The Edge Choosability of $C_n \times P_m$ *

Guoping Wang and Qiongxiang Huang
The College of Mathematics and Systems Sciences,
Xinjiang University, Urumqi, Xinjiang 830046, P.R.China
Email: wanggpxd@163.com

Abstract. Let $G_{n,m} = C_n \times P_m$ be the cartesian product of an n-cycle C_n and a path P_m of length $m-1$. We prove that $\chi'_l(G_{n,m}) = \chi'(G_{n,m}) = 4$ if $m \geq 3$, which implies that the list-edge-coloring conjecture (LECC) holds for all graphs $G_{n,m}$.

Key words: List-edge-coloring, Edge-L-colorable, Edge-k-choosable

MR Classification: 05C15

1. Introduction

Let $G = (V, E)$ be a graph or multigraph. An *edge list assignment* L of G is a mapping that assigns to each $e \in E$ a set $L(e)$ of colors. An *edge-L-coloring* of G is a proper edge coloring c such that $c(e) \in L(e)$ for each $e \in E$. G is *edge-L-colorable* if it has an edge-L-coloring. For a positive integer k, G is *edge-k-choosable* if it is edge-L-colorable whenever $|L(e)| \geq k$ for every $e \in E$, or, equivalently, whenever $|L(e)| = k$ for every $e \in E$. If $|L(e)| = k$ for every $e \in E$ then we call L an *edge k-list assignment*. The *list chromatic index* $\chi'_l(G)$ of G is the smallest integer k such that G is edge-k-choosable. Clearly $\chi'_l(G) \geq \chi'(G)$, the (ordinary) chromatic index of G, for every multigraph G.

A well-known conjecture, the *list-edge-coloring conjecture* (LECC), is that $\chi'_l(G) = \chi'(G)$ for every multigraph G. Up to now, the LECC has been proved only for a few special classes, such as planar graphs with maximum degree at least 12 [1], d-regular d-edge-colorable planar multigraphs [2],

*Project 10171085 supported by Natural Science Foundation of China

bipartite multigraphs [3], complete graphs of odd order [4], line-perfect multigraphs [5], and multicircuits [6].

If X and Y are two graphs, their *cartesian product* $X \times Y$ has vertex-set $V(X) \times V(Y)$, and two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are adjacent in $X \times Y$ if $x_1 = x_2$ and $y_1y_2 \in E(Y)$, or if $y_1 = y_2$ and $x_1x_2 \in E(X)$. Let $G_{n,m} = C_n \times P_m$, where C_n is an n-cycle and P_m is a path of length $m - 1$. Then $G_{n,2}$ is a 3-regular edge-3-colorable planar graph, and so $\chi'_l(G_{n,2}) = \chi'(G_{n,2}) = 3$ by a result of Ellingham and Goddyn [2]. If n is even then $G_{n,m}$ is bipartite, and so $\chi'_l(G_{n,m}) = \chi'(G_{n,m})$ by a result of Galvin [3]. In this paper we will prove that $\chi'_l(G_{n,m}) = \chi'(G_{n,m}) = 4$ if $m \geq 3$ and n is odd, and this will complete the proof of the LECC for all the graphs $G_{n,m}$.

2. The main results

We start with some lemmas; Lemmas 1 and 2 will be used to prove Lemmas 3 and 4, which in turn will be used to prove our main results.

Lemma 1. *Let C_n be an n-cycle, and let L be an edge list assignment of C_n such that $|L(e)| \geq 2$ for every edge $e \in E(C_n)$, and $|\bigcup_{e \in E(C_n)} L(e)| \geq 3$. Then C_n has at least two different edge-L-colorings.*

Proof We may assume that the lists are minimal subject to these conditions, so that they are all of size 2 and not all identical. We can label the edges $e_1, \ldots, e_n$ around C_n in such a way that $L(e_1) \neq L(e_n)$. Now we can color e_1 with a color $c_1 \in L(e_1) \setminus L(e_n)$, and then color each edge e_i in turn with a color $c_i \in L(e_i) \setminus \{c_{i-1}\}$ $(2 \leq i \leq n)$, to obtain an edge-L-coloring of C_n.

If at any stage in this process we have a choice of more than one possible color for some edge e_i, then we can find two different edge-L-colorings in this way. So we may assume that $L(e_i) = \{c_{i-1}, c_i\}$ for $2 \leq i \leq n$, and $L(e_1) = \{c_1, c_{n-1}\}$ or $\{c_1, c_n\}$. If $L(e_1) = \{c_1, c_n\}$ then we obtain a different coloring by recoloring every edge e_i with c_{i-1} (taking subscripts modulo n); and if $L(e_1) = \{c_1, c_{n-1}\}$ and $c_{n-1} \neq c_2$ then we can simply recolor e_1 with c_{n-1}. So we may assume that $c_2 = c_{n-1}$ and $L(e_1) = L(e_2) = \{c_1, c_2\}$. Now let l be the smallest index such that $c_l \notin \{c_1, c_2\}$, which exists since $c_n \notin \{c_1, c_2\}$. Then $L(e_i) = \{c_1, c_2\}$ for $1 \leq i \leq l - 1$, and so we obtain a second edge-L-coloring by interchanging the colors c_1 and c_2 on all of $e_1, \ldots, e_{l-1}$. □

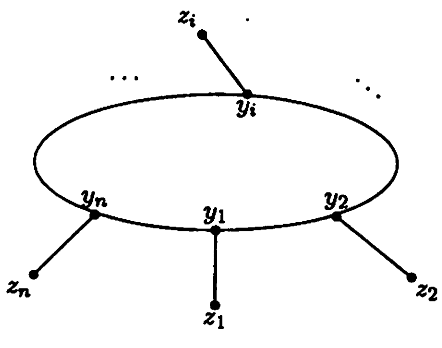

Figure 1: C^*

Let C^* be the graph formed from an odd cycle $C = y_1y_2 \dots y_ny_1$ by adding n new mutually nonadjacent vertices z_i and n new edges y_iz_i $(1 \leq i \leq n)$, as in Fig. 1.

Lemma 2. *Suppose that each edge y_iz_i of C^* is colored with color c_i, where $c_i \neq c_{i+1}$ $(1 \leq i \leq n$, subscripts taken modulo $n)$, and that an edge 4-list assignment of the cycle C is given by $L(y_iy_{i+1}) = \{a, b, c_i, c_{i+1}\}$ $(1 \leq i \leq n)$. If every edge y_iz_i is recolored with c'_i so that $c'_k \neq c_k$ for at least one index k, then $|\bigcup_{1\leq i\leq n}(L(y_iy_{i+1}) \setminus \{c'_i, c'_{i+1}\})| \geq 3$.*

Proof Suppose on the contrary that $L(y_iy_{i+1}) = \{g, h, c'_i, c'_{i+1}\}$ $(1 \leq i \leq n)$. Since n is odd, every color $c \notin \{a, b\}$ is equal to c_i for at most $\frac{1}{2}(n-1)$ values of i, and so there is some i such that $c \notin L(y_iy_{i+1})$. It follows that $\{g, h\} = \{a, b\}$. Thus $\{c'_i, c'_{i+1}\} = \{c_i, c_{i+1}\}$ for each i, and so $c_i = c'_{i+1}$ and $c'_i = c_{i+1}$ (since by hypothesis there is a k such that $c'_k \neq c_k$). But this implies that $c_1 = c_3 = \dots = c_n$, which is impossible since $c_n \neq c_1$. □

If X is a subgraph of a graph G, and L is an edge list assignment of G, let $L|X$ denote L restricted to the edges of X. For $v \in V(G)$, let $E_G(v)$ denote the set of edges of G incident with v, and let $d_G(v) = |E_G(v)|$. For $e \in E(G)$, let $N(e)$ denote the set of edges adjacent to e.

Let H be a graph, and let $v_1, v_2, \dots, v_n$ be $n \geq 3$ vertices of H such that $d_H(v_i) \leq 3$ $(1 \leq i \leq n)$. Let $C = u_1u_2 \dots u_nu_1$ be a cycle disjoint from H. We denote by H_C the graph obtained by joining v_i to u_i for $1 \leq i \leq n$, as in Fig. 2. If $k \geq 5$, or if $k = 4$ and n is even, it is easy to see that H_C is edge-k-choosable if H is edge-k-choosable.

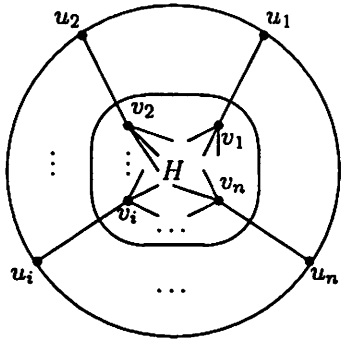

Figure 2: H_C

Lemma 3. *Let H be edge-4-choosable and $n \geq 3$ be odd. Assume that either*

(i) *there exists $l \in [1, n]$ such that $d_H(v_l) \leq 2$, or*

(ii) *$d_H(v_i) = 3$ $(1 \leq i \leq n)$ and for every edge 4-list assignment L_H of H, H has at least two edge-L_H-colorings c_H and c'_H such that $\{c_H(e) : e \in E_H(v_l)\} \neq \{c'_H(e) : e \in E_H(v_l)\}$ for at least one $l \in [1, n]$.*

Then H_C is edge-4-choosable.

Proof Suppose on the contrary that L is an edge 4-list assignment of H_C for which H_C has no edge-L-coloring. Since H is edge-4-choosable, H has an edge-$L|H$-coloring c_H. Color each edge u_iv_i with a color $c_i \in L(u_iv_i) \setminus \{c_H(e) : e \in N(u_iv_i) \cap E(H)\}$, and then define an edge list assignment L_C of the cycle C by setting $L_C(u_iu_{i+1}) = L(u_iu_{i+1}) \setminus \{c_i, c_{i+1}\}$ $(1 \leq i \leq n$, subscripts taken modulo n). Clearly $|L_C(u_iu_{i+1})| \geq 2$ for each i, and C has no edge-L_C-coloring since otherwise H_C would have an edge-L-coloring. By Lemma 1, $|\bigcup_{e \in C} L_C(e)| \leq 2$, so that $L_C(u_iu_{i+1})$ is the same set of two colors, say $L_C(u_iu_{i+1}) = \{a, b\}$, for each i. This implies that $c_i \neq c_{i+1}$ for each i (subscripts taken modulo n), so that $|\{c_i : 1 \leq i \leq n\}| \geq 3$ since n is odd.

If (i) holds, then we can recolor the edge u_lv_l with a different color taken from its list. If (ii) holds, then we can switch to a different edge-$L|H$-coloring of H and then recolor edges u_iv_i as necessary so that at least one edge u_lv_l is colored differently from before. In either case, let c'_i denote the new color of u_iv_i $(1 \le i \le n)$, so that $c_i' \ne c_i$. Define an edge list assignment L_C' of the cycle C by setting $L_C'(u_iu_{i+1}) = L(u_iu_{i+1}) \setminus \{c_i', c_{i+1}'\}$ $(1 \le i \le n)$. Clearly $|L_C'(u_iu_{i+1})| \ge 2$ for each i, and $|\bigcup_{1\le i\le n} L_C'(u_iu_{i+1})| \ge 3$ by Lemma 2. By Lemma 1, the cycle C has an edge-L_C'-coloring, and this, together with colors already assigned, gives an edge-L-coloring of H_C. This contradiction completes our proof. □

Lemma 4. *Let $n \ge 3$ be odd and let L be an edge 4-list assignment of H_C. If H_C has an edge-L-coloring c, then H_C must have another edge-L-coloring c' such that $c'(e) = c(e)$ for every $e \in E(H_C) \setminus E(C)$ and $\{c'(e), c'(e')\} \ne \{c(e), c(e')\}$ for some two adjacent edges e and e' of the cycle C.*

Proof Let $c(u_iv_i) = c_i$ and $c(u_iu_{i+1}) = \bar{c}_i$ $(1 \le i \le n)$, where subscripts are taken modulo n. Define an edge list assignment L_C of the cycle C by setting $L_C(u_iu_{i+1}) = L(u_iu_{i+1}) \setminus \{c_i, c_{i+1}\}$ $(1 \le i \le n)$. Clearly $|L_C(u_iu_{i+1})| \ge 2$, for each i. Since $\bar{c}_i \in L_C(u_iu_{i+1})$ and $|\{\bar{c}_i : 1 \le i \le n\}| \ge 3$ (since C is an odd cycle), it follows that $|\bigcup_{1\le i\le n} L_C(u_iu_{i+1})| \ge 3$. By Lemma 1, the cycle C has another edge-L_C-coloring, which together with the existing coloring c of edges in $E(H_C) \setminus E(C)$ gives an edge-L-coloring c' of H_C that is different from c.

Let $c'(u_iu_{i+1}) = \bar{c}_i'$ $(1 \le i \le n)$. Note that $\bar{c}_i \ne \bar{c}_i'$ for at least one i. If $\{\bar{c}_i, \bar{c}_{i+1}\} = \{\bar{c}_i', \bar{c}_{i+1}'\}$ for every i, then $\bar{c}_1 = \bar{c}_3 = \cdots = \bar{c}_n$ as in the proof of Lemma 2, whereas $\bar{c}_n \ne \bar{c}_1$. This contradiction shows that $\{c'(e), c'(e')\} \ne \{c(e), c(e')\}$ for some two adjacent edges e and e' of the cycle C, as required. □

As remarked in the Introduction, the final part of the following result follows from a result of Ellingham and Goddyn [2].

Lemma 5. *Let $G_{n,2} = C_n \times P_2$. Then $G_{n,2}$ is a 3-regular edge-3-colorable planar graph, and $\chi_l'(G_{n,2}) = \chi'(G_{n,2}) = 3$.*

Lemma 6. *Let $G_{n,m} = C_n \times P_m$, where n is odd and $m \ge 2$. Then $G_{n,m}$ is edge-4-choosable.*

Proof We argue by induction on m. By Lemma 5, the result is true if $m = 2$; so suppose $m \ge 3$. Let $C_n = x_1x_2\ldots x_nx_1$ and $P_m = x^1x^2\ldots x^m$. Denote the element (x_i, x^j) of $V(C_n) \times V(P_m)$ by x_i^j, and for each j let C_n^j be the cycle $x_1^jx_2^j\ldots x_n^jx_1^j$ in $G_{n,m}$. Let $C = C_n^m$, $C' = C_n^{m-1}$, $H = G_{n,m} - C \cong G_{n,m-1}$ and $H' = H - C' \cong G_{n,m-2}$, so that $G_{n,m} = H_C$ and $H = H_{C'}'$. By the induction hypothesis, H is edge-4-choosable. By

Lemma 4, if L_H is any edge 4-list assignment of H, then H $(= H'_{C'})$ has two edge-L_H-colorings c and c' that are identical on all edges of $E(H) \setminus E(C')$ but such that $\{c'(e), c'(e')\} \neq \{c(e), c(e')\}$ for some two adjacent edges e and e' of the cycle C'. It follows from Lemma 3 that $H_C = G_{n,m}$ is edge-4-choosable. □

Theorem 7. *Let $G_{n,m} = C_n \times P_m$, where n is odd and $m \geq 3$. Then $\chi'_l(G_{n,m}) = \chi'(G_{n,m}) = 4$.*

Proof Since $\Delta(G_{n,m}) = 4$, $\chi'_l(G_{n,m}) \geq \chi'(G_{n,m}) \geq 4$. On the other hand, by Lemma 6, $4 \geq \chi'_l(G_{n,m}) \geq \chi'(G_{n,m})$. Thus, $\chi'_l(G_{n,m}) = \chi'(G_{n,m}) = 4$. □

Combining Lemma 5 and Theorem 7 with the known result in [3], we can obtain

Corollary 8. *Let $G_{n,m} = C_n \times P_m$, where $m \geq 2$. Then we have*

$$\chi'_l(G_{n,m}) = \chi'(G_{n,m}) = \begin{cases} 3, & m = 2; \\ 4, & m \geq 3. \end{cases}$$

Let $G_{n,m} = C_n \times P_m$, where $C_n = x_1x_2\ldots x_nx_1$, $P_m = x^1x^2\ldots x^m$ and x_i^j denotes the element (x_i, x^j) of $V(C_n) \times V(P_m)$. Suppose that $G_{n,m}$ is disjoint from H_C; then we denote by $G_{n,m} \uplus H_C$ the graph obtained by joining x_i^1 to u_i for $1 \leq i \leq n$.

Theorem 9. *Suppose that H_C is edge-4-choosable. Then $G_{n,m} \uplus H_C$ is also edge-4-choosable, and further, $\chi'_l(G_{n,m} \uplus H_C) = \chi'(G_{n,m} \uplus H_C) = 4$.*

Proof If n is even, then it is clear that $G_{n,m} \uplus H_C$ is edge-4-choosable. If n is odd, then by the inductive proof that is essentially the same as the proof of Lemma 6, we can know that $G_{n,m} \uplus H_C$ is also edge-4-choosable. Noting that $\Delta(G_{n,m} \uplus H_C) = 4$ we have $\chi'_l(G_{n,m} \uplus H_C) = \chi'(G_{n,m} \uplus H_C) = 4$. □

3. Acknowledgment

We thank the referee very much for help in improving the presentation of the paper.

References

[1] O.V.Borodin, A.V.Kostochka, D.R.Woodall, List edge and list total coloring of multigraphs, J. Combin. Theory Ser. B 71 (1997) 184-204.

[2] M.N.Ellingham and Luis Goddyn, List edge colorings of some 1-factorable multigraphs, Combinatorica 16(3) (1996) 343-352.

[3] F.Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153-158.

[4] R.Häggkvist, J.Janssen, New bounds on the list-chromatic index of the complete graph and other simple graph, Combin. Probab. Comput. 6 (1997) 295-313.

[5] D.Peterson, D.R.Woodall, Edge-choosability in line-perfect multigraphs, Discrete Math. 202 (1999) 191-199.

[6] D.R.Woodall, Edge-choosability of multicircuits, Discrete Math. 202 (1999) 271-277.