The generalized exponent sets of primitive, minimally strong digraphs(II)*

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Abstract

Let $D = (V, E)$ be a primitive digraph. The exponent of D at a vertex $u \in V$, denoted by $\exp_D(u)$, is defined to be the least integer k such that there is a walk of length k from u to v for each $v \in V$. Let $V = \{v_1, v_2, \cdots, v_n\}$. The vertices of V can be ordered so that $\exp_D(v_{i_1}) \leq \exp_D(v_{i_2}) \leq \cdots \leq \exp_D(v_{i_n})$. The number $\exp_D(v_{i_k})$ is called k-exponent of D, denoted by $\exp_D(k)$. In this paper, we completely characterize 1-exponent set of primitive, minimally strong digraphs with n vertices.

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1 Introduction

We consider only the digraphs without multiple arcs. Let $D = (V, E)$ be a digraph with n vertices. A walk uWv of length p from u to v in D is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs (u, u_1), (u_1, u_2), $\ldots$, (u_{p-1}, v), where the vertices and arcs need not to be distinct, and denoted by $uWv = (u, u_1, \ldots, u_{p-1}, v)$. The initial vertex of uWv is u, the terminal vertex is v, and $u_1, u_2, \ldots, u_{p-1}$ are the internal vertices of uWv.

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If $u = v$, then uWv is a circuit (or a closed walk). A path is a walk with distinct vertices. A cycle(an elementary circuit) is a circuit with distinct vertices except for $u = v$. For convenience, we treat a cycle as a path (a closed path) in this paper. The girth s of D is the length of a shortest cycle in D. An r-cycle is a cycle of length r. By $L(D)$ we denote the set of distinct lengths of the cycles of D, and $|L(D)|$ the number of distinct lengths of the cycles of D.

For the sake of simplicity, we use notation $[a, \ldots, b]$ to denote the set of all integers between a and b, namely $[a, \ldots, b] = \{m \mid m \in Z \text{ and } a \leq m \leq b\}$. We use notation $\lfloor a \rfloor$ and $\lceil a \rceil$, respectively, to denote the greatest integer which is not greater than a and the least integer which is not less than a.

The digraph D is called strongly connected(or strong) if for each ordered pair of distinct vertices u, v there is a walk from u to v. A strongly connected digraph D is called minimally strong(or ministrong) provided each digraph obtained from D by removing an arc is not strongly connected. A digraph D is primitive if there exists an integer $k > 0$ such that for each ordered pair of vertices $u, v \in V(D)$ (not necessarily distinct), there is a walk of length k from u to v in D, and the least such k is called the exponent of D, denoted by $\exp(D)$. It is well known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1.

In 1990, from the background of memoryless communication system, R. A. Brualdi and Bolian Liu [1] generalized the concept of the exponent for a primitive digraph and introduced the concept of k-exponent. Let $D = (V, E)$ be a primitive digraph with n vertices $v_1, v_2, \ldots, v_n$. For any $v_i, v_j \in V$, let $\exp_D(v_i, v_j) :=$ the smallest integer p such that there is a walk of length t from v_i to v_j for each integer $t \geq p$. Let the exponent of vertex v_i be defined by $\exp_D(v_i) := \max\{\exp_D(v_i, v_j) : v_j \in V\}$. Then $\exp_D(v_i)$ is the smallest integer p such that there is a walk of length p from v_i to each vertex of D. We arrange the vertices of D in such a way that $\exp_D(v_{i_1}) \leq \exp_D(v_{i_2}) \leq \cdots \leq \exp_D(v_{i_n})$, and we call the number $\exp_D(v_{i_k})$ the k-point exponent of D (the k-exponent for short), which is denoted by $\exp_D(k)$.

Let $PMSD_n$ be the set of all primitive, ministrong digraphs of order n. We define $ME_n(k) : = \{\exp_D(k) : D \in PMSD_n\}$ $(k = 1, 2, \ldots, n)$. Jiayu Shao [9] characterized $ME_n(n)$. Bolian Liu[5] obtained the maximum value of the k-exponent for $PMSD_n$. Bo Zhou [11] characterized primitive, ministrong digraphs with n vertices whose k-exponent $(1 \leq k \leq n)$ achieve the maximum value. In 2002, Bo Zhou [11] pointed out that the complete determination of $ME_n(k)$ $(1 \leq k \leq n-1)$ is an interesting and difficult problem. Recently Yahui Hu,etal.[3] characterized 1-exponent sets of primitive, ministrong digraphs with n vertices and $L(D) = \{p, q\}$, where

$3 \le p < q$ and $p + q > n$.

In this paper, we shall completely characterize $ME_n(1)$ for $n \ge 14$ (see Theorem 4.1).

2 $\exp_D(1) \le \frac{1}{2}(n^2 - 7n + 16)$ when $n \ge 14$ and $|L(D)| \ge 3$

Let $D = (V, E)$ be a digraph. $D' = (V', E')$ is called a subdigraph of D if $V' \subseteq V$ and $E' \subseteq E$, and denoted by $D' \subseteq D$. We call D' a proper subdigraph of D (write $D' \subset D$) if $D' \subseteq D$ and $D' \neq D$. Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be two subdigraphs of D. We call the digraphs $D_1 \cap D_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $D_1 \cup D_2 = (V_1 \cup V_2, E_1 \cup E_2)$ the intersection and the union of D_1, D_2, respectively.

Let $D = (V, E)$ be a digraph. We use $|V|$ to denote the number of the vertices in V, $R_t(u)$ the set of vertices of D that can be reached by a walk with initial vertex u of length t (for $t = 0$, we define $R_t(u) = \{u\}$). Let uWv be a walk from vertex u to vertex v. We use $\eta(uWv)$ to denote the length of the walk uWv. Let $vW'\omega$ be a walk from vertex v to vertex ω. For convenience, we also use $uWvW'\omega$ to denote the walk $uWv + vW'\omega$ from u to ω.

Let D be a digraph, C a cycle of D with length at least 2. Let u and v be two vertices in $V(C)$. We define $uC^{(0)}u = u$, $uC^{(0)}v$ the path from u to v in C for $u \neq v$, and $uC^{(k)}v (k \ge 1)$ the walk $uC^{(0)}v + \underbrace{C + \cdots + C}_{k \text{ times}}$ from u to v.

Let $D = (V, E)$ be a primitive digraph, and $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$ the set of distinct lengths of the cycles of D, where $r_1 > r_2 > \cdots > r_\lambda$ and $\gcd(r_1, r_2, \ldots, r_\lambda) = 1$. For $u, v \in V(D)$, the distance $d(u, v)$ from u to v is defined to be the length of shortest walk from u to v in D, the relative distance $d_{L(D)}(u, v)$ from u to v is defined to be the length of the shortest walk from u to v that meets at least one cycle of each length r_i for $i = 1, 2, \ldots, \lambda$.

Let $a_1, a_2, \ldots, a_k$ be distinct positive integers with $\gcd(a_1, a_2, \ldots, a_k) = 1$. The Frobenius number $\phi(a_1, a_2, \ldots, a_k)$ is defined to be the smallest integer m such that every integer with $t \ge m$ can be represented in the form $t = z_1 a_1 + z_2 a_2 + \cdots + z_k a_k$, where $z_1, z_2, \ldots, z_k$ are nonnegative integers. It is well known that $\phi(a_1, a_2, \ldots, a_k)$ is finite if $\gcd(a_1, a_2, \ldots, a_k) = 1$ and that $\phi(a_1, a_2) = (a_1 - 1)(a_2 - 1)$.

Lemma 2.1 ([7]) *Let $D = (V, E)$ be a primitive digraph and $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$. Let $L_1(D) = \{r_{i_1}, r_{i_2}, \ldots, r_{i_k}\} \subseteq L(D)$ and $\gcd(r_{i_1}, r_{i_2}, \ldots, r_{i_k}) = 1$. Then for any $u, v \in V$, we have $\exp_D(u, v) \le d_{L_1(D)}(u, v) +$*

$\phi_{L_1(D)}$, *where* $\phi_{L_1(D)} = \phi(r_{i_1}, r_{i_2}, \ldots, r_{i_\lambda})$. *Furthermore, we have* $\exp_D(u) \leq \max\{d_{L_1(D)}(u,v) : v \in V\} + \phi_{L_1(D)}$.

Lemma 2.2 ([4], Theorem 1.1) *Let* $D \in PMSD_n$ *with girth* s. *Then*

$$\exp_D(k) \leq \begin{cases} k+1+s(n-3) & \text{for} \quad 1 \leq k \leq s, \\ k+s(n-3) & \text{for} \quad s+1 \leq k \leq n. \end{cases}$$

Lemma 2.3 ([10], Theorem 4) *Let* $a_1, a_2, \ldots, a_s$ *be positive integers with* $a_1 > a_2 > \cdots > a_s$ *and* $\gcd(a_1, a_2, \ldots, a_s) = 1$. *Let* i *be the greatest subscript such that* $a_i \neq ka_s$ *for integral* k. *If there is an* a_j *such that* $a_j \neq \mu a_s + \nu a_i$ *for all nonnegative integers* μ, ν, *then* $\phi(a_1, a_2, \ldots, a_s) \leq \lfloor \frac{1}{2} a_s \rfloor (a_1 - 2)$. *Otherwise* $\phi(a_1, a_2, \ldots, a_s) = (a_s - 1)(a_i - 1)$.

Lemma 2.4 ([10], Page 83) *Let* $r(\geq 3)$ *be a positive integer,* k *a positive integer with* $k \mid (r-2)$. *Then* $\phi(\frac{r-2}{k}, r-1, r) = \lfloor \frac{r-2}{2k} \rfloor (r-2)$. *In the case* $k = 1$, $\phi(r-2, r-1, r) = \lfloor \frac{r-2}{2} \rfloor (r-2)$.

Lemma 2.5 *Let* $D \in PMSD_n$ *and* $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$, *where* $\lambda \geq 3, r_\lambda + r_{\lambda-1} > n$ *and* $r_1 > r_2 > \cdots > r_\lambda$. *Let* C_{r_λ} *and* $C_{r_{\lambda-1}}$, *respectively, be* r_λ*-cycle and* $r_{\lambda-1}$*-cycle in* D. *If* u *is a vertex in* $V(C_{r_\lambda}) \cap V(C_{r_{\lambda-1}})$, *then* $d_{L(D)}(u,v) \leq n-2$ *for any* $v \in V(D)$.

Proof. Clearly $r_1 \leq n-2$ and $r_\lambda \geq 2$ by D ministrong and $\lambda \geq 3$. Let uPv be the shortest path from u to v. We first prove that $\eta(uPv) \leq n-2$.

If $v \in V(C_{r_{\lambda-1}})$, then there exists a path from u to v in $C_{r_{\lambda-1}}$ whose length is not greater than $r_{\lambda-1}$, and so $\eta(uPv) \leq r_{\lambda-1} < r_1 \leq n-2$. If $v \in V(C_{r_\lambda})$, then there exists a path from u to v in C_{r_λ} whose length is not greater than r_λ, and so $\eta(uPv) \leq r_\lambda < r_1 \leq n-2$.

Now we suppose that $v \notin V(C_{r_\lambda} \cup C_{r_{\lambda-1}})$. Let ω be a vertex in $V(C_{r_\lambda} \cup C_{r_{\lambda-1}})$ such that $d(\omega, v) = \min\{d(x,v) : x \in V(C_{r_\lambda} \cup C_{r_{\lambda-1}})\}$ and let ωPv be a shortest path from ω to v. If $\omega \in V(C_{r_{\lambda-1}})$, note that the path $uC^{(0)}_{r_{\lambda-1}}\omega Pv$ contains no vertex in $V(C_{r_\lambda}) \backslash V(C_{r_\lambda} \cap C_{r_{\lambda-1}})$, and $|V(C_{r_\lambda}) \backslash V(C_{r_\lambda} \cap C_{r_{\lambda-1}})| \geq 1$ by D ministrong, it follows that $\eta(uPv) \leq \eta(uC^{(0)}_{r_{\lambda-1}}\omega Pv) \leq n-2$. If $\omega \in V(C_{r_\lambda})$, note that the path $uC^{(0)}_{r_\lambda}\omega Pv$ from u to v contains no vertex in $V(C_{r_{\lambda-1}}) \backslash V(C_{r_\lambda} \cap C_{r_{\lambda-1}})$, and $|V(C_{r_{\lambda-1}}) \backslash V(C_{r_\lambda} \cap C_{r_{\lambda-1}})| \geq 2$ by D ministrong and $r_{\lambda-1} > r_\lambda$, it follows that $\eta(uPv) \leq \eta(uC^{(0)}_{r_\lambda}\omega Pv) \leq n-3 < n-2$.

To sum up, we have $\eta(uPv) \leq n-2$.

Now we prove that $d_{L(D)}(u,v) \leq n-2$. If $\eta(uPv) \geq n - r_{\lambda-2} + 1$, then for $i = 1, 2, \ldots, \lambda-2$, uPv must meet at least one cycle of length r_i by $r_i + \eta(uPv) \geq n+1$, and so uPv meets at least one cycle of length r_i for $i = 1, 2, \ldots, \lambda$. Hence $d_{L(D)}(u,v) = \eta(uPv) \leq n-2$. If $\eta(uPv) \leq n - r_{\lambda-2}$, note that the walk $uC_{r_\lambda}^{(1)}uPv$ from u to v meets at least one cycle of length r_i for $i = 1, 2, \ldots, \lambda$ by $r_\lambda + r_{\lambda-1} > n$, then $d_{L(D)}(u, v) \le \eta(uC_{r_\lambda}^{(1)}uPv) = r_\lambda + \eta(uPv) \le n - r_{\lambda-2} + r_\lambda \le n - 2$. The proof of Lemma 2.5 is complete. □

Lemma 2.6 *Let $D = (V, E)$ be a primitive digraph with n vertices and $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$, where $n - 2 \ge r_1 > r_2 > \cdots > r_\lambda \ge (n-3)/2$ and $r_1 + r_\lambda > n$. Let $u \in V(C_{r_1}) \cap V(C_{r_\lambda})$. Then $d_{L(D)}(u, v) \le (3n-3)/2$ for any $v \in V$.*

Proof. Let uPv be the shortest path from u to v in D. If $\eta(uPv) \ge n - r_{\lambda-1} + 1$, then for $i = 1, 2, \ldots, \lambda - 1$, uPv meets at least one cycle of length r_i since $r_i + \eta(uPv) \ge n+1$, and so uPv meets at least one cycle of length r_i for $i = 1, 2, \ldots, \lambda$. Therefore $d_{L(D)}(u, v) = \eta(uPv) \le n - 1 \le (3n-3)/2$. If $\eta(uPv) \le n - r_{\lambda-1}$, note that the walk $uC_{r_1}^{(1)}uPv$ meets at least one cycle of length r_i for $i = 1, 2, \ldots, \lambda$ by $r_1 + r_\lambda > n$, then $d_{L(D)}(u, v) \le \eta(uC_{r_1}^{(1)}uPv) = \eta(uPv) + r_1 \le n - r_{\lambda-1} + r_1 \le n - (\frac{n-3}{2} + 1) + n - 2 = \frac{3n-3}{2}$. The proof of Lemma 2.6 is complete. □

Lemma 2.7 *Let $D = (V, E)$ be a primitive digraph with n vertices and $L(D) = \{r_1, r_2, \ldots, r_\lambda\}$, where $(n+3)/2 \ge r_1 > r_2 > \cdots > r_\lambda$, $r_\lambda \le (n-1)/2$ and $3r_\lambda > n$. Let C_{r_λ} be a r_λ-cycle and $u \in V(C_{r_\lambda})$. Then $d_{L(D)}(u, v) \le (5n-3)/2$ for any $v \in V$.*

Proof. Let uPv be the shortest path from u to v in D. If for each $i \in \{1, 2, \ldots, \lambda\}$, C_{r_λ} meets at least one cycle of length r_i, then $d_{L(D)}(u, v) \le \eta(uC_{r_\lambda}^{(1)}uPv) = \eta(uPv) + r_\lambda \le n - 1 + (n-1)/2 = (3n-3)/2 \le (5n-3)/2$.

If there exists some $j \in \{1, 2, \ldots, \lambda - 1\}$ such that C_{r_λ} does not meet any cycle of length r_j, let C_{r_j} be a cycle of length r_j, then $V(C_{r_j}) \cap V(C_{r_\lambda}) = \emptyset$. From $3r_\lambda > n$, $C_{r_\lambda} \cup C_{r_j}$ must meet at least one cycle of length r_i for $i = 1, 2, \ldots, \lambda$. Let z be a vertex in $V(C_{r_j})$ such that $d(u, z) = \min\{d(u, y) : y \in V(C_{r_j})\}$, and let uP_1z, zP_2v, respectively, be a shortest path from u to z and from z to v. We can check that $d(u, z) \le n - r_j$. Note that the walk $uC_{r_\lambda}^{(1)}uP_1zC_{r_j}^{(1)}zP_2v$ from u to v meets at least one cycle of length r_i for $i = 1, 2, \ldots, \lambda$. Hence $d_{L(D)}(u, v) \le \eta(uC_{r_\lambda}^{(1)}uP_1zC_{r_j}^{(1)}zP_2v) = r_\lambda + \eta(uP_1z) + r_j + \eta(zP_2v) \le r_\lambda + (n - r_j) + r_j + (n-1) = 2n + r_\lambda - 1 \le 2n + \frac{n-1}{2} - 1 = \frac{5n-3}{2}$. The proof of Lemma 2.7 is complete. □

Lemma 2.8 *Let $D = (V, E)$ be a primitive digraph which contains precisely three cycles C_{r_1}, C_{r_2} and C_{r_3}, where $r_1 \ge r_2 \ge r_3$, C_{r_i} is a cycle of length r_i for $i = 1, 2, 3$, and $V(C_{r_1}) \cap V(C_{r_2}) \cap V(C_{r_3}) \ne \emptyset$. For any nonnegative integer t, write $Y = \{a \mid t = k_1r_1 + k_2r_2 + k_3r_3 + a,\ k_i (i = 1, 2, 3)$ and a are nonnegative integers and $a \le r_1 - 1\}$. If $u \in V(C_{r_1}) \cap V(C_{r_2}) \cap V(C_{r_3})$. Then $R_t(u) = \bigcup_{a \in Y} R_a(u)$.*

Proof. First we prove that $R_t(u) \supseteq \bigcup_{a\in Y} R_a(u)$. From the definition of Y, for any $a \in Y$, there is nonnegative integers k_1, k_2, k_3 and $a \in [0,\dots,r_1-1]$ such that $t = k_1r_1 + k_2r_2 + k_3r_3 + a$. If the vertex $v \in R_a(u)$, namely there is a walk from u to v of length a, then there is a walk from u to v of length t since $u \in V(C_{r_1}) \cap V(C_{r_2}) \cap V(C_{r_3})$, and so $v \in R_t(u)$. Therefore $R_a(u) \subseteq R_t(u)$, and thus $\bigcup_{a\in Y} R_a(u) \subseteq R_t(u)$.

Now we prove that $R_t(u) \subseteq \bigcup_{a\in Y} R_a(u)$. If $v \in R_t(u)$, since D is primitive, then there must exist $i \in \{1,2,3\}$ such that $v \in V(C_{r_i})$. Let uWv be a walk from u to v of length t. Since D is primitive and $u \in V(C_{r_1}) \cap V(C_{r_2}) \cap V(C_{r_3})$, then uWv can be expressed as

$$uWv = \underbrace{C_{r_1} + \cdots + C_{r_1}}_{k_1 \text{ times}} + \underbrace{C_{r_2} + \cdots + C_{r_2}}_{k_2 \text{ times}} + \underbrace{C_{r_3} + \cdots + C_{r_3}}_{k_3 \text{ times}} + uC_{r_i}^{(0)}v,$$

where k_1, k_2, k_3 are nonnegative integers. Write $b = \eta(uC_{r_i}^{(0)}v)$ (then $v \in R_b(u)$). Clearly $0 \le b \le r_i-1 \le r_1-1$ and $t = k_1r_1+k_2r_2+k_3r_3+b$, namely $b \in Y$, and so $v \in R_b(u) \subseteq \bigcup_{a\in Y} R_a(u)$. Therefore $R_t(u) \subseteq \bigcup_{a\in Y} R_a(u)$. The proof of Lemma 2.8 is complete. $\square$

Lemma 2.9 (i) *Let n be an odd with $n \ge 14$, $Y_1 = \{a \mid (n^2-9n+22)/2 = k_1(n-3)/2 + k_2(n-2) + a,\ k_1, k_2$ are nonnegative integers and $a \in [0,\dots,n-3]\}$, $Y_2 = \{a \mid (n^2-9n+24)/2 = k_1(n-3)/2 + k_2(n-2) + a,\ k_1, k_2$ are nonnegative integers and $a \in [0,\dots,n-3]\}$. Then $Y_1 = [0,\dots,n-3]\backslash\{3, (n+3)/2, (n+5)/2\}$, $Y_2 = [0,\dots,n-3]\backslash\{4, (n+5)/2, (n+7)/2\}$.*

(ii) *Let n be an even with $n \ge 14$, $Y_3 = \{a \mid (n^2-9n+22)/2 = k_1(n-2)/2 + k_2(n-3) + a,\ k_1, k_2$ are nonnegative integers and $a \in [0,\dots,n-3]\}$, $Y_4 = \{a \mid (n^2-9n+24)/2 = k_1(n-2)/2 + k_2(n-3) + a,\ k_1, k_2$ are nonnegative integers and $a \in [0,\dots,n-3]\}$, $Y_5 = \{a \mid (n^2-7n+16)/2 = k_1(n-2)/2 + k_2(n-3) + a,\ k_1, k_2$ are nonnegative integers and $a \in [0,\dots,n-3]\}$. Then $Y_3 = [0,\dots,n-3]\backslash\{3, (n+2)/2, (n+4)/2\}$, $Y_4 = [0,\dots,n-3]\backslash\{4, (n+4)/2, (n+6)/2\}$, $Y_5 = [0,\dots,n-3]\backslash\{(n+2)/2\}$.*

Proof. Since

$$\begin{aligned}\frac{n^2-9n+22}{2} &= (2k+1)\frac{n-3}{2} + (\frac{n-7}{2} - k)(n-2) + (k - \frac{n-11}{2})\\ &= 2k\frac{n-3}{2} + (\frac{n-7}{2} - k)(n-2) + (k+4)\\ &= (2k-1)\frac{n-3}{2} + (\frac{n-7}{2} - k)(n-2) + (k + \frac{n+5}{2}).\end{aligned}$$

Hence

$$\begin{aligned}&\{k-\frac{n-11}{2}\mid k=\frac{n-11}{2},\frac{n-9}{2},\frac{n-7}{2}\}\cup\{k+4\mid k=\\&\quad 0,1,2,\ldots,\frac{n-7}{2}\}\cup\{k+\frac{n+5}{2}\mid k=1,2,\ldots,\frac{n-11}{2}\}\\&=\{0,1,2\}\cup\{4,5,\ldots,\frac{n+1}{2}\}\cup\{\frac{n+7}{2},\frac{n+9}{2}\ldots,n-3\}\subseteq Y_1.\end{aligned}$$

By the definition of Frobenius number, $\phi((n-3)/2,n-2)-1=(n^2-8n+13)/2$ can not be represented in the form $k_1(n-3)/2+k_2(n-2)$, where k_1,k_2 are nonnegative integers. Hence every one of the numbers $(n^2-8n+13)/2$ $-(n-3)/2=(n^2-9n+16)/2$, $(n^2-8n+13)/2-(n-3)=(n^2-10n+19)/2$, and $(n^2-8n+13)/2-(n-2)=(n^2-10n+17)/2$ can not be represented in the form $k_1(n-3)/2+k_2(n-2)$ (k_1,k_2 are nonnegative integers). It follows that $(n^2-9n+22)/2$ can not be represented in any one of the forms $k_1(n-3)/2+k_2(n-2)+3$, $k_1(n-3)/2+k_2(n-2)+(n+3)/2$ and $k_1(n-3)/2+k_2(n-2)+(n+5)/2$. In other words, $3\notin Y_1$, $(n+3)/2\notin Y_1$, and $(n+5)/2\notin Y_1$. Therefore $Y_1=[0,\ldots,n-3]\setminus\{3,(n+3)/2,(n+5)/2\}$.

For $Y_i (i=2,3,4,5)$, we can prove the results in the similar method. The proof of Lemma 2.9 is complete. □

Theorem 2.1 *Let $D\in PMSD_n$ with $n\geq 14$ and $|L(D)|\geq 3$. Then*

$$\exp_D(1)\leq\frac{1}{2}(n^2-7n+16).$$

Proof. Let $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$ with $r_1>r_2>\cdots>r_\lambda$. Then $r_\lambda\geq 2$ and $r_1\leq n-2$ by D ministrong and $\lambda=|L(D)|\geq 3$. We divide the proof into the following five cases.

Case 1: $r_\lambda\leq(n-4)/2$. It follows from Lemma 2.2 that

$$\exp_D(1)\leq 2+\frac{n-4}{2}(n-3)=\frac{n^2-7n+16}{2}.$$

Case 2: $n/2\leq r_\lambda\leq n-5$. Then $\phi_{L(D)}\leq\lfloor r_\lambda/2\rfloor(r_1-2)$ by Lemma 2.3. Let C_{r_λ} be a r_λ-cycle and $C_{r_{\lambda-1}}$ a $r_{\lambda-1}$-cycle in D. Then $V(C_{r_\lambda})\cap V(C_{r_{\lambda-1}})\neq\emptyset$ since $r_\lambda+r_{\lambda-1}>n$. Let u be a vertex in $V(C_{r_\lambda})\cap V(C_{r_{\lambda-1}})$. By Lemmas 2.1 and 2.5,

$$\begin{aligned}\exp_D(u)&\leq\max\{d_{L(D)}(u,v):v\in V\}+\phi_{L(D)}\\&\leq n-2+\lfloor\frac{r_\lambda}{2}\rfloor(r_1-2)\leq n-2+\lfloor\frac{n-5}{2}\rfloor(n-4)\\&\leq(n-2)+\frac{n-5}{2}(n-4)=\frac{1}{2}(n^2-7n+16).\end{aligned}$$

Case 3: $r_\lambda = n-4$. Then $L(D) = \{n-4, n-3, n-2\}$, and so $\phi_{L(D)} = \lfloor (n-4)/2 \rfloor (n-4)$ by Lemma 2.4. Write

$$E' = \{(v_i, v_{i+1}) : i = 1, 2, \ldots, n-3\} \cup \{(v_{n-2}, v_1), (v_{n-4}, v_n), (v_n, v_1)\}.$$

We can check that D must be isomorphic to one of the digraphs $D_1 \sim D_7$: $D_i = (V_i, E_i) (i = 1, 2, \ldots, 7)$, where

$$\begin{aligned}
V_i &= \{v_1, v_2, \ldots, v_n\} (i = 1, 2, \ldots, 7),\\
E_1 &= E' \cup \{(v_k, v_{n-1}), (v_{n-1}, v_{k+3})\} (1 \le k \le n-7),\\
E_2 &= E' \cup \{(v_{n-7}, v_{n-1}), (v_{n-1}, v_{n-3})\},\\
E_3 &= E' \cup \{(v_{n-6}, v_{n-1}), (v_{n-1}, v_{n-2})\},\\
E_4 &= E' \cup \{(v_{n-5}, v_{n-1}), (v_{n-1}, v_1)\},\\
E_5 &= E' \cup \{(v_{n-4}, v_{n-1}), (v_{n-1}, v_2)\},\\
E_6 &= E' \cup \{(v_{n-3}, v_{n-1}), (v_{n-1}, v_3)\},\\
E_7 &= E' \cup \{(v_{n-2}, v_{n-1}), (v_{n-1}, v_4)\}.
\end{aligned}$$

Subcase 3.1: $D \cong D_1$. We can check that for any positive integer t, $R_t(v_k) = R_{t+k+1}(v_{n-3})$ and $R_t(v_{n-4}) = R_{t+n-k-5}(v_{k+1})$. It follows that $\exp_{D_1}(v_k) = \exp_{D_1}(v_{n-3}) - (k+1)$ and $\exp_{D_1}(v_{n-4}) = \exp_{D_1}(v_{k+1}) - (n-k-5)$. Hence

$$\begin{aligned}
\exp_{D_1}(1) &\le \min\{\exp_{D_1}(v_k), \exp_{D_1}(v_{n-4})\}\\
&= \min\{\exp_{D_1}(v_{n-3}) - (k+1), \exp_{D_1}(v_{k+1}) - (n-k-5)\}.
\end{aligned}$$

By Lemma 2.1, $\exp_{D_1}(v_{n-3}) \le \max\{d_{L(D_1)}(v_{n-3}, v) : v \in V\} + \phi_{L(D_1)} = d_{L(D_1)}(v_{n-3}, v_{n-2}) + \phi_{L(D_1)} = n-2+\lfloor \frac{n-4}{2} \rfloor (n-4)$ and $\exp_{D_1}(v_{k+1}) \le \max\{d_{L(D_1)}(v_{k+1}, v) : v \in V\} + \phi_{L(D_1)} = d_{L(D_1)}(v_{k+1}, v_{k+2}) + \phi_{L(D_1)} = n-2+\lfloor \frac{n-4}{2} \rfloor (n-4)$. Hence

$$\exp_{D_1}(1) \le (n-2) + \lfloor \frac{n-4}{2} \rfloor (n-4) - \max\{k+1, n-k-5\}.$$

Since $k+1+(n-k-5) = n-4$, then $\max\{k+1, n-k-5\} \ge (n-4)/2$, and so

$$\begin{aligned}
\exp_D(1) &= \exp_{D_1}(1) \le n-2+\lfloor \frac{n-4}{2} \rfloor (n-4) - \frac{n-4}{2}\\
&\le \frac{n}{2} + \frac{n-4}{2}(n-4) = \frac{1}{2}(n^2 - 7n + 16).
\end{aligned}$$

Subcase 3.2: $D \cong D_2$. Clearly for any positive integer t, $R_t(v_{n-7}) = R_{t+(n-8)}(v_1)$. Hence $\exp_{D_2}(v_{n-7}) = \exp_{D_2}(v_1) - (n-8)$. By Lemma 2.1, we have

$$\begin{aligned}\exp_{D_2}(v_1) &\le \max\{d_{L(D_2)}(v_1,v) : v \in V\} + \phi_{L(D_2)}\\ &= d_{L(D_2)}(v_1,v_1) + \phi(n-4,n-3,n-2)\\ &= n-4+\lfloor\frac{n-4}{2}\rfloor(n-4).\end{aligned}$$

Hence

$$\begin{aligned}\exp_{D_2}(v_{n-7}) &\le (n-4)+\lfloor\frac{n-4}{2}\rfloor(n-4)-(n-8)\\ &\le 4+\frac{n-4}{2}(n-4) = \frac{1}{2}(n^2-8n+24)\\ &\le \frac{1}{2}(n^2-7n+16) \ \text{ since } \ n \ge 14.\end{aligned}$$

Therefore $\exp_D(1) = \exp_{D_2}(1) \le \exp_{D_2}(v_{n-7}) \le \frac{1}{2}(n^2-7n+16)$.

Subcase 3.3: D is isomorphic to one of the the digraphs $D_3 \sim D_7$. By using the same method as in the proof of Subcase 3.2, we obtain that

$$\begin{aligned}\exp_{D_3}(v_{n-6}) &= \exp_{D_3}(v_1)-(n-7)\\ &\le (n-4)+\lfloor\frac{n-4}{2}\rfloor(n-4)-(n-7)\\ &\le \frac{1}{2}(n^2-7n+16),\end{aligned}$$

$$\begin{aligned}\exp_{D_4}(v_{n-5}) &= \exp_{D_4}(v_1)-(n-6)\\ &\le (n-3)+\lfloor\frac{n-4}{2}\rfloor(n-4)-(n-6)\\ &\le \frac{1}{2}(n^2-7n+16),\end{aligned}$$

$$\begin{aligned}\exp_{D_5}(v_{n-4}) &= \exp_{D_5}(v_2)-(n-6)\\ &\le (n-4)+\lfloor\frac{n-4}{2}\rfloor(n-4)-(n-6)\\ &\le \frac{1}{2}(n^2-7n+16),\end{aligned}$$

$$\begin{aligned}\exp_{D_6}(v_{n-4}) &= \exp_{D_6}(v_3)-(n-7)\\ &\le (n-4)+\lfloor\frac{n-4}{2}\rfloor(n-4)-(n-7)\\ &\le \frac{1}{2}(n^2-7n+16),\end{aligned}$$

and

$$\begin{aligned}\exp_{D_7}(v_{n-4}) &= \exp_{D_7}(v_4) - (n-8)\\ &\le (n-4) + \lfloor\frac{n-4}{2}\rfloor(n-4) - (n-8)\\ &\le \frac{1}{2}(n^2 - 7n + 16).\end{aligned}$$

Therefore

$$\exp_D(1) \le \frac{1}{2}(n^2 - 7n + 16) \text{ for } D \cong D_i \ \ (i = 3, 4, \ldots, 7).$$

Case 4: $(n-3)/2 \le r_\lambda \le (n-1)/2$ and there exists $r_j \in L(D)$ such that $r_j \ne \mu r_i + \nu r_\lambda$ for all nonnegative integers μ, ν, where i is the greatest subscript such that $r_i \ne k r_\lambda$ for integral k.

Subcase 4.1: $(n+3)/2 < r_1 \le n-2$. Then $r_1 + r_\lambda > n$. Let C_{r_1} be a r_1-cycle in D and C_{r_λ} a r_λ-cycle in D. It follows that $V(C_{r_1}) \cap V(C_{r_\lambda}) \ne \emptyset$. Let u be a vertex in $V(C_{r_1}) \cap V(C_{r_\lambda})$ and v any vertex in D. By Lemmas 2.1, 2.3 and 2.6, we have

$$\begin{aligned}\exp_D(1) &\le \exp_D(u) \le \max\{d_{L(D)}(u,v) : v \in V\} + \phi_{L(D)}\\ &\le \frac{3n-3}{2} + [\frac{r_\lambda}{2}](r_1 - 2) \le \frac{3n-3}{2} + \frac{n-1}{4}(n-4)\\ &= \frac{n^2+n-2}{4} \le \frac{1}{2}(n^2 - 7n + 16) \quad \text{since} \quad n \ge 14.\end{aligned}$$

Subcase 4.2: $r_1 \le (n+3)/2$. Note that $3r_\lambda > (3n-9)/2 > n$ by $n \ge 14$. Let $u \in V(C_{r_\lambda})$, where C_{r_λ} is a r_λ-cycle of D. Then for any $v \in V$, we have $d_{L(D)}(u,v) \le (5n-3)/2$ by Lemma 2.7. By Lemmas 2.1, 2.3 and $n \ge 14$,

$$\begin{aligned}\exp_D(1) &\le \exp_D(u) \le \max\{d_{L(D)}(u,v) : v \in V\} + \phi_{L(D)}\\ &\le (5n-3)/2 + \lfloor r_\lambda/2 \rfloor (r_1 - 2) \le \frac{5n-3}{2} + \frac{(n-1)^2}{8}\\ &= \frac{n^2 + 18n - 11}{8} \le \frac{1}{2}(n^2 - 7n + 16).\end{aligned}$$

Case 5: $(n-3)/2 \le r_\lambda \le (n-1)/2$ and for each $j \in \{1, 2, \ldots, \lambda\}, r_j$ can be expressed as $r_j = \mu r_i + \nu r_\lambda$, where μ, ν are nonnegative integers and i is the greatest subscript such that $r_i \ne k r_\lambda$ for integral k. Then $\phi_{L(D)} = \phi(r_i, r_\lambda)$ by Lemma 2.3. Let j_1 be any number in $\{1, 2, \ldots, \lambda\} \backslash \{i, \lambda\}$. Then $r_{j_1} \ge 2r_\lambda$.

Subcase 5.1: If $r_i + r_\lambda \le n$, then

$$\phi_{L(D)} = \phi(r_i, r_\lambda) = (r_i - 1)(r_\lambda - 1) \le (\frac{r_i + r_\lambda - 2}{2})^2 \le (\frac{n-2}{2})^2.$$

Let C_{r_1} be a r_1-cycle in D and C_{r_λ} be a r_λ-cycle in D. Since

$$r_1 + r_\lambda \geq r_{j_1} + r_\lambda \geq 3r_\lambda \geq \frac{3n-9}{2} > n \text{ by } n \geq 14,$$

then $V(C_{r_1}) \cap V(C_{r_\lambda}) \neq \emptyset$. Let vertex $u \in V(C_{r_1}) \cap V(C_{r_\lambda})$. Then by Lemma 2.6, $d_{L(D)}(u,v) \leq (3n-3)/2$ for any vertex $v \in V$.
By Lemma 2.1 and $n \geq 14$,

$$\begin{aligned} \exp_D(1) &\leq \exp_D(u) \leq \max\{d_{L(D)}(u,v) : v \in V\} + \phi_{L(D)} \\ &\leq \frac{3n-3}{2} + \frac{(n-2)^2}{4} = \frac{n^2+2n-2}{4} \\ &\leq \frac{1}{2}(n^2 - 7n + 16). \end{aligned}$$

Subcasse 5.2: If $r_i + r_\lambda > n$, then $2r_\lambda \leq r_{j_1} < 3r_\lambda$ by $3r_\lambda \geq 3(n-3)/2 > n$. We claim that $r_{j_1} = 2r_\lambda$. Otherwise, r_{j_1} can not be expressed as $r_{j_1} = \mu r_i + \nu r_\lambda$ (μ, ν are nonnegative integers), which is a contradiction. Therefore $L(D) = \{r_i, r_\lambda, 2r_\lambda\}$. For the sake of simplicity, we denote r_i by r, namely $L(D) = \{r, r_\lambda, 2r_\lambda\}$. Since $2r_\lambda \leq n-2$, namely $r_\lambda \leq (n-2)/2$, Then $r_\lambda \in \{(n-2)/2, (n-3)/2\}$ by $r_\lambda \geq (n-3)/2$.

(i) Suppose that $r_\lambda = (n-3)/2$ and $r < 2r_\lambda = n-3$. If C_r is a r-cycle in D and C_{r_λ} a r_λ-cycle in D, then $V(C_r) \cap V(C_{r_\lambda}) \neq \emptyset$ by $r + r_\lambda > n$. Let u be a vertex in $V(C_r) \cap V(C_{r_\lambda})$. Then $d_{L(D)}(u,v) \leq n-2$ for any vertex $v \in V$ by Lemma 2.5. It follows from Lemma 2.1 and $n \geq 14$ that

$$\begin{aligned} \exp_D(u) &\leq \max\{d_{L(D)}(u,v) : v \in V\} + \phi_{L(D)} \\ &\leq n - 2 + (\frac{n-3}{2} - 1)(r-1) \\ &< n - 2 + \frac{n-5}{2}(n-4) = \frac{n^2 - 7n + 16}{2}. \end{aligned}$$

(ii) Suppose that $r_\lambda = (n-3)/2$ and $r > 2r_\lambda = n-3$. Then $r = n-2$, and so $L(D) = \{(n-3)/2, n-3, n-2\}$. Write

$$\tilde{E} = \{(v_i, v_{i+1}) : i = 1, 2, \ldots, n-3\} \cup \{(v_{n-2}, v_1), (v_{\frac{n-5}{2}}, v_{n-1}), (v_{n-1}, v_1)\}.$$

Then D is isomorphic to one of the digraphs $D_i = (V_i, E_i)(i = 8, 9, \ldots, 13)$, where $V_i = \{v_1, v_2, \ldots, v_n\}(i = 8, 9, \ldots, 13)$,

$E_8 = \tilde{E} \cup \{(v_j, v_n), (v_n, v_{j+3})\}((n-5)/2 \leq j \leq n-5)$,
$E_9 = \tilde{E} \cup \{(v_{n-4}, v_n), (v_n, v_1)\}$, $E_{10} = \tilde{E} \cup \{(v_{n-3}, v_n), (v_n, v_2)\}$,
$E_{11} = \tilde{E} \cup \{(v_{n-2}, v_n), (v_n, v_3)\}$, $E_{12} = \tilde{E} \cup \{(v_{\frac{n-9}{2}}, v_n), (v_n, v_{\frac{n-3}{2}})\}$,
$E_{13} = \tilde{E} \cup \{(v_{\frac{n-7}{2}}, v_n), (v_n, v_{\frac{n-1}{2}})\}$.

Now we prove that

$$\exp_{D_i}(1) \le \frac{1}{2}(n^2 - 7n + 16) \quad \text{for } i = 8, 9, \ldots, 13.$$

Clearly for $D_i (i = 8, 9, 10, 12, 13)$, $v_{\frac{n-5}{2}} \in V(C_{\frac{n-3}{2}}) \cap V(C_{n-2})$, where $C_{\frac{n-3}{2}}$ and C_{n-2} are respectively the $\frac{n-3}{2}$-cycle and $(n-2)$-cycle of D_i. Write $L_1(D_i) = \{\frac{n-3}{2}, n-2\} (i = 8, 9, 10, 12, 13)$. We can check that for each $i \in \{8, 9, 10, 12, 13\}$,

$$\max\{d_{L_1(D_i)}(v_{\frac{n-5}{2}}, v) : v \in V\} = d_{L_1(D_i)}(v_{\frac{n-5}{2}}, v_{n-2}) \le \frac{n+1}{2}.$$

By Lemma 2.1,

$$\begin{aligned} \exp_{D_i}(v_{\frac{n-5}{2}}) &\le \max\{d_{L_1(D_i)}(v_{\frac{n-5}{2}}, v) : v \in V\} + \phi_{L_1(D_i)} \\ &\le \frac{n+1}{2} + (n-3)\frac{n-5}{2} \\ &= \frac{n^2 - 7n + 16}{2} \quad (i = 8, 9, 10, 12, 13). \end{aligned}$$

Hence

$$\exp_{D_i}(1) \le \frac{n^2 - 7n + 16}{2} \qquad (i = 8, 9, 10, 12, 13).$$

To prove that $\exp_{D_{11}}(1) \le (n^2 - 7n + 16)/2$, we first prove that $\exp_{D_{11}}(v_{\frac{n-3}{2}})$ $(n^2 - 7n + 16)/2$. Since

$$\begin{aligned} R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-3}{2}}) &= R_{\frac{n^2-8n+17}{2}}(R_{\frac{n-1}{2}}(v_{\frac{n-3}{2}})) = R_{\frac{n^2-8n+17}{2}}(v_{n-2}) \\ &= R_{\frac{n^2-8n+15}{2}}(R_1(v_{n-2})) = R_{\frac{n^2-8n+15}{2}}(v_n) \cup R_{\frac{n^2-8n+15}{2}}(v_1) \\ &= R_{\frac{n^2-9n+24}{2}}(v_{\frac{n-5}{2}}) \cup R_{\frac{n^2-9n+22}{2}}(v_{\frac{n-5}{2}}), \end{aligned}$$

and by Lemma 2.9, we have

$$\begin{aligned} &\{a \mid \frac{n^2 - 9n + 24}{2} = k_1\frac{n-3}{2} + k_2(n-3) + k_3(n-2) + a, \ \ k_1, k_2, k_3 \\ &\quad \text{are nonnegative integers and } 0 \le a \le n-3\} \\ &= \{a \mid \frac{n^2 - 9n + 24}{2} = k_1\frac{n-3}{2} + k_2(n-2) + a, \ \ k_1, k_2 \\ &\quad \text{are nonnegative integers and } 0 \le a \le n-3\} \\ &= \{0, 1, 2, \ldots, n-3\} - \{4, \frac{n+5}{2}, \frac{n+7}{2}\}, \end{aligned}$$

$$\{a \mid \frac{n^2-9n+22}{2} = k_1\frac{n-3}{2} + k_2(n-3) + k_3(n-2) + a, \ \ k_1, k_2, k_3$$
$$\text{are nonnegative integers and } 0 \le a \le n-3\}$$
$$= \{a \mid \frac{n^2-9n+22}{2} = k_1\frac{n-3}{2} + k_2(n-2) + a, \ \ k_1, k_2$$
$$\text{are nonnegative integers and } 0 \le a \le n-3\}$$
$$= \{0, 1, 2, \ldots, n-3\} - \{3, \frac{n+3}{2}, \frac{n+5}{2}\}.$$

Note that $v_{\frac{n-5}{2}} \in V(C_{\frac{n-3}{2}}) \cap V(C_{n-3}) \cap V(C_{n-2})$, where $C_{\frac{n-3}{2}}$, C_{n-3}, and C_{n-2}, respectively, are the $\frac{n-3}{2}$-cycle, the $(n-3)$-cycle and the $(n-2)$-cycle of D_{11}. By Lemma 2.8, we have

$$R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-3}{2}}) = \bigcup_{\substack{0 \le i \le n-3 \\ i \ne \frac{n+5}{2}}} R_i(v_{\frac{n-5}{2}}).$$

We can check that $\bigcup_{i=0}^{\frac{n+3}{2}} R_i(v_{\frac{n-5}{2}}) = V$. It follows that $R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-3}{2}}) = V$, and so

$$\exp_{D_{11}}(v_{\frac{n-3}{2}}) \le \frac{1}{2}(n^2 - 7n + 16).$$

Therefore

$$\exp_{D_{11}}(1) \le \exp_{D_{11}}(v_{\frac{n-3}{2}}) \le \frac{1}{2}(n^2 - 7n + 16).$$

(iii) Suppose that $r_\lambda = (n-2)/2$. Then $L(D) = \{(n-2)/2, r, n-2\}$. We can check that $r < n-2$ and $r + (n-2)/2 = r + r_\lambda > n$, and so $(n+4)/2 \le r \le n-3$. Write

$$\hat{E} = \{(v_j, v_{j+1}) \colon j = 1, 2, \ldots, n-3\} \cup \{(v_{n-2}, v_1), (v_{\frac{n-4}{2}}, v_{n-1}), (v_{n-1}, v_1)\}.$$

Then D is isomorphic to one of the digraph $D_i = (V_i, \ E_i)$ $(i = 14, 15, 16)$, where $V_i = \{v_1, v_2, \ldots, v_n\}$ $(i = 14, 15, 16)$,

$$E_{14} = \hat{E} \cup \{(v_j, v_n), (v_n, v_{n-r+j})\} \ \ (\frac{n-4}{2} \le j \le r-2),$$
$$E_{15} = \hat{E} \cup \{(v_j, v_n), (v_n, v_{j+2-r})\} \ \ (r-1 \le j \le n-2),$$
$$E_{16} = \hat{E} \cup \{(v_j, v_n), (v_n, v_{n-r+j})\} \ \ (r - \frac{n+2}{2} \le j \le \frac{n-6}{2}).$$

First we consider D_{14}. We have

$$\begin{aligned}&\max\ \{d_{L(D_{14})}(v_{\frac{n-4}{2}}, v) : v \in V_{14}\}\\&= \max\{d(v_{\frac{n-4}{2}}, v_{\frac{n-4}{2}}), d(v_{\frac{n-4}{2}}, v_{n-2}), d(v_{\frac{n-4}{2}}, v_{n-r+j-1})\}\\&= \max\{\frac{n-2}{2},\ r-\frac{n-4}{2},\ \frac{n+2}{2}-r+j\}.\end{aligned}$$

Since

$$r-\frac{n-4}{2} \leq n-3-\frac{n-4}{2} = \frac{n-2}{2}$$

and

$$\frac{n+2}{2}-r+j \leq \frac{n+2}{2}-r+(r-2) = \frac{n-2}{2},$$

then $\max\{d_{L(D_{14})}(v_{\frac{n-4}{2}}, v) : v \in V_{14}\} = \frac{n-2}{2}$. By Lemma 2.1, we have

$$\begin{aligned}\exp_{D_{14}}(1) &\leq \exp_{D_{14}}(v_{\frac{n-4}{2}})\\&\leq \max\{d_{L(D)}(v_{\frac{n-4}{2}}, v) : v \in V\} + \phi_{L(D)}\\&= \frac{n-2}{2}+\frac{n-4}{2}(r-1) \leq \frac{n-2}{2}+\frac{n-4}{2}(n-4)\\&= \frac{n^2-7n+14}{2} < \frac{n^2-7n+16}{2}.\end{aligned}$$

Next we consider D_{15}. We have

$$\begin{aligned}&\max\ \{d_{L(D_{15})}(v_{\frac{n-4}{2}}, v) : v \in V_{15}\}\\&= \max\{d(v_{\frac{n-4}{2}}, v_{\frac{n-4}{2}}), d(v_{\frac{n-4}{2}}, v_{n-2}), d(v_{\frac{n-4}{2}}, v_n)\}\\&= \max\{\frac{n-2}{2}, \frac{n}{2}, j-\frac{n-6}{2}\} = \begin{cases} n/2, & \text{if } j \leq n-3,\\ (n+2)/2, & \text{if } j = n-2.\end{cases}\end{aligned}$$

If $j \leq n-3$, it follows from Lemma 2.1 that

$$\begin{aligned}\exp_{D_{15}}(1) &\leq \exp_{D_{15}}(v_{\frac{n-4}{2}})\\&\leq \max\{d_{L(D_{15})}(v_{\frac{n-4}{2}}, v) : v \in V_{15}\} + \phi_{L(D_{15})}\\&= \frac{n}{2}+\frac{n-4}{2}(r-1) \leq \frac{n}{2}+\frac{n-4}{2}(n-4)\\&= \frac{1}{2}(n^2-7n+16).\end{aligned}$$

If $j = n-2$ and $r \leq n-4$, it follows from Lemma 2.1 that

$$\begin{aligned} \exp_{15}(1) &\leq \exp_{D_{15}}(v_{\frac{n-4}{2}}) \\ &\leq \max\{d_{L(D_{15})}(v_{\frac{n-4}{2}}, v) : v \in V_{15}\} + \phi_{L(D_{15})} \\ &= \frac{n+2}{2} + \frac{n-4}{2}(r-1) \leq \frac{n+2}{2} + \frac{n-4}{2}(n-5) \\ &= \frac{n^2-8n+22}{2} \leq \frac{n^2-7n+16}{2}. \end{aligned}$$

If $j = n-2$ and $r = n-3$, we come to prove that

$$\exp_{D_{15}}(v_{\frac{n-2}{2}}) \leq \frac{n^2-7n+16}{2}.$$

We have

$$\begin{aligned} R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-2}{2}}) &= R_{\frac{n^2-6n+14}{2}-j}(R_{j-\frac{n-2}{2}}(v_{\frac{n-2}{2}})) \\ &= R_{\frac{n^2-6n+14}{2}-j}(v_j) = R_{\frac{n^2-6n+12}{2}-j}(R_1(v_j)) \\ &= R_{\frac{n^2-6n+12}{2}-j}(v_n) \cup R_{\frac{n^2-6n+12}{2}-j}(v_1) \\ &= R_{\frac{n-9n+24}{2}}(v_{\frac{n-4}{2}}) \cup R_{\frac{n^2-9n+22}{2}}(v_{\frac{n-4}{2}}). \end{aligned}$$

By Lemma 2.9, we have

$$\begin{aligned} &\{a \mid \frac{n^2-9n+24}{2} = k_1\frac{n-2}{2} + k_2(n-3) + k_3(n-2) + a, \quad k_1, k_2, k_3 \\ &\quad \text{are nonnegative integers and } 0 \leq a \leq n-3\} \\ &= \{a \mid \frac{n^2-9n+24}{2} = k_1\frac{n-2}{2} + k_2(n-3) + a, \quad k_1, k_2 \\ &\quad \text{are nonnegative integers and } 0 \leq a \leq n-3\} \\ &= \{0, 1, 2, \ldots, n-3\} \backslash \{4, \frac{n+4}{2}, \frac{n+6}{2}\}, \end{aligned}$$

$$\begin{aligned} &\{a \mid \frac{n^2-9n+22}{2} = k_1\frac{n-2}{2} + k_2(n-3) + k_3(n-2) + a, \quad k_1, k_2, k_3 \\ &\quad \text{are nonnegative integers and } 0 \leq a \leq n-3\} \\ &= \{a \mid \frac{n^2-9n+22}{2} = k_1\frac{n-2}{2} + k_2(n-3) + a, \quad k_1, k_2 \\ &\quad \text{are nonnegative integers and } 0 \leq a \leq n-3\} \\ &= \{0, 1, 2, \ldots, n-3\} \backslash \{3, \frac{n+2}{2}, \frac{n+4}{2}\}. \end{aligned}$$

Note that $v_{\frac{n-4}{2}} \in V(C_{\frac{n-2}{2}}) \cap V(C_{n-3}) \cap V(C_{n-2})$, where $C_{\frac{n-2}{2}}, C_{n-3}$ and C_{n-2}, respectively, denote the $\frac{n-2}{2}$-cycle, $(n-3)$-cycle, and $(n-2)$-cycle of D_{15}. By Lemma 2.8, we have

$$R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-2}{2}}) = \bigcup_{\substack{i=0 \\ i \neq \frac{n+4}{2}}}^{n-3} R_i(v_{\frac{n-4}{2}}).$$

We can check that $\bigcup_{i=0}^{\frac{n+2}{2}} R_i(v_{\frac{n-4}{2}}) = V_{15}$. Hence $R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-2}{2}}) = V_{15}$. Therefore

$$\exp_{D_{15}}(1) \leq \exp_{D_{15}}(v_{\frac{n-2}{2}}) \leq \frac{1}{2}(n^2 - 7n + 16).$$

Now we consider D_{16}. Since

$$n - r - 1 \leq n - \frac{n+4}{2} - 1 = \frac{n-6}{2} < \frac{n-2}{2},$$

we have

$$\begin{aligned}
&\max\{d_{L(D_{16})}(v_j, v) : v \in V_{16}\} \\
&= \max\{d(v_j, v_j), d(v_j, v_{n-2}), d(v_j, v_{n-r+j-1})\} \\
&= \max\{\frac{n-2}{2}, r-j, n-r-1\} \\
&= \max\{\frac{n-2}{2}, r-j\}.
\end{aligned}$$

Suppose that $r \leq n-4$. Note that $r - j \leq (n+2)/2$. Then by Lemma 2.1 and $n \geq 14$,

$$\begin{aligned}
\exp_{D_{16}}(1) &\leq \exp_{D_{16}}(v_j) \leq \max\{d_{L(D_{16})}(v_j, v) : v \in V_{16}\} + \phi_{L(D_{16})} \\
&\leq \frac{n+2}{2} + \frac{n-4}{2}(r-1) \leq \frac{n+2}{2} + \frac{n-4}{2}(n-5) \\
&= \frac{1}{2}(n^2 - 8n + 22) \leq \frac{1}{2}(n^2 - 7n + 16).
\end{aligned}$$

Suppose that $r = n-3$ and $r - j \leq n/2$. Then $\max\{d_{L(D_{16})}(v_j, v) : v \in V_{16}\} \leq n/2$. By Lemma 2.1,

$$\begin{aligned}
\exp_{D_{16}}(1) &\leq \exp_{D_{16}}(v_j) \leq \max\{d_{L(D_{16})}(v_j, v) : v \in V_{16}\} + \phi_{L(D_{16})} \\
&\leq \frac{n}{2} + \frac{n-4}{2}(n-4) = \frac{n^2 - 7n + 16}{2}.
\end{aligned}$$

Suppose that $r = n-3$ and $r-j = (n+2)/2$. Then $E_{16} = \hat{E} \cup \{(v_{\frac{n-8}{2}}, v_n), (v_n, v_{\frac{n-2}{2}})\}$. We come to prove that

$$\exp_{D_{16}}(v_j)(= \exp_{D_{16}}(v_{\frac{n-8}{2}})) \leq \frac{1}{2}(n^2 - 7n + 16).$$

By Lemmas 2.8 and 2.9,

$$R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-8}{2}}) = \bigcup_{\substack{i=0 \\ i \neq \frac{n+2}{2}}}^{n-3} R_i(v_{\frac{n-8}{2}}).$$

We can check that

$$\begin{aligned} R_{\frac{n+4}{2}}(v_{\frac{n-8}{2}}) &= \{v_{\frac{n-2}{2}}, v_{\frac{n}{2}}, v_{n-2}, v_{n-1}, v_1\}, \\ R_{\frac{n+2}{2}}(v_{\frac{n-8}{2}}) &= \{v_{\frac{n-4}{2}}, v_{\frac{n-2}{2}}, v_{n-3}, v_{n-2}\}, \\ R_{\frac{n}{2}}(v_{\frac{n-8}{2}}) &= \{v_{\frac{n-6}{2}}, v_n, v_{n-4}, v_{n-3}\}, \\ R_2(v_{\frac{n-8}{2}}) &= \{v_{\frac{n-4}{2}}, v_{\frac{n-2}{2}}\}. \end{aligned}$$

It follows that

$$R_{\frac{n+2}{2}}(v_{\frac{n-8}{2}}) \subseteq R_2(v_{\frac{n-8}{2}}) \cup R_{\frac{n}{2}}(v_{\frac{n-8}{2}}) \cup R_{\frac{n+4}{2}}(v_{\frac{n-8}{2}}).$$

We still can check that

$$\bigcup_{i=0}^{\frac{n+2}{2}} R_i(v_{\frac{n-8}{2}}) = V_{16}.$$

Thus

$$V_{16} = \bigcup_{i=0}^{\frac{n+2}{2}} R_i(v_{\frac{n-8}{2}}) \subseteq R_{\frac{n+4}{2}}(v_{\frac{n-8}{2}}) \cup \bigcup_{i=0}^{\frac{n}{2}} R_i(v_{\frac{n-8}{2}}) \subseteq \bigcup_{\substack{i=0 \\ i \neq \frac{n+2}{2}}}^{n-3} R_i(v_{\frac{n-8}{2}}).$$

Hence

$$R_{\frac{n^2-7n+16}{2}}(v_{\frac{n-8}{2}}) = V_{16}.$$

Therefore

$$\exp_{D_{16}}(1) \leq \exp_{D_{16}}(v_{\frac{n-8}{2}}) \leq \frac{1}{2}(n^2 - 7n + 16).$$

The proof of the Theorem 2.1 is complete. □

3 $[4, \ldots, \frac{1}{2}(n^2 - 7n + 16)] \subseteq ME_n(1)$ *for* $n \geq 14$

Let D be a strong digraph. The vertex v is called an antinode of D if both the indegree $d^-(v)$ and the outdegree $d^+(v)$ equal to 1.

Lemma 3.1 ([8], Corollary of Lemma 2.1) *Every ministrong digraph D contains an antinode.*

Lemma 3.2 ([8], Lemma 2.2) *Let $D = (V, E)$ be a ministrong digraph, v an antinode of D with $(u_1, v) \in E$ and $(v, u_2) \in E$. Define $\tilde{D} = (\tilde{V}, \tilde{E})$ to be a new digraph with $\tilde{V} = V \cup \{\tilde{v}\}$ (where $\tilde{v} \notin V$) and $\tilde{E} = E \cup \{(u_1, \tilde{v}), (\tilde{v}, u_2)\}$. Then $\tilde{D}$ is also ministrong.*

Lemma 3.3 $ME_n(1) \subseteq ME_{n+1}(1)$ $(n \geq 4)$.

Proof. If $m \in ME_n(1)$, then there exists a primitive, minimally strong digraph $D = (V, E)$ with n vertices such that $\exp_D(1) = m$. By Lemma 3.1, we may suppose that v is an antinode of D, and $(u_1, v) \in E$, $(v, u_2) \in E$. Let $\tilde{D} = (\tilde{V}, \tilde{E})$ with $\tilde{V} = V \cup \{\tilde{v}\}$ $(\tilde{v} \notin V)$ and $\tilde{E} = E \cup \{(u_1, \tilde{v}), (\tilde{v}, u_2)\}$. Then by Lemma 3.2, $\tilde{D}$ is a primitive, minimally strong digraph with $n+1$ vertices. We use $R_t(u)$, $\tilde{R}_t(u)$ respectively to denote the set of vertices which can be reached in D, $\tilde{D}$ by a walk with the initial vertex u of length t. Then for any positive integer x,

$$R_x(u) = V \iff \tilde{R}_x(u) = \tilde{V} \text{ for } u \notin \{v, \tilde{v}\},$$
$$R_x(v) = V \iff \tilde{R}_x(v) = \tilde{V} \iff \tilde{R}_x(\tilde{v}) = \tilde{V},$$

and so

$$\exp_D(u) = \exp_{\tilde{D}}(u) \text{ for } u \notin \{v, \tilde{v}\},$$
$$\exp_D(v) = \exp_{\tilde{D}}(v) = \exp_{\tilde{D}}(\tilde{v}).$$

It follows that

$$\exp_{\tilde{D}}(1) = \exp_D(1) = m,$$

and thus $m \in ME_{n+1}(1)$. Therefore $ME_n(1) \subseteq ME_{n+1}(1)$. The proof of Lemma 3.3 is complete. □

Lemma 3.4 [3] *Let S be the set of 1-exponent of all primitive, minimally strong digraphs with n vertices and $L(D) = \{p, q\}$, where $3 \leq p < q$, $p + q > n$.*

(i)*If $q + \lceil \frac{q-2}{p-2} \rceil \leq n$, then $S = [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p]$.*

(ii)*If $q + \lceil \frac{q-2}{p-2} \rceil > n$, then $S = [p(q-1) - (n-q)(p-2), \ldots, (p-1)(q-1) + n - p]$.*

For the sake of simplicity, we write $f_n = \frac{1}{2}(n^2 - 7n + 16)$.

Theorem 3.1 *Let* $n \equiv 0 \pmod 4$ *and* $n \geq 12$. *Then* $[f_{n-1}+1, \ldots, f_n] \subseteq ME_n(1)$.

Proof. Let $n = 4k$ $(k \geq 3)$.

(i) Let $p = (n-2)/2$, $q = n-4$, i,e., $p = 2k-1$, $q = 2(2k-2)$. Then $(p,q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil\frac{q-2}{p-2}\rceil = n-2 < n$. By Lemma 3.4,

$$[(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).$$

Equivalently,

$$[\frac{1}{2}(n^2 - 9n + 22), \ldots, \frac{1}{2}(n^2 - 8n + 22)] \subseteq ME_n(1).$$

(ii) Let $p = (n-2)/2$, $q = n-3$, i,e., $p = 2k-1$, $q = 4k-3 = (2k-1)+(2k-2)$. Then $(p,q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil\frac{q-2}{p-2}\rceil = n$. By Lemma 3.4,

$$[(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).$$

Equivalently,

$$[\frac{1}{2}(n^2 - 8n + 18), \ldots, \frac{1}{2}(n^2 - 7n + 18)] \subseteq ME_n(1).$$

By (i)(ii), we have

$$[f_{n-1}+1, \ldots, f_n] = [\frac{1}{2}(n^2 - 9n + 26), \ldots, \frac{1}{2}(n^2 - 7n + 16)] \subseteq ME_n(1).$$

The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Let* $n \equiv 2 \pmod 4$ *and* $n \geq 14$. *Then* $[f_{n-1}+1, \ldots, f_n] \subseteq ME_n(1)$.

Proof. Let $n = 4k+2$ $(k \geq 3)$.

(i) Let $p = (n-4)/2$, $q = n-2$, i,e., $p = 2k-1$, $q = 4k = 2 \cdot 2k$. Then $(p,q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil\frac{q-2}{p-2}\rceil > n$. By Lemma 3.4,

$$[p(q-1) - (n-q)(p-2), \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).$$

Equivalently,

$$[\frac{1}{2}(n^2 - 9n + 28), \ldots, \frac{1}{2}(n^2 - 8n + 22)] \subseteq ME_n(1).$$

(ii) Let $p = (n-2)/2$, $q = n-3$, i,e., $p = 2k$, $q = 4k-1 = 2k+(2k-1)$. Then $(p,q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil = n$. By Lemma 3.4,

$$[(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).$$

Namely

$$[\frac{1}{2}(n^2-8n+18), \ldots, \frac{1}{2}(n^2-7n+18)] \subseteq ME_n(1).$$

(iii) Let $p = (n-4)/2$, $q = n-3$, i,e., $p = 2k-1$, $q = 4k-1 = 2k+(2k-1)$. Then $(p,q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil = n$. By Lemma 3.4,

$$[(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).$$

Equivalently,

$$[\frac{1}{2}(n^2-10n+26), \ldots, \frac{1}{2}(n^2-9n+28)] \subseteq ME_n(1).$$

By (i)(ii)(iii), we have

$$[f_{n-1}+1, \ldots, f_n] = [\frac{1}{2}(n^2-9n+26), \ldots, \frac{1}{2}(n^2-7n+16)] \subseteq ME_n(1).$$

The proof of Theorem 3.2 is complete. □

Lemma 3.5 *Let $t_1 = 3$, $t_2 = 7$, $t_3 = 11$, $t_4 = 19$, $t_5 = 23$, $\ldots$, be a sequence of prime numbers which can be represented in the form $4k+3$, where k is nonnegative integer. Then*

(i) $t_{i+1} \leq 2t_i$ $(i \geq 2)$.

(ii) $t_l \leq \sqrt{2t_3t_4\cdots t_{l-1}-1}$ $(l \geq 6)$.

Proof. (i) This is a result of Lemma 4.3 of [9].

(ii) We prove the conclusion by induction. Clearly the conclusion holds for $l = 6$. If the conclusion holds for $l = k$ $(k \geq 6)$, i,e., $t_k \leq \sqrt{2t_3t_4\cdots t_{k-1}-1}$. Then by (i), we have

$$\begin{aligned} t_{k+1}^2 &\leq 4t_k^2 \leq 4(2t_3t_4\cdots t_{k-1}-1) \leq t_k(2t_3t_4\cdots t_{k-1}-1) \\ &= 2t_3t_4\cdots t_k - t_k \leq 2t_3t_4\cdots t_k - 1. \end{aligned}$$

Namely the conclusion holds for $l = k+1$. Therefore, for each each integer $l \geq 6$, $t_l \leq \sqrt{2t_3t_4\cdots t_{l-1}-1}$. The proof of Lemma 3.5 is complete. □

Lemma 3.6 *Let n be a positive integer with $n \geq 135$. Then there must exist a prime number t such that $t \equiv 3 \pmod 4$, $n \not\equiv \frac{t+5}{2} \pmod t$ and $11 \leq t \leq \sqrt{4n-11}$.*

Proof. We prove this lemma by using the same method as in the proof of Lemma 4.4 of [9]. Let $\{t_i\}$ be a sequence defined in Lemma 3.5,

$$l = \min\{i : i \geq 3 \text{ and } n \not\equiv \frac{t_i + 5}{2} \pmod{t_i}\} \text{ and } t = t_l.$$

Clearly $t \equiv 3 \pmod 4$, $n \not\equiv \frac{t+5}{2} \pmod{t}$ and $t \geq 11$. Now we prove that $t \leq \sqrt{4n-11}$. If $l \leq 5$, then

$$t = t_l \leq t_5 = 23 \leq \sqrt{4n-11} \quad \text{since } n \geq 135.$$

If $l \geq 6$, then for each $i \in \{3, 4, \ldots, l-1\}$, $n \equiv \frac{t_i+5}{2} \pmod{t_i}$ by the definition of l, and so $t_i \mid (2n-5)$ for each $i \in \{3, 4, \ldots, l-1\}$. Since t_i $(i = 3, 4, \ldots, l-1)$ are prime numbers, then $t_3 t_4 \cdots t_{l-1} \mid (2n-5)$, and so

$$\frac{t_3 t_4 \cdots t_{l-1} + 5}{2} \leq n.$$

By Lemma 3.5, we have $\sqrt{4n-11} \geq \sqrt{2t_3 t_4 \cdots t_{l-1} - 1} \geq t_l$. The proof of Lemma 3.6 is complete. □

Theorem 3.3 *Let* $n \equiv 1 \pmod 4$ *and* $n \geq 17$. *Then* $[\frac{1}{2}(n^2 - 9n + 26), \ldots, \frac{1}{2}(n^2 - 8n + 7)] \cup [\frac{1}{2}(n^2 - 8n + 23), \ldots, \frac{1}{2}(n^2 - 7n + 16)] \subseteq ME_n(1)$.

Proof. Let $n = 4k + 1$ $(k \geq 4)$.

(i) Let $p = (n+1)/2$, $q = n - 7$, i,e., $p = 2k + 1$, $q = 4k - 6 = 2(2k-3)$. Then $(p, q) = 1$, $3 < p < q < n - 1$, $p + q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil \leq n$. By Lemma 3.4,

$$\begin{aligned}&[\frac{1}{2}(n^2 - 9n + 10), \ldots, \frac{1}{2}(n^2 - 8n + 7)]\\&= [(p-1)(q-1) + 1, \ldots, (p-1)(q-1) + n - p] \subseteq ME_n(1).\end{aligned}$$

(ii) Let $p = (n-3)/2$, $q = n-2$, i,e., $p = 2k-1$, $q = 4k-1 = 2k+(2k-1)$. Then $(p, q) = 1$, $3 < p < q < n - 1$, $p + q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil > n$. By Lemma 3.4,

$$\begin{aligned}&[\frac{1}{2}(n^2 - 8n + 23), \ldots, \frac{1}{2}(n^2 - 7n + 18)]\\&= [p(q-1) - (n-q)(p-2), \ldots, (p-1)(q-1) + n - p]\\&\subseteq ME_n(1).\end{aligned}$$

By(i) and (ii),

$$\begin{aligned}&[\frac{1}{2}(n^2 - 9n + 26), \ldots, \frac{1}{2}(n^2 - 8n + 7)]\\&\qquad \cup [\frac{1}{2}(n^2 - 8n + 23), \ldots, \frac{1}{2}(n^2 - 7n + 16)] \subseteq ME_n(1).\end{aligned}$$

The proof of Theorem 3.3 is complete. □

Theorem 3.4 *Let $n \geq 15$. We have*

(i) *If $n \equiv 1 \pmod 4$, then* $[\frac{1}{2}(n^2-8n+9), \ldots, \frac{1}{2}(n^2-8n+21)] \subseteq ME_n(1)$.

(ii) *If $n \equiv 3 \pmod 4$, then* $\frac{1}{2}(n^2-8n+21) \in ME_n(1)$.

Proof. (a) If $n \geq 135$ and n is odd, then by Lemma 3.6, there exists a prime number t such that $t \equiv 3 \pmod 4$, $n \not\equiv \frac{t+5}{2} \pmod t$ and $11 \leq t \leq \sqrt{4n-11}$. Let $p = \frac{1}{2}(n + \frac{t-5}{2})$, $q = n - \frac{t+5}{2}$. Then $q = 2p - t$. Since t is a prime number and $t \nmid (n - \frac{t+5}{2})$ (namely $t \nmid q$), then $(p,q) = 1$. We can check from $11 \leq t \leq \sqrt{4n-11}$ that

$$p+q = \frac{3n}{2} - \frac{t+15}{4} \geq \frac{3n}{2} - \frac{\sqrt{4n-11}+15}{4} > n,$$

$$3 < p < q < n-1 \text{ and } q + \lceil \frac{q-2}{p-2} \rceil < n.$$

By $11 \leq t \leq \sqrt{4n-11}$ and Lemma 3.4,

$$\begin{aligned}
&[\frac{1}{2}(n^2-8n+9), \ldots, \frac{1}{2}(n^2-8n+21)] \\
&\subseteq [\frac{1}{2}(n^2-8n-\frac{(t+7)(t-9)}{4})+1, \ldots, \frac{1}{2}(n^2-7n-\frac{t^2-73}{4})] \\
&= [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).
\end{aligned}$$

(b) If $n \in [15, \ldots, 133] \backslash \{27, 41, 55, 69, 83, 97, 111, 125\}$ and n is odd, we take $p = (n+1)/2$ and $q = n-6$, then $q = 2p-7$ and $7 \nmid p$, and so $(p,q) = 1$. It is easy to check that $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil < n$. By $n \geq 15$ and Lemma 3.4,

$$\begin{aligned}
&[\frac{1}{2}(n^2-8n+9), \ldots, \frac{1}{2}(n^2-8n+21)] \\
&\subseteq [\frac{1}{2}(n^2-8n+9), \ldots, \frac{1}{2}(n^2-7n+6)] \\
&= [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).
\end{aligned}$$

(c) If $n \in \{55, 69, 83, 97, 111, 125\}$, we take $p = (n+3)/2$ and $q = n-8$, then $q = 2p-11$ and $11 \nmid p$, and so $(p,q) = 1$. It is easy to check that $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil < n$. By $n \geq 55$ and Lemma 3.4,

$$\begin{aligned}
&[\frac{1}{2}(n^2-8n+9), \ldots, \frac{1}{2}(n^2-8n+21)] \\
&\subseteq [\frac{1}{2}(n^2-8n-7), \ldots, \frac{1}{2}(n^2-7n-12)] \\
&= [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).
\end{aligned}$$

(d) If $n = 41$, we take $p = 25$ and $q = 29$, then $(p, q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil < n$. By Lemma 3.4,

$$\begin{aligned}&[\frac{1}{2}(n^2 - 8n + 9), \ldots, \frac{1}{2}(n^2 - 8n + 21)]\\&= [681, \ldots, 687] \subseteq [673, \ldots, 688]\\&= [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).\end{aligned}$$

(e) If $n = 27$, we take $p = 13$ and $q = 23$, then $(p, q) = 1$, $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil < n$. By Lemma 3.4,

$$\begin{aligned}&\frac{1}{2}(n^2 - 8n + 21) = 267 \in [265, 278]\\&= [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).\end{aligned}$$

By (a), (b), (c) and (d), (i)holds. By (a), (b), (c) and (e), (ii) holds. The proof of Theorem 3.4 is complete. □

Theorem 3.5 *Let* $n \equiv 1 \pmod 4$ *and* $n \geq 17$. *Then* $[f_{n-1}+1, \ldots, f_n] \subseteq ME_n(1)$.

Proof. By Theorems 3.3 and 3.4(i), we have

$$[f_{n-1}+1, \ldots, f_n] = [\frac{1}{2}(n^2 - 9n + 26), \ldots, \frac{1}{2}(n^2 - 7n + 16)] \subseteq ME_n(1).$$

The proof of Theorem 3.5 is complete. □

Theorem 3.6 *Let* $n \equiv 3 \pmod 4$ *and* $n \geq 15$. *Then* $[f_{n-1}+1, \ldots, f_n] \subseteq ME_n(1)$.

Proof. Let $n = 4k + 3$ $(k \geq 3)$.

(i) Let $p = (n-1)/2$, $q = n - 5$, i,e., $p = 2k+1$, $q = 4k - 2 = 2(2k-1)$. Then $(p, q) = 1$. It is easy to check that $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil < n$. By Lemma 3.4,

$$\begin{aligned}&[\frac{1}{2}(n^2 - 9n + 20), \ldots, \frac{1}{2}(n^2 - 8n + 19)]\\&= [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p] \subseteq ME_n(1).\end{aligned}$$

(ii) Let $p = (n-3)/2$, $q = n - 2$, i,e., $p = 2k$, $q = 4k+1 = 2k + (2k+1)$. Then $(p, q) = 1$. It is easy to check that $3 < p < q < n-1$, $p+q > n$ and $q + \lceil \frac{q-2}{p-2} \rceil > n$. By Lemma 3.4,

$$\begin{aligned}&[\frac{1}{2}(n^2 - 8n + 23), \ldots, \frac{1}{2}(n^2 - 7n + 18)]\\&= [p(q-1) - (n-q)(p-2), \ldots, (p-1)(q-1)+n-p]\\&\subseteq ME_n(1).\end{aligned}$$

(iii) By Theorem 3.4(ii), $\frac{1}{2}(n^2-8n+21) \subseteq ME_n(1)$. By (i), (ii) and (iii), we have

$$[f_{n-1}+1,\ldots,f_n]$$
$$= [\frac{1}{2}(n^2-9n+26),\ldots,\frac{1}{2}(n^2-7n+16)] \subseteq ME_n(1).$$

The proof of Theorem 3.6 is complete. □

Theorem 3.7 *Let* $n \geq 14$. *Then* $[4,\ldots,\frac{1}{2}(n^2-7n+16)] \subseteq ME_n(1)$.

Proof. We first prove that $\{4,5,6\} \subseteq ME_n(1)$. Consider the digraphs $\Gamma_k = (V_k, E_k)$ $(k=1,2,3)$, where

$$V_1 = \{v_1, v_2, \ldots, v_6\}, V_2 = \{v_1, v_2, \ldots, v_5\}, V_3 = \{v_1, v_2, v_3, v_4\},$$
$$E_1 = \{(v_i, v_{i+1}) : i = 1,2,3,4,5\} \cup \{(v_2,v_1),(v_4,v_2),(v_6,v_4)\},$$
$$E_2 = \{((v_i, v_{i+1}) : i = 1,2,3,4\} \cup \{(v_2,v_1),(v_4,v_2),(v_5,v_3)\},$$
$$E_3 = \{((v_i, v_{i+1}) : i = 1,2,3\} \cup \{(v_2,v_1),(v_4,v_2)\}.$$

Clearly, Γ_k $(k=1,2,3)$ are all primitive, minimally strong digraphs with $L(\Gamma_k) = \{2,3\}$ $(k=1,2,3)$. It is not difficult to prove that $\exp_{\Gamma_1}(1) = \exp_{\Gamma_1}(v_2) = 6$, $\exp_{\Gamma_2}(1) = \exp_{\Gamma_2}(v_2) = 5$, $\exp_{\Gamma_3}(1) = \exp_{\Gamma_3}(v_2) = 4$. By Lemma 3.3, $\{4,5,6\} \subseteq ME_n(1)$ for $n \geq 14$.

Next we prove that $[7,\ldots,47] \subseteq ME_n(1)$. By Lemma 3.4, we have Table 3.1, where S is the set of 1-exponent of primitive, minimally strong digraph with n vertices and $L(D) = \{p,q\}$.

Table 3.1

n	p	q	S
14	5	11	[41,...,49]
13	5	9	[33,...,40]
10	5	8	[29,...,34]
10	5	7	[25,...,29]
10	4	7	[19,...,24]
9	3	7	[16,...,18]
8	4	5	[13,...,16]
7	3	5	[10,...,12]
6	3	4	[7,...,9]

By Table 3.1 and Lemma 3.3, $[7,..,47] \subseteq ME_n(1)$ for $n \geq 14$.

Finally we prove that $[48,\ldots,\frac{1}{2}(n^2-7n+16)] \subseteq ME_n(1)$. By Theorems 3.1, 3.2, 3.5 and 3.6, we have

$$[f_{n-1}+1,\ldots,f_n] \subseteq ME_n(1) \quad \text{for } n \geq 17,$$
$$[f_{13}+1,\ldots,f_{14}] \subseteq ME_{14}(1), \quad [f_{14}+1,\ldots,f_{15}] \subseteq ME_{15}(1),$$
$$[f_{15}+1,\ldots,f_{16}] \subseteq ME_{16}(1).$$

By Lemma 3.3, $[48, \ldots, \frac{1}{2}(n^2 - 7n + 16)] = [f_{13} + 1, \ldots, f_n] \subseteq ME_n(1)$ for $n \geq 14$.

In conclusion, $[4, \ldots, \frac{1}{2}(n^2 - 7n + 16)] \subseteq ME_n(1)$ for $n \geq 14$. The proof of Theorem 3.7 is complete. □

4 Characterization of $ME_n(1)$

Lemma 4.1 ([6], Theorem 3.5) *Let D be a primitive digraph with n vertices and $L(D) = \{p, q\}$ with $p + q \leq n$. Then* $\exp_D(1) \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1$.

Lemma 4.2 ([2], Lemma 2.5)*Let D be a minimally strong digraph, and let xWy be a walk from vertex x to vertex y of length $k(\geq 2)$ and xPy be a path from x to y of length $k - 1$. Then some arc of xPy is not an arc of xWy.*

Lemma 4.3 *Let $D \in PMSD_n$. Then* $\exp_D(1) \geq 4$.

Proof. Since there is no loop in D, then for any $v \in V(D)$, there exists no walk v to v of length 1, and so $\exp_D(1) \geq 2$.

If $\exp_D(1) = 2$, let v be a vertex such that $\exp_D(v) = 2$ and (v, u) be an arc beginning at vertex v, then there exists a walk vWu from v to u of length 2. By D minimally strong, the walk vWu contains the arc (v, u). This contradicts that Lemma 4.2. Hence $\exp_D(1) \geq 3$.

If $\exp_D(1) = 3$, let v be a vertex such that $\exp_D(v) = 3$, then there exists a walk $vWv = (v, u, w, v)$ from v to v of length 3. Since D contains no loop and $\exp_D(v) = 3$, then v, u,w are distinct and there exists a walk vW_1u from v to u of length 3. We claim that vW_1u contain the arc (v, u) by D minimally strong. So vW_1u can be expressed as either $vW_1u = (v, u, z, u)$ or $vW_1u = (v, z, v, u)$, where $z \notin \{u, v, w\}$ by D ministrong. Without loss of generality we assume that the former holds. Since $\exp_D(v) = 3$, then there exists a walk $vW_2z = (v, x, y, z)$ from v to z of length 3, where $x \neq v$ by D ministrong. It follows that $x = u$ (otherwise, there exists the walk (v, x, y, z, u) not containing the arc (v, u). This contradicts that D ministrong). Then $vW_2z = (v, u, y, z)$, and so $y \notin \{u, z\}$ by the walk (v, u, y, z) containing no loops. Thus there exists the walk (u, y, z) not containing the arc (u, z). However, $(u, z) \in E(vW_1u) \subseteq E(D)$. This contradicts that D ministrong. Consequently $\exp_D(1) \geq 4$. The proof of Lemma 4.3 is complete. □

Theorem 4.1 *Let $n \geq 14$. Then $ME_n(1) = S_1 \cup S_2 \cup S_3$, where S_1 =*

$[4, \ldots, \frac{1}{2}(n^2 - 7n + 16)]$,

$$S_2 = \bigcup_{\substack{6 \le p < q \le n-1 \\ \gcd(p,q)=1 \\ p+q>n \\ q + \lceil \frac{q-2}{p-2} \rceil \le n}} [(p-1)(q-1)+1, \ldots, (p-1)(q-1)+n-p]$$

$$S_3 = \bigcup_{\substack{6 \le p < q \le n-1 \\ \gcd(p,q)=1 \\ p+q>n \\ q + \lceil \frac{q-2}{p-2} \rceil > n}} [p(q-1)-(n-q)(p-2), \ldots, (p-1)(q-1)+n-p].$$

Proof. By Theorem 3.7, $S_1 \subseteq ME_n(1)$. If D contains at least three distinct lengths of cycles, it follows from Theorem 2.1 and Lemma 4.3 that $\exp_D(1) \in S_1 \subseteq ME_n(1)$.

If $L(D) = \{p, q\}$ and $p+q \le n$, it follows from Lemmas 4.1 and 4.3 that

$$4 \le \exp_D(1) \le \lfloor \frac{(n-1)^2}{4} \rfloor + 1 < \frac{1}{2}(n^2 - 7n + 16) \text{ by } n \ge 14,$$

and so $\exp_D(1) \in S_1$.

If $L(D) = \{p, q\}$, $p < q$ and $p \le 5$, it follows from Lemmas 2.2 and 4.3 that

$$4 \le \exp_D(1) \le 2 + 5(n-3) \le \frac{1}{2}(n^2 - 7n + 16) \text{ by } n \ge 14,$$

and so $\exp_D(1) \in S_1$. Therefore, by Lemma 3.4,

$$ME_n(1) = ME_n(1) = S_1 \cup S_2 \cup S_3.$$

The proof of Theorem 4.1 is complete. □

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