$ and then $d(x_4, Q) = d(x_6, Q) = 2$ as $e(Q, P_1) = 6$. So $G[V(Q \cup P)]$ contains two independent quadrilaterals $Q_1 = a_3 x_4 x_5 x_6 a_3$ and $Q_2 = x_1 a_2 a_1 a_4 x_1$. The lemma is proved. □

Lemma 2.6. *Let C be a quadrilateral and P a path with two end-vertices $u \in V_1$ and $v \in V_2$ in G such that C and P are independent. If $d(u, C) + d(v, C) \geq 3$, then G has a cycle C' such that $V(C') = V(C \cup P)$.*

3 Proof of Theorem 1

Let k be a positive integer and $G = (V_1, V_2; E)$ a bipartite graph with $|V_1| = |V_2| = n \geq 2k + 1$. We assume that $\sigma_{1,1}(G) \geq \lceil (4n + k)/3 \rceil$. Suppose, for a contradiction, that there exist k distinct vertices $v_1, v_2, \cdots, v_k$ of G such that G does not contain k independent quadrilaterals with respect to $\{v_1, v_2, \cdots, v_k\}$. Without loss of generality, we may assume that G is an edge-maximal counterexample. That is, for any two nonadjacent vertices $u \in V_1$ and $v \in V_2$, $G + uv$ contains k independent quadrilaterals $Q_1, Q_2, \cdots, Q_k$ such that $v_i \in V(Q_i)$ for each $i \in \{1, 2, \cdots, k\}$. We distinguish the following two cases: $k = 1$ and $k \geq 2$.

Case 1. $k = 1$.

Since $d(x) + d(y) \geq (4n + 1)/3 \geq n + 1$ for any two nonadjacent vertices x and y with $x \in V_1$ and $y \in V_2$, G has a hamiltonian cycle by Lemma 2.2. Let $C = x_1 y_1 \cdots x_n y_n x_1$ be the hamiltonian cycle of G. Without loss of generality, we may assume that $v_1 = x_2$ (Otherwise, we may renumber the vertices of C). Let $P = x_1 y_1 x_2 y_2 x_3 y_3$ be a subpath of G. Since G does not contain a v_1-quadrilateral, we must have that

$$N(x_2, G) \cap N(x_1, G) = \{y_1\}.$$

$$N(x_2, G) \cap \{N(x_3, G) = \{y_2\}.$$

$$N(y_1, G) \cap \{N(y_2, G) = \{x_2\}.$$

It follows that

$$d(x_2) + d(x_1) = |N(x_2, G) \cup N(x_1, G)| + |N(x_2, G) \cap N(x_1, G)| \leq n + 1.$$

Similarly, we have $d(x_2) + d(x_3) \leq n+1$ and $d(y_1) + d(y_2) \leq n+1$. Since G does not contain v_1-quadrilateral, we have $d(y_3) \leq n-1$. Clearly, $d(x_2) \geq 2$. Then we have

$$\sum_{u \in V(P)} d(u) \leq 3n + 3 - d(x_2) + d(y_3) \leq 4n.$$

On the other hand, $\{x_1, y_2\}, \{x_2, y_3\}, \{x_3, y_1\}$ are three pairs of nonadjacent vertices in different bipartition of P as G does not contain a v_1-quadrilateral. Therefore, we have $\sum_{u \in V(P)} d(u) \geq 4n + 1$, a contradiction.

Case 2. $k \geq 2$.

Since G is an edge-maximal counterexample, there exists at least one vertex of $\{v_1, v_2, \cdots, v_k\}$, say v, such that G contains $k-1$ independent quadrilaterals $Q_1^v, Q_2^v, \cdots, Q_{k-1}^v$ such that $v \notin V(\cup_{i=1}^{k-1} Q_i^v)$. We choose $v \in \{v_1, v_2, \cdots, v_k\}$ and $Q_1^v, Q_2^v, \cdots, Q_{k-1}^v$ such that

$$G - V(\cup_{i=1}^{k-1} Q_i^v) \text{ has the longest } v-\text{path}. \qquad (1)$$

Let P be the longest v-path of $G - V(\cup_{i=1}^{k-1} Q_i^v)$. Subject to (1), we choose $v \in \{v_1, v_2, \cdots, v_k\}$, $Q_1^v, Q_2,^v, \cdots, Q_{k-1}^v$ and P such that

$$\lambda(v, P) \text{ is maximum.} \qquad (2)$$

Without loss of generality, suppose that $v = v_k$. Let $Q_i = Q_i^v$ and $v_i \in V(Q_i)$ for all $i \in \{1, 2, \cdots, k-1\}$. Set $H = \cup_{i=1}^{k-1} Q_i$, $D = G - V(H)$ and $|V(D)| = 2t$. Clearly, $n = 2(k-1) + t$ and $l(Q_i) = 4$ for all $i \in \{1, 2, \cdots, k-1\}$. Since $n \geq 2k+1$, we see that $t \geq 3$. Let $P = x_1 x_2 \cdots x_p$ be the longest v_k-path with $x_1 \in V_1$. By the assumption on G, D contains a v_k-path of length at least 3. Thus, $p \geq 4$.

Claim 1. $p = 2t$.

Proof of Claim 1. On the contrary, suppose that $p < 2t$. Set $p = 2r + \delta$, where $\delta = 0$ or 1. Let $u \in D - V(P)$ be a vertex with $\{u, x_p\} \nsubseteq V_i$ for each $i \in \{1, 2\}$. We claim that $d(u, Q_i) + d(x_p, Q_i) \leq 3$ for every $i \in \{1, 2, \cdots k-1\}$. If it is not true, then there exists $i \in \{1, 2, \cdots, k-1\}$ such that $d(u, Q_i) + d(x_p, Q_i) = 4$. By Lemma 2.4, there exists a vertex w in $V(Q_i)$ such that $Q_i - w + u$ contains a v_i-quadrilateral and $wx_p \in E$. Thus, $P + w$ has a v_k-path longer than P, contradicting (1). So $d(u, Q_i) + d(x_p, Q_i) \leq 3$ for each $i \in \{1, 2, \cdots, k-1\}$. It follows that $d(u, H) + d(x_p, H) \leq 3(k-1)$.

Clearly, $d(u, D - V(P)) \le t - r$ and $d(x_p, D - V(P)) = 0$. Since $n = 2(k-1)+t$, we have

$$\begin{aligned} d(u,P) + d(x_p,P) &\ge \lceil \frac{4n+k}{3} \rceil - 3(k-1) - (t-r) \\ &= \lceil \frac{8(k-1)+4t+k}{3} \rceil - 3(k-1) - (t-r) \\ &\ge r+1. \end{aligned}$$

By Lemma 2,3, $G[V(P) \cup \{u\}]$ has a hamiltonian path, this contradicts (1). So Claim 1 holds.

Claim 2. If $\lambda(v_k, P) = 0$ or 1, then D has a hamiltonian cycle.

Proof of Claim 2. If $x_1x_{2t} \in E$, then we have nothing to prove. In the following, we assume that $x_1x_{2t} \notin E$. By symmetry, we assume that $v_k = x_1$ if $\lambda(v_k, P) = 0$ and $v_k = x_2$ if $\lambda(v_k, P) = 1$. We consider the endvertices x_1 and x_{2t} of P. If there exists Q_i in H such that $d(x_1, Q_i) = d(x_{2t}, Q_i) = 2$, then, by Lemma 2.4, there is a vertex in $V(Q_i)$, say w, such that $Q_i - w + x_{2t}$ contains a v_i-quadrilateral and $wx_1 \in E$. Then we obtain a v_k-path $P' = P + w - x_{2t}$ and $\lambda(v_k, P') = \lambda(v_k, P) + 1$ as $t \ge 3$, which contradicts (2) while (1) is maintained. Therefore, we have $d(x_1, Q_i) + d(x_{2t}, Q_i) \le 3$ for each $i \in \{1, 2, \cdots, k-1\}$. It follows that

$$d(x_1, D) + d(x_{2t}, D) \ge \frac{4n+k}{3} - 3(k-1) = \frac{4t+1}{3} \ge t+1.$$

By Lemma 2.1, D contains a hamiltonian cycle. So Claim 2 holds.

Now we are in the position to complete the proof. As $t \ge 3$, by Claim 1 and Claim 2, we may choose a subpath P' of length 5 of P such that $\lambda(v_k, P) = 2$. Let $P' = y_1y_2y_3y_4y_5y_6$ be such a path with $y_1 \in V_1$. Then $v_k = y_3$ or y_4. As D does not contain a v_k-quadrilateral, we obtain

$$N(y_3, D) \cap N(y_5, D) = \{y_4\}$$

and

$$N(y_2, D) \cap N(y_4, D) = \{y_3\}.$$

If $v_k = y_3$, then $N(y_1, D) \cap N(y_3, D) = \{y_2\}$. If $v_k = y_4$, then $N(y_4, D) \cap N(y_6, D) = y_5$. So we obtain $d(y_3, D) + d(y_5, D) = |N(y_3, D) \cup N(y_5, D)| + |N(y_3, D) \cap N(y_5, D)| \le t+1$. Similarly, we have $d(y_2, D) + d(y_4, D) \le t+1$, either $d(y_1, D) + d(y_3, D) \le t+1$ or $d(y_4, D) + d(y_6, D) \le t+1$. Suppose that $v_k = y_3$. As D does not contain a v_k-quadrilateral, we have $d(y_6, D) \le t-1$. Then

$$\sum_{i=1}^{6} d(y_i, D) \le 3t + 3 - d(y_3, D) + d(y_6, D) \le 4t.$$

As D does not contain a v_k-quadrilateral, we have three pairs of non-adjacent vertices $\{y_1, y_4\}, \{y_3, y_6\}, \{y_5, y_2\}$ in different bipartition in P'. Therefore,

$$\sum_{y_i \in V(P')} d(y_i, H) \geq 4n + k - 4t = 8(k-1) + 4t + k - 4t = 9(k-1) + 1.$$

This implies that there exists Q_j in H such that $\sum_{y_i \in V(P')} d(y_i, Q_j) \geq 10$. By Lemma 2.5, $G[V(Q_j \cup P')]$ contains two independent quadrilaterals Q_j and Q'_j such that $v_j \in V(Q_j)$ and $v_k \in V(Q'_j)$, a contradiction. Similarly, if $v_k = y_4$, we can obtain the same contradiction. The proof is completed.□

4 Proof of Theorem 2

Let $G = (V_1, V_2; E)$ be a bipartite graph satisfying the conditions of Theorem 2. If $k = 1$, by Lemma 2.2 and the degree condition, we have that G has a hamiltonian cycle. Thus the theorem holds. In the following we assume that $k \geq 2$. Suppose, for a contradiction, that there exist k distinct vertices $v_1, v_2, \cdots, v_k$ of G such that G does not have a 2-factor with k required cycles $C_1, C_2, \cdots, C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$. By Theorem 1, G contains k independent quadrilaterals $C_1, \cdots, C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$. We choose such k quadrilaterals $C_1, \cdots, C_k$ such that

$$\text{The length of the longest path of } G - V(\cup_{i=1}^{k} C_i) \text{ is maximum.} \qquad (3)$$

We may assume that $v_i \in V(C_i)$ for all $i \in \{1, 2, \cdots, k\}$. Set $H = \cup_{i=1}^{k} C_i$ and $D = G - V(H)$. Let $|V(D)| = 2t$, then $n = 2k + t$. Clearly, $l(C_i) = 4$ for all $i \in \{1, 2, \cdots, k\}$. Let $P = x_1 x_2 \cdots x_p$ be the longest path of D with $x_1 \in V_1$. Let $p = 2r + q$, where $q = 0$ or 1.

Claim 3. D is hamiltonian.

Proof of Claim 3. First, we show that $p = 2t$. Suppose, for a contradiction, that $p < 2t$. We choose an arbitrary vertex x_0 in $D - V(P)$ such that $\{x_0, x_p\} \not\subseteq V_i$ for each $i \in \{1, 2\}$. If there exists C_i in H such that $d(x_0, C_i) = d(x_p, C_i) = 2$, By Lemma 2.4, there is a vertex $z \in V(C_i)$ such that $C_i - z + x_0$ is an v_i-quadrilateral and $x_p z \in E$. Then $P + z$ is a path of D longer than P, a contradiction with (3). So $d(x_0, C_i) + d(x_p, C_i) \leq 3$ for each $i \in \{1, 2, \cdots, k\}$. Clearly, for any $x_j \in D - V(P)$, $x_j x_p \notin E$ and $d(x_j, D - V(P)) \leq t - r$. Therefore,

$$d(x_0, P) + d(x_p, P) \geq \frac{4n + k}{3} - 3k - (t - r) \geq r + 1.$$

By Lemma 2.3, $G[V(P) \cup \{x_0\}]$ contains a hamiltonian path, this contradicts (3). So D has a hamiltonian path $P = x_1 x_2 \cdots x_{2t}$.

If $x_1x_{2t} \in E$, then we have nothing to prove. So $x_1x_{2t} \notin E$. By Lemma 2.6, $d(x_1, C_i) + d(x_{2t}, C_i) \leq 2$ for all $i \in \{1, \cdots, k\}$ since G does not has a 2-factor with k required cycles. Therefore,

$$d(x_1, P) + d(x_{2t}, P) \geq \frac{4n + k}{3} - 2k > t + 1.$$

So D is hamiltonian by Lemma 2.1. The claim holds.

By Claim 3, we may assume that $x_1x_{2t} \in E$. Without loss of generality, suppose that $d(x_1, C_1) \geq d(x_j, C_i)$ for all $j \in \{1, 2, \cdots, 2t\}$ and $i \in \{1, 2, \cdots, k\}$. Let $C_1 = a_1a_2a_3a_4a_1$ with $a_1 \in V_1$. Since G is connected, we see that $d(x_1, C_1) \geq 1$. We may assume that $x_1a_2 \in E$. If $d(x_2, C_1) = 1$ or $d(x_{2t}, C_1) = 1$, then we have a hamiltonian cycle of $G[V(C_1 \cup P)]$, a contradiction. So $d(x_2, C_1) = 0$ and $d(x_{2t}, C_1) = 0$. If $d(a_1, P - x_1) + d(x_{2t}, P - x_1) \geq (t-1) + 2$, By Lemma 2.3, $G[V(P - x_1 + a_1)]$ has a hamiltonian path P' from a_1 to x_2, and then $G[V(C_1 \cup P)]$ has a hamiltonian cycle $x_1a_2a_3a_4a_1P'x_2x_1$, a contradiction. Therefore, we have $d(a_1, P) + d(x_{2t}, P) \leq t + 1$.

Claim 4. $d(x_1, C_1) = 2$.

Proof of Claim 4. On the contrary, suppose that $d(x_1, C_1) = 1$. As $t \geq 1$, $d(a_1, P) + d(x_{2t}, P) \leq t + 1$, we have

$$d(a_1, H - C_1) + d(x_{2t}, H - C_1) \geq \frac{4n + k}{3} - (t + 3) \geq 3(k - 1) + 1.$$

This implies that there exists C_i in $H - V(C_1)$ such that $d(x_{2t}, C_i) = 2$, this contradicts the maximality of $d(x_1, C_1)$. So Claim 4 holds.

We continue to prove the theorem. We assume that $v_1 = a_3$ if $v_1 \in V_1$ and $v_1 = a_4$ if $v_1 \in V_2$. Then $C_1^* = C_1 - a_1 + x_1$ is a v_1-quadrilateral. Note that $d(a_1, P) + d(x_{2t}, P) \leq t + 1$ and $d(x_{2t}, C_1) = 0$. By the same argument as the proof of Claim 4, we have $d(a_1, H - V(C_1)) + d(x_{2t}, H - V(C_1)) \geq 3(k - 1) + 1$. This implies that there exists C_s in $H - V(C_1)$, say C_2, such that $d(a_1, C_2) = d(x_{2t}, C_2) = 2$. Let $C_2 = b_1b_2b_3b_4b_1$ with $b_1 \in V_1$. We assume that $v_2 = b_1$ if $v_2 \in V_1$ and $v_2 = b_2$ if $v_2 \in V_2$. If $t = 1$, then $C_2^* = C_2 + a_1 + x_{2t}$ has a hamiltonian cycle $b_1b_2a_1b_4b_3x_{2t}b_1$. Hence, G has a 2-factor with k independent cycles $C_1^*, C_2^*, C_3, \cdots, C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$ such that $k - 1$ of them are quadrilaterals. So $t \geq 2$. Let $C_2^* = C_2 - b_4 + x_{2t}$ and $R = \{a_1, b_4, x_2, x_{2t-1}\}$.

In the following, we show that $\sum_{z \in R} d(z, V(C_1 \cup C_2 \cup P)) \leq 2t + 12$. If $d(a_1, P - x_1) + d(x_2, P - x_1) \geq (t - 1) + 2$, then, by Lemma 2.3, we have a hamiltonian path P_1 of $G[V(P - x_1 + a_1)]$ from x_{2t} to a_1. Consequently, we see that $G_1 = G[V(C_2 \cup P - x_1) \cup \{a_1\}]$ has an v_2-hamiltonian cycle $C_2' = a_1P_1x_{2t}b_1b_2b_3b_4a_1$. Therefore, G has a 2-factor with k independent cycles $C_1^*, C_2', C_3, \cdots C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$ such that $k-1$ of the cycles are quadrilaterals, a contradiction. Hence, $d(a_1, P) + d(x_2, P) \le t+1$. Since $x_1x_{2t} \in E$, we renumber the hamiltonian path $P = x_{2t}x_1x_2 \cdots x_{2t-1}$. Similarly, if $d(b_4, P) + d(x_{2t-1}, P) \ge t+2$, then $G[V(P) \cup \{b_4\}]$ has a hamiltonian path P_2 from x_{2t} to b_4. Therefore, $G[V(C_2 \cup P)]$ has a v_2-hamiltonian cycle $C_2' = b_4P_2x_{2t}b_3b_2b_1b_4$, and then G has a 2-factor with k independent cycles $C_1, C_2', C_3, \cdots C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$ such that $k-1$ of the cycles are quadrilaterals, a contradiction. Hence, we have $d(b_4, P) + d(x_{2t-1}, P) \le t+1$. On the other hand, as C_1^* is v_1-hamiltonian, $G[V(C_2 \cup P - x_1) \cup \{a_1\}]$ is not hamiltonian, and then $d(x_2, C_2) = 0$. Otherwise, if $d(x_2, C_2) \ge 1$, say $x_2b_3 \in E$, Since $d(a_1, C_2) = d(x_{2t}, C_2) = 2$, then $G[V(C_2 \cup P - x_1) \cup \{a_1\}]$ has a hamiltonian cycle $x_2b_3b_2a_1b_4b_1x_{2t} \cdots x_2$, a contradiction. Note that $d(x_2, C_1) = 0$. Clearly, $d(x_{2t-1}, C_1) \le 2$, $d(a_1, C_1 \cup C_2) \le 4$ and $d(b_4, C_1 \cup C_2) \le 4$. Let $P' = x_{2t}x_1x_2 \cdots x_{2t-1}$. Since $G[V(C_2 \cup D)]$ is not v_2-hamiltonian and $d(x_{2t}, C_2) = 2$, we have $d(x_{2t-1}, C_2) = 0$ by Lemma 2.6. Then $\sum_{z \in R} d(z, V(C_1 \cup C_2 \cup P)) \le 2t + 12$.

As $d(a_1, C_2) = 2$, we have $a_1b_4 \in E$. Note that $a_1x_2 \notin E$. Since $d(x_{2t-1}, C_2) = 0$, $b_4x_{2t-1} \notin E$. By the degree condition on G, we have that

$$\sum_{z \in R} d(z, H - V(C_1 \cup C_2)) \ge \frac{8n + 2k}{3} - (2t + 12) = 6(k-2) + \frac{2t}{3}.$$

This implies that there exists C_s in $H - V(C_1 \cup C_2)$, say C_3, such that $\sum_{z \in R} d(z, C_3) \ge 7$. Thus, $e(a_1b_4, C_3) \ge 3$ and $e(\{x_2, x_{2t-1}\}, C_3) \ge 3$. By Lemma 2.6, $C_3 + a_1 + b_4$ contains a hamiltonian cycle C^*. Let $C^* = c_1c_2c_3c_4c_5c_6c_1$ with $c_1 \in V_1$. Since $d(\{x_2, x_{2t-1}\}, C_3) \ge 3$, there exist c_i and c_{i+1} in C^* such that $e(\{x_2, x_{2t-1}\}, \{c_i, c_{i+1}\}) = 2$, where $c_7 = c_1$. We may assume that $\{x_2c_1, x_{2t-1}c_2\} \subseteq E$. It follows that $G[V(C^* \cup P - x_1 - x_{2t})]$ has a v_3-hamiltonian cycle $C_3^* = c_1x_2x_3 \cdots x_{2t-1}c_2c_3c_4c_5c_6c_1$. Then G has a 2-factor with k independent cycles $C_1^*, C_2^*, C_3^*, C_4, \cdots, C_k$ with respect to $\{v_1, v_2, \cdots, v_k\}$ such that $k-1$ of them are quadrilaterals, a contradiction. The theorem is proved. □

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