Extremal bipartite graphs with high girth

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Abstract

Let us denote by $EX(m, n; \{C_4, \dots, C_{2t}\})$ the family of bipartite graphs G with m and n vertices in its classes that contain no cycles of length less than or equal to $2t$ and have maximum size. In this paper the following question is proposed: does always such an extremal graph G contain a $(2t+2)$-cycle? The answer is shown to be affirmative for $t = 2, 3$ or whenever m and n are large enough in comparison with t. The latter asymptotical result needs two preliminary theorems. First we prove that the diameter of an extremal bipartite graph is at most $2t$, and afterwards we show that its girth is equal to $2t+2$ when the minimum degree is at least 2 and the maximum degree is at least $t+1$.

Key words. extremal graph, bipartite graph, girth.

1 Introduction

Throughout this paper only undirected simple graphs without loops or multiple edges are considered. Unless stated otherwise, we follow the book

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by Bollobás [2] for undefined terminology and definitions.

Let $\mathcal{G}(m,n)$ denote the family of bipartite graphs $G = G(m,n) = (X,Y)$ with m vertices in the class X and n vertices in the class Y. We will denote by C_{2t} a cycle of length $2t$, $t \geq 2$. The girth of G, $g(G) = g$, is the length of a shortest cycle in G. Clearly a tree is an acyclic bipartite graph, thus we say that its girth is infinity. Let $ex(m,n;\{C_4,\dots,C_{2t}\})$ denote the maximum size of a bipartite graph $G = G(m,n)$ with girth at least $2t+2$, and let $EX(m,n;\{C_4,\dots,C_{2t}\})$ be the corresponding family of extremal graphs.

Erdös and Sachs [3] showed that a δ-regular graph of girth at least $r+1$ of the smallest order must have girth equal to $r+1$. (A proof of this result can be found in Lovász [8], pp. 66, 384, 385, see also the references therein.) In this paper we consider a similar problem:

What is the girth of an extremal bipartite graph G of $EX(m,n;\{C_4,\dots,C_{2t}\})$? Is it always $2t+2$ or can it be greater?

This problem has been studied for general graphs [5, 6, 7]. Some of the most important results contained in these references are listed below.

Theorem 1.1 *Let G be a $\{C_3,C_4,\dots,C_r\}$-free graph of order ν and maximum size.*

(i) [5, 6] For $\nu \geq 7$ and $r = 4$, the girth of G is 5.

(ii) [7] For $\nu \geq 8$ and $r = 5$, the girth of G is 6.

(iii) [7] If the maximum degree of G is $\Delta \geq r$ then the girth is necessarily $r+1$.

(iv) [7] Let $r \geq 12$, $a = r - 3 - \lfloor (r-2)/4 \rfloor$, $\nu \geq 2^{a^2+a+1} r^a$. Then the girth is $r+1$.

In this paper we state several results which are similar to those in Theorem 1.1, concerning the extremal bipartite family $EX(m,n;\{C_4,\dots,C_{2t}\})$. In Section 2 we present our main theorems and prove them in Section 3.

2 Main Results

We study the extremal function $ex(m,n;\{C_4,\dots,C_{2t}\})$ assuming $\min\{m,n\} \geq t+1$, because in other case the problem becomes trivial.

Then $ex(m,n;\{C_4,\dots,C_{2t}\}) \geq m+n$, and a cycle of length $2t+2$ is an extremal graph for which the equality holds when $m=n=t+1$. So every graph $G \in EX(m,n;\{C_4,\dots,C_{2t}\})$ contains some cycle and the degree of every vertex is clearly at least 1.

Our first result concerns the diameter of extremal $\{C_4,\dots,C_{2t}\}$-free bipartite graphs. We know that the diameter D of a bipartite graph with girth g satisfies $D \geq g/2$. In the following theorem we obtain an upper bound for the diameter of such an extremal graph.

Theorem 2.1 *Let $G = (X,Y)$ be a bipartite graph of the family $EX(m,n;\{C_4,\dots,C_{2t}\})$. Then the diameter is $D(G) \leq 2t$. Furthermore, $D(G) \leq 2t-1$ if there is one vertex in the class X and one vertex in the class Y both of degree 1.*

A graph G is called *connected* if every pair of vertices is joined by a path; that is, if $D(G) < \infty$. If $G - S$ is not connected for certain $S \subset V$, then S is said to be a *cut set*. A (noncomplete) connected graph is called *k-connected* if every cut set has cardinality at least k. The *connectivity* $\kappa(G)$ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k-connected. The *connectivity* of a complete graph K_{k+1} on $k+1$ vertices is defined as $\kappa(K_{k+1}) = k$. Connectivity has an edge analogue. An *edge-cut* in a graph G is a set W of edges of G such that $G - W$ is nonconnected. The *edge-connectivity* $\lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G. A classic result due to Whitney is that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for every graph G of minimum degree $\delta(G)$. A graph is *maximally connected* if $\kappa(G) = \delta(G)$, and maximally edge-connected if $\lambda(G) = \delta(G)$.

Sufficient conditions for a bipartite graph G with minimum degree $\delta(G)$ to be maximally connected have been given in terms of its diameter and its girth. In this regard, the following result is contained in [4]:

$$\begin{aligned} \lambda(G) = \delta(G) \quad &\text{if} \quad D(G) \leq g(G) - 1; \\ \kappa(G) = \delta(G) \quad &\text{if} \quad D(G) \leq g(G) - 2. \end{aligned} \tag{1}$$

Clearly, every extremal bipartite graph $G \in EX(m,n;\{C_4,\dots,C_{2t}\})$ must be connected. By Theorem 2.1, we have $D(G) \leq 2t \leq g(G) - 2$, because $g(G) \geq 2t+2$. Hence next result concerning the connectivities of an extremal bipartite graph follows immediately from (1).

Corollary 2.1 *Every bipartite graph $G \in EX(m,n;\{C_4,\dots,C_{2t}\})$ has $\kappa(G) = \lambda(G) = \delta(G)$.*

Based on Theorem 2.1 we deduce the following result in which we prove that the girth of every extremal $\{C_4\}$-free bipartite graph is 6.

Theorem 2.2 *Every bipartite graph $G \in EX(m,n;\{C_4\})$ has girth 6.*

It is known [2] (pp. 312-313) that $ex(n,n;\{C_4\}) \leq (n+n\sqrt{4n-3})/2$ and equality holds when $n = q^2+q+1$ for a prime power q. More precisely, for a prime power q all generalized triangles $PG(2,q)$ provide examples proving $ex(n,n;\{C_4\}) = (n+n\sqrt{4n-3})/2$, where $n = q^2+q+1$.

Our next theorem states that an extremal $\{C_4,\dots,C_{2t}\}$-free bipartite graph with maximum degree $\Delta \geq t+1$ has necessarily girth $2t+2$.

Theorem 2.3 *Let $G \in EX(m,n;\{C_4,\dots,C_{2t}\})$ be a bipartite graph with maximum degree $\Delta \geq t+1$. Suppose that the degree of every vertex adjacent to any vertex of maximum degree is at least 2. Then $g(G) = 2t+2$.*

As a consequence of Theorem 2.1 and Theorem 2.3 the girth of an extremal $\{C_4,\dots,C_{2t}\}$-free bipartite graph is proved to be equal to $2t+2$ provided that $m+n$ is large in comparison with t.

Theorem 2.4 *Let $G \in EX(m,n;\{C_4,\dots,C_{2t}\})$. If the minimum degree is $\delta \geq 2$, $t \geq 3$ and $m+n > 2\Big((t-1)^{2t}-1\Big)/(t-2)$, then $g(G) = 2t+2$.*

This result can be compared with item (iv) of Theorem 1.1. Both results give a sufficient condition on the order of an extremal graph to contain a cycle of minimum length $2t+2$. When $r = 2t+1$ Theorem 1.1 gives $\nu \geq 2^{a^2+a+1}(2t+1)^a$ for $t \geq 6$, where $a = \lceil 3(t-1)/2 \rceil$. We have for $t \geq 6$ that $2^a > t$, hence $2^{a^2+a} > t^{a+1}$, and so $\nu > 2t^{a+1}(2t+1)^a > 2t^{2(a+1)} \geq 2t^{3t-1}$, which is much larger than the requirement on the order obtained in Theorem 2.4, $m+n > 2\Big((t-1)^{2t}-1\Big)/(t-2)$.

In what follows we will prove that the girth of $G \in EX(m,n;\{C_4,C_6\})$ is 8. We need first to compute some exact extremal numbers for the bipartite case. In order to do that we will use Theorem 5 of [7] which proves for general graphs that $ex(2r+2;\{C_3,\dots,C_r\}) = 2r+4$ if $r \geq 12$.

Theorem 2.5 *Let t be an integer.*

(i) If $t \geq 2$ then $ex(2t,2t;\{C_4,\dots,C_{2t}\}) = 4t+1$.

(ii) If $t \geq 6$ then $ex(2t+1, 2t+1; \{C_4, \dots, C_{2t}\}) = 4t+4$ and every $G \in EX(2t+1, 2t+1; \{C_4, \dots, C_{2t}\})$ has girth $2t+2$.

Figure 1 depicts an extremal graph of the family $EX(2t, 2t; \{C_4, \dots, C_{2t}\})$, and Figure 2 shows an extremal graph of the family $EX(2t+1, 2t+1; \{C_4, \dots, C_{2t}\})$, which is also of the family $EX(4t+2; \{C_3, C_4, \dots, C_{2t}\})$.

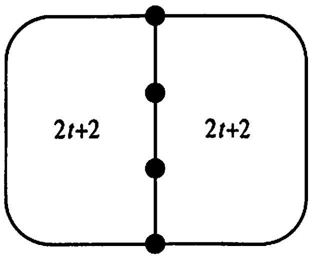

Figure 1: A graph belonging to the family $EX(2t, 2t; \{C_4, \dots, C_{2t}\})$

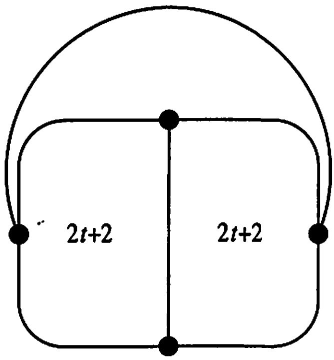

Figure 2: An extremal graph of both $EX(2t+1, 2t+1; \{C_4, \dots, C_{2t}\})$ and $EX(4t+2; \{C_3, \dots, C_{2t}\})$

Using Theorem 2.3 and Theorem 2.5, every extremal bipartite graph free of cycles of length 4 and 6 is shown to have girth 8.

Theorem 2.6 *Every bipartite graph $G \in EX(m, n; \{C_4, C_6\})$ has girth 8 unless $G = C_{10}$.*

3 Proofs

The degree of a vertex w in a graph G is denoted by $d_G(w)$ and $N_G(w)$ is the set of vertices adjacent to w in G. We also use $e(G)$ to denote the number of edges of G, and $G[V']$ stands for the induced subgraph in G by the set of vertices $V' \subseteq V(G)$. Moreover, let $[U, W]$ denote the set of edges of G with one end vertex in $U \subseteq V(G)$ and the another end vertex in $W \subseteq V(G)$. A uv-path of shortest length is called a *uv-geodesic*.

Proof of Theorem 2.1. Let u, v be two vertices of $G = (X, Y)$ at distance $d_G(u, v) = D(G) \geq 3$. First, suppose $D(G)$ is even, then the vertices u and v belong to the same class, say X. Take $w \in N_G(u)$ and let us consider the bipartite graph G^* obtained from G by adding the edge vw. Clearly $g(G^*) \leq 2t$ because $G \in EX(m, n; \{C_4, \ldots, C_{2t}\})$. Let us denote by C any cycle of length at most $2t$ in G^*. Since $g(G) \geq 2t + 2$ it is clear that the edge vw must belong to $E(C)$. But then $D(G) = d_G(u, v) \leq 1 + d_G(w, v) \leq 1 + |V(C)| - 1 \leq 2t$ and the result holds. We reason similarly for $D(G)$ odd, by considering G^* as the result of adding the edge uv to G.

In order to finish the proof, let $x_1 \in X$ and $y_1 \in Y$ be two vertices of degree 1. As $D(G) \leq 2t - 1$ for $D(G)$ odd, we suppose $D(G)$ even, and $u, v \in X$. The graph G' obtained from G by deleting the edge incident with y_1 and adding the edge $y_1 u$ is also an extremal graph of $EX(m, n; \{C_4, \ldots, C_{2t}\})$, and thus $2t \geq D(G') \geq d_{G'}(y_1, v) = 1 + D(G)$. Then $D(G) \leq 2t - 1$, and the result is valid. ■

Lemma 3.1 *If there exists a bipartite graph $G \in EX(m, n; \{C_4, \ldots, C_{2t}\})$ with girth $g(G) \geq 2t + 4$, then there exists another bipartite graph in $EX(m, n; \{C_4, \ldots, C_{2t}\})$ having at least one vertex of degree 1 in each vertex class.*

Proof. Let $G = (X, Y) \in EX(m, n; \{C_4, \ldots, C_{2t}\})$ be with girth $g(G) \geq 2t + 4$. Let C be a shortest cycle in G and take any path $x_1 y_1 x_2 y_2$ of length 3 in C with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then if we denote by $G^* = (X^*, Y^*)$ the bipartite graph obtained from G by identifying the pairs of vertices (x_1, x_2) to one vertex x and (y_1, y_2) to one vertex y, we have $G^* \in \mathcal{G}(m-1, n-1)$, $e(G^*) = e(G) - 2$ and $g(G^*) \geq g(G) - 2 \geq 2t + 2$. Now we consider the bipartite graph $G' = (X', Y')$ obtained from G^* by adding two new vertices x' and y' to the classes X^* and Y^* respectively and two edges $x'y^*$ and x^*y' for some $y^* \in Y^*$ and $x^* \in X^*$. Clearly, $G' \in \mathcal{G}(m, n)$, $g(G') = g(G^*) \geq 2t + 2$ and $e(G') = e(G^*) + 2 = e(G) =$

$ex\,(m,n;\{C_4,\ldots,C_{2t}\})$. Moreover, the vertices $x' \in X'$ and $y' \in Y'$ have degree 1, so the result follows. ■

Proof of Theorem 2.2. We reason by contradiction assuming $g(G) \geq 8$. By applying Lemma 3.1, there exists another bipartite graph $G^* = (X^*,Y^*) \in EX(m,n;\{C_4\})$ having one vertex of degree 1 in each class, $x \in X^*$ and $y \in Y^*$. Since $g(G^*) \geq 6$, it is clear that we can consider a cycle C of G^* and two vertices $u,v \in V(C)$ at distance 3 having degree at least 2 in G^*, so they belong to different classes, say $u \in X^*$ and $v \in Y^*$. Now, let us consider the bipartite graph G' obtained from G^* by removing the edges incident with x and y and adding the edges xv and uy. Clearly, $e(G') = e(G^*) = e(G) = ex(m,n;\{C_4\})$ and $g(G') = g(G^*) \geq 6$. Then G' is extremal, but $D(G') \geq d_{G'}(x,y) = 1 + d_{G^*}(u,v) + 1 = 5$, against Theorem 2.1. Hence, the result holds. ■

Proof of Theorem 2.3. Suppose that $G = (X,Y) \in EX\,(m,n;\{C_4,\ldots,C_{2t}\})$ satisfies the hypotheses of the theorem, and assume $g(G) \geq 2t{+}4$. Let x be a vertex of X of degree Δ and let $y_1,y_2,\ldots,y_\Delta$ be all the neighbors of x. Since $d_G(y_i) \geq 2$ for each $i = 1,\ldots,\Delta$, there exists $x_i \in X - x$ adjacent to y_i. Notice also that $x_i \neq x_j$ for all $i \neq j$, since $g(G) > 4$. Taking into account that $g(G) \geq 2t+4$, we deduce that $d_{G-x}(x_i,x_j) \geq g(G) - 4 \geq 2t$, $d_{G-x}(y_i,y_j) \geq g(G) - 2 \geq 2t+2$ and $d_{G-x}(x_i,y_j) \geq g(G) - 3 \geq 2t+1$ for all $i,j = 1,\ldots,\Delta$ with $i \neq j$. Let G^* be the bipartite graph obtained from G by first deleting the $\Delta - 1$ edges $xy_2,\ldots,xy_\Delta$ and second adding the new Δ edges $y_1x_2,\ldots,y_{\Delta-1}x_\Delta,y_\Delta x_1$. Then $G^* = (X,Y) \in \mathcal{G}(m,n)$ and $e(G^*) = e(G) + 1$. Since G is extremal, G^* must contain a cycle of length at most $2t$. Let us denote by C^* a shortest cycle in G^* (notice that $x \notin V(C^*)$, since x has degree 1 in G^*). We denote by C the cycle $x_1y_1x_2y_2\ldots x_\Delta y_\Delta x_1$ which has length $2\Delta \geq 2t+2$. Observe that $G^*[V(C)] = C$, since x_i is non adjacent to y_j in G, for any $i \neq j$ and the only newly introduced edges are y_ix_{i+1} for $i = 1,\ldots,\Delta-1$ and $y_\Delta x_1$. Moreover, $C^* \neq C$, since $g(C) \geq 2t+2$ and $g(C^*) \leq 2t$. Hence we may express $C^* = P_1 \cup P_2$, where P_1 is the longest path whose edges belong to the set $E(C^*)\backslash E(C) \subseteq E(G{-}x)$, and P_2 is the rest of C^*. Notice that the end vertices of P_1 must belong to $\{x_1,\ldots,x_\Delta\} \cup \{y_1,\ldots,y_\Delta\}$ by the construction of P_1. Observe also that P_2 contains at least one edge of $E(C)$, because otherwise the cycle C^* would be contained in G, against the assumption $g(G) \geq 2t+4$. If the end vertices of P_1 are x_i and y_i, for some $i = 1,\ldots,\Delta$, then $e(P_1) \geq d_{G-x-\{x_iy_i\}}(x_i,y_i) \geq g(G) - 1 \geq 2t+3$ and hence $|V(C^*)| = e(C^*) = e(P_1) + e(P_2) \geq 2t+3+1 \geq 2t+4$, a contradiction. Otherwise,

$$\begin{aligned}
& e(P_1) \\
& \geq \min\{d_{G-x}(x_i, x_j), d_{G-x}(y_i, y_j), d_{G-x}(x_i, y_j) : i, j = 1, \ldots, \Delta, i \neq j\} \\
& = 2t,
\end{aligned}$$

which implies that $|V(C^*)| = e(C^*) = e(P_1) + e(P_2) \geq 2t+1 > 2t$, arriving at a contradiction. Hence, $g(G) = 2t + 2$. ■

Proof of Theorem 2.4 By applying Theorem 2.1, we know that $D(G) \leq 2t$. Assume $\Delta(G) \leq t$. As it is widely known, the order of a bipartite graph with $D(G) \leq 2t$ and maximum degree $\Delta(G) \leq t$ is upper bounded by the Moore bound as

$$\begin{aligned}
|V(G)| &\leq 1 + \sum_{i=0}^{2t-2} t(t-1)^i + (t-1)^{2t-1} \\
&= \frac{2(t-1)^{2t} - 2}{t-2},
\end{aligned}$$

which contradicts our assumptions. Then, $\Delta(G) \geq t + 1$, and we are done following Theorem 2.3. ■

Proof of Theorem 2.5. Let G be a graph formed by two cycles of length $2t + 2$ that share a path of length 3 (see Figure 1). The new graph belongs to $\mathcal{G}(2t, 2t)$, clearly has girth $2t + 2$ and size $4t + 1$. Hence $ex\,(2t, 2t; \{C_4, \ldots, C_{2t}\}) \geq 4t + 1$. In order to see the another inequality, any extremal graph $G \in EX\,(2t, 2t; \{C_4, \ldots, C_{2t}\})$ is shown to be planar by repeating the same reasoning contained in Theorem 5 of [7]. Moreover, it is well known that for a planar graph, $e(G) \leq (|V(G)| - 2)g(G)/(g(G) - 2)$. Therefore,

$$\begin{aligned}
e(G) &\leq (4t-2)g(G)/(g(G)-2) = (4t-2)\,(1 + 2/(g(G)-2)) \\
&\leq (4t-2)(1 + 1/t) < 4t + 2.
\end{aligned}$$

Hence $ex\,(2t, 2t; \{C_4, \ldots, C_{2t}\}) \leq 4t+1$ and item (i) of the theorem is valid.

Theorem 5 of [7] proves for general graphs that $ex(2r + 2; \{C_3, \ldots, C_r\}) = 2r + 4$ if $r \geq 12$. If we take $r = 2t$ then the graph G formed by two cycles of length $2t + 2$ that share an edge, plus one edge connecting two vertices at maximum distance $2t + 1$ (see Figure 2) is a bipartite graph with $2t + 1$ vertices in each class and size

$e(G) = 4t + 4 = ex(4t + 2; \{C_3, \ldots, C_{2t}\})$. Moreover, G has girth $2t + 2$, hence $ex(2t + 1, 2t + 1; \{C_4, \ldots, C_{2t}\}) \geq e(G) = 4t + 4 = ex(4t + 2; \{C_3, C_4, \ldots, C_{2t}\})$. The another inequality is simply obtained because $ex(m, n; \{C_4, \ldots, C_{2t}\}) \leq ex(m + n; \{C_3, C_4, \ldots, C_{2t}\})$. As far as the girth of any extremal graph of the family is concerned, we reason by contradiction supposing that there exists $G \in EX(2t+1, 2t+1; \{C_4, \ldots, C_{2t}\})$ with girth $g(G) \geq 2t + 4$. Then, by applying Lemma 3.1, there exists another bipartite graph $G^* \in EX(2t+1, 2t+1; \{C_4, \ldots, C_{2t}\})$ having at least one vertex with degree 1 in each vertex class. Then the bipartite graph G' obtained from G^* by removing one vertex of degree 1 in each class satisfies that $G' \in \mathcal{G}(2t, 2t)$, $g(G') = g(G^*) \geq 2t + 2$ and $e(G') = e(G^*) - 2 = 4t + 2$ and this is a contradiction with item (i). So the result holds. ■

Lemma 3.2 *Let G be a bipartite graph with diameter D and girth $2D$. Then:*

(i) $d_G(u) = d_G(v)$ *for every* $u, v \in V(G)$ *such that* $d_G(u, v) = D$.

(ii) *If the diameter is $D = 4$ and the minimum degree is $\delta = 2$ then G is formed by α internally disjoint paths of length 4 joining two vertices of degree α, or G is a subdivision of a complete bipartite graph $K_{\alpha,\beta}$, with $\alpha, \beta \geq 3$.*

Proof. (i) Let $u, v \in V(G)$ be such that $d_G(u, v) = D$, then it is clear that $d_G(u_i, v) = D - 1$ holds for every $u_i \in N_G(u)$ because G is bipartite. Therefore we can find $d_G(u)$ paths of length D from u to v which must be internally disjoint as $g = 2D$. Hence $d_G(u) = d_G(v)$.

(ii) Notice that if G is a cycle of length $2D$ the lemma holds. So assume that G is different from a cycle of length $2D$, hence we can consider $u, v \in V(G)$ such that $d_G(u, v) = 4$, with $d_G(u) = \alpha \geq 3$. By item (i) we know $d_G(u) = d_G(v) = \alpha \geq 3$. Clearly the graph G contains an induced subgraph H formed by α internally disjoint paths of length 4, say $uu_{i1}u_{i2}u_{i3}v$, with $u_{i1} \in N_G(u)$, $u_{i3} \in N_G(v)$, $i = 1, 2, \ldots, \alpha$. Notice that for $i \neq j$, $d_G(u_{i1}, u_{j3}) = d_G(u_{i2}, u_{j2}) = 4$. Therefore by item (i) we have that $d_G(u_{i1}) = d_G(u_{j3})$ and $d_G(u_{i2}) = d_G(u_{j2})$ for all $i, j \in \{1, 2, \ldots, \alpha\}$.

First let us see that $d_G(u_{i1}) = d_G(u_{j3}) = 2$. If not, $d_G(u_{i1}) = d_G(u_{j3}) \geq 3$ and every $t \in N_G(u_{i1}) \setminus \{u, u_{i2}\}$ satisfies $d_G(t, v) = 4$, so $d_G(t) = d_G(v) = \alpha \geq 3$ because of item (i); and every $t \in N_G(u_{j3}) \setminus \{v, u_{j2}\}$ satisfies $d_G(t, u) = 4$ hence $d_G(t) = d_G(u) = \alpha \geq 3$. Moreover, if

$j \neq i$, $d_G(t, u_{j2}) = 4$ or $d_G(t, u_{i2}) = 4$, so $d_G(u_{j2}) = d_G(t) = \alpha \geq 3$ or $d_G(u_{i2}) = d_G(t) = \alpha \geq 3$. This implies that u, v, $N_G(u)$, $N_G(v)$ and all the vertices at distance two from both vertices u, v have degree at least three. Therefore there exists a vertex w such that $d_G(w, u) \geq 3$ and $d_G(w, v) \geq 3$ with $d_G(w) = 2$ because $\delta = 2$. Hence by taking into account again item (i) we have $d_G(w, u) = d_G(w, v) = 3$. This means that G contains a path such as $u_{i1}twt'u_{j3}$, with $j \neq i$, but then $d_G(w, u_{j1}) = 4$, so $d_G(w) = d_G(u_{j1}) \geq 3$ which is a contradiction.

Thus, $d_G(u_{i1}) = d_G(u_{j3}) = 2$ for all $i, j \in \{1, 2, \ldots, \alpha\}$. If $d_G(u_{i2}) = d_G(u_{j2}) = 2$ we have finished. So assume $d_G(u_{i2}) = d_G(u_{j2}) = \beta \geq 3$. Then every $z \in N_G(u_{j2}) - u_{j1}$ has degree $d_G(z) = 2$ because $d_G(z, u_{i1}) = 4$ for $i \neq j$, and every $v' \in N_G(z) - u_{j2}$ has degree $d_G(v') = \alpha$ because $d_G(v', u) = 4$. Consequently, if we consider the sets $B_k(u) = \{w \in V(G) : d_G(w, u) = k\}$ for $1 \leq k \leq 4$, then $V(G) - u$ can be partitioned as $V(G) - u = B_1(u) \cup B_2(u) \cup B_3(u) \cup B_4(u)$. Observe that $B_3(u) = \cup_{j=1}^{\alpha}(N_G(u_{j2}) - u_{j1})$, for which $|B_3(u)| = \alpha(\beta - 1)$ holds because otherwise a cycle of length at most 6 is formed. Taking into account that both the induced subgraphs $G[B_3(u)]$ and $G[B_4(u)]$ contain no edges and that every vertex in $B_3(u)$ has exactly 1 neighbor in $B_2(u)$, it follows that every vertex in $B_3(u)$ has 1 neighbor in $B_4(u)$, and every vertex in $B_4(u)$ has α neighbors in $B_3(u)$. Hence $|B_3(u)| = |B_4(u)| \cdot \alpha$, so $|B_4(u)| = \beta - 1$. Therefore, there exist $|B_2(u)| = \alpha$ vertices of degree β and $|\{u\} \cup B_4(u)| = \beta$ vertices of degree α in G. Thus the graph G must be a subdivision of a complete bipartite graph $K_{\alpha,\beta}$, with $\alpha, \beta \geq 3$. ■

Proof of Theorem 2.6. We reason by contradiction assuming that $G = (X, Y) \neq C_{10}$ belongs to the family $EX(m, n; \{C_4, C_6\})$ and it has girth $g(G) \geq 10$, which implies that the diameter is $D(G) \geq 5$. First, let us see that $\delta(G) \geq 2$. Otherwise, there is some vertex with degree 1, say $w \in X$. If we consider the graph G^* obtained from G by deleting the vertex w, it is clear that $G^* \in EX(m-1, n; \{C_4, C_6\})$, $g(G^*) \geq 10$ and $D(G^*) \geq 5$. Then by applying Lemma 3.1, there exists another graph $H^* \in EX(m-1, n; \{C_4, C_6\})$ having at least one vertex of degree 1 in each vertex class, say $x \in X - w$, $y \in Y$. Observe that $g(H^*) \geq 8$ and hence, $D(H^*) \geq 4$. Then, given any cycle C of length at least 8, we can take two vertices, $u, v \in V(C) \cap Y$ at distance 4 in H^*. In this case, it is enough to consider the graph H' obtained from H^* by deleting the edges incident with x and y, respectively and adding the vertex $w \in X$ and the edges of the path $wyxu$. Clearly, $H' = (X, Y)$ has girth $g(H') = g(H^*) \geq 8$ and $e(H') = e(G)$, so $H \in EX(m, n; \{C_4, C_6\})$. But its diameter is $D(H') \geq d_{H'}(w, v) = 3 + d_{H^*}(u, v) = 7$, against Theorem 2.1. Hence, $\delta(G) \geq 2$.

Furthermore by applying Theorem 2.3 we know that maximum degree of G is $\Delta(G) \leq 3$.

If G is 2-regular then G is a cycle C_r of length at least 12 because by hypothesis $G \neq C_{10}$. But, $G = C_r$ with $r \geq 14$ is not in $EX(m,n;\{C_4,C_6\})$, since $D(G) \geq 7$ and this contradicts Theorem 2.1. And $G = C_{12}$ is not an extremal graph by item (i) of Theorem 2.5. Hence there exists some vertex $x_1 \in X$ with degree 3 in G. Let us denote $N_G(x_1) = \{y_1, z_1, z_2\}$ and let us take a path $x_1y_1x_2y_2$ of length 3 in G, with $x_i \in X$, $y_i \in Y$ for $i = 1,2$. Then we consider the graph $G^* = (X^*, Y^*)$ obtained from G by identifying the pairs of vertices (x_1, x_2) to one vertex x and (y_1, y_2) to one vertex y. Then $G^* \in \mathcal{G}(m-1, n-1)$, $e(G^*) = e(G) - 2$, $g(G^*) \geq g(G) - 2 \geq 8$ and any cycle containing both vertices z_1, z_2 must have length at least 10. Moreover, $d_{G^*}(x) \geq 3$, since $d_G(x_1) = 3$ and $d_G(x_2) \geq 2$. Besides $d_{G^*}(u) = d_G(u) \leq 3$, for each $u \in V(G^*)\backslash\{x,y\}$. Let us see that $g(G^*) = 8$ and $D(G^*) = 4$. Otherwise, we can find two vertices $u \in X$ and $v \in Y$ at distance 5 in G^* and hence, the graph G' obtained from G^* by adding two new vertices $x' \in X^*$, $y' \in Y^*$ and the edges uy' and $x'v$ satisfies that $G' \in \mathcal{G}(m,n)$, $e(G') = e(G^*) + 2 = e(G)$ and $g(G') = g(G^*) \geq 8$. Then $G' \in EX(m,n;\{C_4,C_6\})$, but $D(G') \geq d_{G'}(x',y') = 1 + d_{G^*}(u,v) + 1 = 7$, against Theorem 2.1. Thus, $g(G^*) = 8$ and $D(G^*) = 4$. If G^* has some vertex with degree 4, this vertex must belong to $\{x = (x_1,x_2), y = (y_1,y_2)\}$. But then, given any vertex $w \in G^*$ at distance 4 from x, by applying Lemma 3.2 we have $4 = d_{G^*}(x) = d_{G^*}(w) = d_G(w)$, contradicting the fact that $\Delta(G) \leq 3$. Hence $\Delta(G^*) = 3$. If $\delta(G^*) = 2$ then by applying Lemma 3.2 there are two possible structures for G^*. Either G^* is formed by $\alpha = 3$ internally disjoint paths of length $D = 4$ connecting two vertices, or G^* is a subdivision of a complete bipartite graph $K_{3,3}$. But the two possibilities imply that some cycle containing both vertices z_1, z_2 must have length 8, which is a contradiction. Finally, if $\delta(G^*) = \Delta(G^*) = 3$, then G^* is the cubic generalized quadrangle on 30 vertices, which also implies that some cycle containing both vertices z_1, z_2 must have length 8, again a contradiction. ■

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