Summation of Reciprocals Related to l-th Power of Generalized Fibonacci Sequences

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Abstract

By applying the method of generating function, the purpose of this paper is to give several summations of reciprocals related to l-th power of generalized Fibonacci sequences. As applications, some identities involving Fibonacci, Lucas numbers are obtained.

Key Words: Recurrent sequence; Fibonacci number; Lucas number; l-th power; Reciprocal

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1 Introduction

In the notation of Horadam [4], write

$$W_n = W_n(a, b; P, Q),$$

so that

$$W_n = PW_{n-1} - QW_{n-2}, \quad (W_0 = a,\ W_1 = b,\ n \geq 2), \tag{1}$$

where a, b, P and Q are integers, with $PQ \neq 0$. In the sequel we shall suppose that $\Delta = P^2 - 4Q > 0$. Then it is easily to obtain the Binet formula [4]:

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

where $\alpha = \frac{P+\sqrt{P^2-4Q}}{2}$, $\beta = \frac{P-\sqrt{P^2-4Q}}{2}$, $A = b - \beta a$, and $B = b - \alpha a$. In particular, we write

$$\begin{cases} U_n = W_n(0,1;P,Q) \\ V_n = W_n(2,P;P,Q) \end{cases} \tag{3}$$

In [1], R. Andre-Jeannin obtained the following series identities:

$$\sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}}, \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k} \left(\sum_{n=1}^{k} \frac{W_{n+1}}{W_n} - k\alpha \right), \tag{5}$$

where $P > 0$ and k and m are nonnegative integers. (4) and (5) are given by Good [3] in the case $Q = -1$. Brousseau [2] proved (5) for $W_n = F_n$. In [5] and [6], T. Mansour used the generating function techniques to get several summations of reciprocals related to generalized Fibonacci numbers.

By applying the method of generating function, the purpose of this paper is to give several summations involving the reciprocals of l-th power of recurrence sequences (1).

Throughout the paper, l, k and t are three positive integers with $t \geq k$.

2 Main Result

Theorem 2.1 *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^l W_{n+1}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^{l-1-i}$$

$$\times \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}\right]^i$$

$$= \frac{(P^2-4Q)^{(l-1)/2}Q^k}{W_{k+1} - W_k\beta}\left[\frac{1}{W_k^l} - \frac{\beta^{l(t+1-k)}}{W_{t+1}^l}\right]. \qquad (6)$$

Proof Let $f(x) = \sum_{n=k}^{\infty} W_n x^n$. From (1) we have

$$f(x) - W_k x^k - W_{k+1}x^{k+1} = Px[f(x) - W_k x^k] - Qx^2 f(x).$$

Hence the following generating function is obtained:

$$f(x) = x^k \frac{W_k + x(W_{k+1} - PW_k)}{1 - Px + Qx^2}.$$

Since $1 - Px + Qx^2 = (1-\alpha x)(1-\beta x)$, we can decompose $f(x)$ into partial fractions:

$$f(x) = \frac{x^k}{\alpha - \beta}\left(\frac{W_{k+1} - \beta W_k}{1 - \alpha x} - \frac{W_{k+1} - \alpha W_k}{1 - \beta x}\right).$$

Comparing the coefficients of x^n in both sides of above equation, we obtain that

$$W_n = \frac{W_{k+1} - W_k\beta}{\alpha - \beta}\alpha^{n-k} - \frac{W_{k+1} - W_k\alpha}{\alpha - \beta}\beta^{n-k}.$$

Let

$$T_n = \frac{\beta^{n-k}}{W_n} = \frac{\beta^{n-k}}{\frac{W_{k+1} - W_k\beta}{\alpha-\beta}\alpha^{n-k} - \frac{W_{k+1} - W_k\alpha}{\alpha-\beta}\beta^{n-k}}.$$

Then

$$T_n^l = \left(\frac{\beta^{n-k}}{W_n}\right)^l = \left(\frac{\beta^{n-k}}{\frac{W_{k+1} - W_k\beta}{\alpha-\beta}\alpha^{n-k} - \frac{W_{k+1} - W_k\alpha}{\alpha-\beta}\beta^{n-k}}\right)^l. \qquad (7)$$

Computing the difference of T_n^l, we have

$$T_n^l - T_{n+1}^l = \left(\frac{\beta^{n-k}}{W_n}\right)^l - \left(\frac{\beta^{n+1-k}}{W_{n+1}}\right)^l$$

$$= \frac{[\beta^{(n-k)}W_{n+1}]^l - [\beta^{(n+1-k)}W_n]^l)}{W_n^l W_{n+1}^l}$$

$$= \frac{1}{(\alpha-\beta)^l W_n^l W_{n+1}^l} \left\{ \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^l \right.$$

$$\left. - \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^l \right\}$$

$$= \frac{(W_{k+1} - W_k\beta)Q^{n-k}}{(\alpha-\beta)^{l-1} W_n^l W_{n+1}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha - (W_{k+1} - W_k\alpha) \right.$$

$$\left. \times \beta^{2(n-k)+1} \right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^i$$

That is

$$T_n^l - T_{n+1}^l = \frac{(W_{k+1} - W_k\beta)Q^{n-k}}{(\alpha-\beta)^{l-1} W_n^l W_{n+1}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha \right.$$
$$\left. -(W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta \right.$$
$$\left. -(W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^i. \qquad (8)$$

Hence we have

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^l W_{n+1}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^{l-1-i}$$
$$\times \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1} \right]^i$$
$$= \frac{(\alpha-\beta)^{l-1}Q^k}{W_{k+1} - W_k\beta} \sum_{n=k}^{t} (T_n^l - T_{n+1}^l)$$
$$= \frac{(P^2-4Q)^{(l-1)/2}Q^k}{W_{k+1} - W_k\beta} \left[\frac{1}{W_k^l} - \frac{\beta^{l(t+1-k)}}{W_{t+1}^l} \right].$$

The proof of theorem2.1 is completed. □

Corollary 2.2 *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^n}{W_n W_{n+1}} = \frac{Q^k}{W_{k+1} - W_k\beta} \left(\frac{1}{W_k} - \frac{\beta^{t+1-k}}{W_{t+1}} \right), \qquad (9)$$

and

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^2 W_{n+1}^2}[P(W_{k+1} - W_k\beta)Q^{n-k} - 2(W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}]$$
$$= \frac{\sqrt{P^2 - 4Q}\, Q^k}{W_{k+1} - W_k\beta}\left[\frac{1}{W_k^2} - \frac{\beta^{2(t+1-k)}}{W_{t+1}^2}\right]. \quad (10)$$

Proof Take $l = 1$ and $l = 2$ in identity (6), respectively. □

Theorem 2.3 *Let $P > 0$. Then*

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+1}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}\right]^{l-1-i}$$
$$\times \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}\right]^i$$
$$= \frac{(P^2 - 4Q)^{(l-1)/2}Q^k}{W_k^l(W_{k+1} - W_k\beta)}. \quad (11)$$

Proof By (6), we have

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+1}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}\right]^{l-1-i}$$
$$\times \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}\right]^i$$
$$= \frac{(P^2 - 4Q)^{(l-1)/2}Q^k}{W_{k+1} - W_k\beta}\left[\frac{1}{W_k^l} - \lim_{t\longrightarrow\infty} \frac{\beta^{l(t+1-k)}}{W_{t+1}^l}\right]$$
$$= \frac{(P^2 - 4Q)^{(l-1)/2}Q^k}{W_k^l(W_{k+1} - W_k\beta)}.$$

where the limiting process has been justified by

$$\left|\frac{\alpha}{\beta}\right| = \left|\frac{P + \sqrt{P^2 - 4Q}}{P - \sqrt{P^2 - 4Q}}\right| > 1, \quad \text{for} \quad P > 0,$$

and

$$\lim_{t\longrightarrow\infty} T_t^l = \left(\lim_{t\longrightarrow\infty} \frac{\beta^{t-k}}{W_t}\right)^l$$
$$= \left[\lim_{t\longrightarrow\infty} \frac{1}{\frac{W_{k+1}-W_k\beta}{\alpha-\beta}(\frac{\alpha}{\beta})^{t-k} - \frac{W_{k+1}-W_k\alpha}{\alpha-\beta}}\right]^l = 0. \quad (12)$$

The proof of theorem2.3 is completed. □

Corollary 2.4 *Let $P > 0$. Then*

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n W_{n+1}} = \frac{Q^k}{W_k(W_{k+1} - W_k\beta)}, \tag{13}$$

and

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^2 W_{n+1}^2}[P(W_{k+1} - W_k\beta)Q^{n-k} - 2(W_{k+1} - W_k\alpha)\beta^{2(n-k)+1}] = \frac{Q^k\sqrt{P^2-4Q}}{W_k^2(W_{k+1} - W_k\beta)}. \tag{14}$$

Proof Take $l = 1$ and $l = 2$ in the identity (11), respectively. □

Theorem 2.5 *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha) \times\beta^{2(n-k)+m}\right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\right]^i$$
$$= \frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \left[\frac{\beta^{li}}{W_{k+i}^l} - \frac{\beta^{l(t+i+1-k)}}{W_{t+i+1}^l}\right]. \tag{15}$$

Proof From (7), we have

$$T_n^l - T_{n+m}^l = \left(\frac{\beta^{n-k}}{W_n}\right)^l - \left(\frac{\beta^{n+m-k}}{W_{n+m}}\right)^l$$
$$= \frac{[\beta^{(n-k)}W_{n+m}]^l - [\beta^{(n+m-k)}W_n]^l}{W_n^l W_{n+m}^l}$$
$$= \frac{1}{(\alpha-\beta)^l W_n^l W_{n+m}^l}\left\{\left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha) \times\beta^{2(n-k)+m}\right]^l - \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m -(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\right]^l\right\}$$

$$= \frac{Q^{n-k}(W_{k+1} - W_k\beta)(\alpha^m - \beta^m)}{(\alpha-\beta)^l W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m \right.$$

$$\left. -(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m \right.$$

$$\left. -(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\right]^i$$

Hence

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha)\right.$$

$$\left. \times \beta^{2(n-k)+m}\right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m \right.$$

$$\left. -(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\right]^i$$

$$= \frac{(\alpha-\beta)^l Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{n=k}^{t} (T_n^l - T_{n+m}^l)$$

$$= \frac{(P^2-4Q)^{l/2} Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \left[\frac{\beta^{li}}{W_{k+i}^l} - \frac{\beta^{l(t+i+1-k)}}{W_{t+i+1}^l}\right],$$

as required.

Corollary 2.6 *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^n}{W_n W_{n+m}} = \frac{Q^k\sqrt{P^2-4Q}}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \times \sum_{i=0}^{m-1} \left[\frac{\beta^i}{W_{k+i}} - \frac{\beta^{t+i+1-k}}{W_{t+i+1}}\right], \qquad (16)$$

and

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^2 W_{n+m}^2} \left[(W_{k+1} - W_k\beta)Q^{n-k}V_m - 2(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\right]$$

$$= \frac{(P^2-4Q)Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \left[\frac{\beta^{2i}}{W_{k+i}^2} - \frac{\beta^{2(t+i+1-k)}}{W_{t+i+1}^2}\right]. \qquad (17)$$

Proof Take $l = 1$ and $l = 2$ in the identity (15), respectively. □

Theorem 2.7 *Let $P > 0$. Then*

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \Big[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha) \times\beta^{2(n-k)+m}\Big]^{l-1-i} \Big[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\Big]^i$$

$$= \frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{W_{k+i}^l}. \tag{18}$$

Proof By (15) and (12), we have

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} \Big[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha) \times\beta^{2(n-k)+m}\Big]^{l-1-i} \Big[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m - (W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\Big]^i$$

$$= \frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \left[\frac{\beta^{li}}{W_{k+i}^l} - \lim_{t\to\infty} \frac{\beta^{l(t+i+1-k)}}{W_{t+i+1}^l}\right]$$

$$= \frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{W_{k+i}^l},$$

as required.

Corollary 2.8 *Let $P > 0$. Then*

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n W_{n+m}} = \frac{Q^k\sqrt{P^2-4Q}}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^i}{W_{k+i}}, \tag{19}$$

and

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^2 W_{n+m}^2} \Big[(W_{k+1} - W_k\beta)Q^{n-k}V_m - 2(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\Big]$$

$$= \frac{(P^2-4Q)Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{2i}}{W_{k+i}^2}. \tag{20}$$

Proof Take $l = 1$ and $l = 2$ in the identity (18), respectively. □

Corollary 2.9 *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^n}{W_n^l W_{n+2}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^2 - (W_{k+1} - W_k\alpha)\right.$$
$$\left.\times\beta^{2(n-k+1)}\right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta^2\right.$$
$$\left.-(W_{k+1} - W_k\alpha)\beta^{2(n-k+1)}\right]^i$$
$$= \frac{(P^2 - 4Q)^{(l-1)/2}Q^k}{P(W_{k+1} - W_k\beta)} \left[\frac{1}{W_k^l} - \frac{\beta^{l(t+1-k)}}{W_{t+1}^l} + \frac{\beta^l}{W_{k+1}^l} - \frac{\beta^{l(t+2-k)}}{W_{t+2}^l}\right],$$

and

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+2}^l} \sum_{i=0}^{l-1} \left[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^2 - (W_{k+1} - W_k\alpha)\right.$$
$$\left.\times\beta^{2(n-k+1)}\right]^{l-1-i} \left[(W_{k+1} - W_k\beta)Q^{n-k}\beta^2\right.$$
$$\left.-(W_{k+1} - W_k\alpha)\beta^{2(n-k+1)}\right]^i$$
$$= \frac{(P^2 - 4Q)^{(l-1)/2}Q^k}{P(W_{k+1} - W_k\beta)} \left(\frac{1}{W_k^l} + \frac{\beta^l}{W_{k+1}^l}\right).$$

Proof Take $m = 2$ in the identity (15) and (18), respectively. □

3 Some Applications

In this section we can obtain some interesting identities involving Fibonacci, Lucas numbers by taking special values for a, b, P and Q.

3.1 Fibonacci Numbers

In this case, $W_n(0, 1; 1, -1) = F_n$, the Fibonacci number. Then according to above theorems and corollaries we obtain

$$\sum_{n=k}^{t}\frac{(-1)^n}{F_n^lF_{n+m}^l}\sum_{i=0}^{l-1}\Big[(F_{k+1}-F_k\beta)(-1)^{n-k}\alpha^m-(F_{k+1}-F_k\alpha)$$
$$\times\beta^{2(n-k)+m}\Big]^{l-1-i}\Big[(F_{k+1}-F_k\beta)(-1)^{n-k}\beta^m$$
$$-(F_{k+1}-F_k\alpha)\beta^{2(n-k)+m}\Big]^i$$
$$=\frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m-\beta^m)(F_{k+1}-F_k\beta)}\sum_{i=0}^{m-1}\left[\frac{\beta^{li}}{F_{k+i}^l}-\frac{\beta^{l(t+i+1-k)}}{F_{t+i+1}^l}\right],$$

and

$$\sum_{n=k}^{\infty}\frac{(-1)^n}{F_n^lF_{n+m}^l}\sum_{i=0}^{l-1}\Big[(F_{k+1}-F_k\beta)(-1)^{n-k}\alpha^m-(F_{k+1}-F_k\alpha)$$
$$\times\beta^{2(n-k)+m}\Big]^{l-1-i}\Big[(F_{k+1}-F_k\beta)(-1)^{n-k}\beta^m$$
$$-(F_{k+1}-F_k\alpha)\beta^{2(n-k)+m}\Big]^i$$
$$=\frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m-\beta^m)(F_{k+1}-F_k\beta)}\sum_{i=0}^{m-1}\frac{\beta^{li}}{F_{k+i}^l}.$$

In particular,

$$\sum_{n=1}^{t}\frac{(-1)^n}{F_nF_{n+2}}=\frac{1-\sqrt5}{2}\left[\frac{3-\sqrt5}{2}-\frac{(\frac{1-\sqrt5}{2})^t}{F_{t+1}}-\frac{(\frac{1-\sqrt5}{2})^{t+1}}{F_{t+2}}\right];$$

$$\sum_{n=1}^{t}\frac{(-1)^n[2(-1)^{n-1}+(\frac{1-\sqrt5}{2})^{2n+3}]}{F_n^2F_{n+3}^2}$$
$$=\frac{5-3\sqrt5}{8}\left[\frac{27-7\sqrt5}{8}-\frac{(\frac{1-\sqrt5}{2})^{2t}}{F_{t+1}^2}-\frac{(\frac{1-\sqrt5}{2})^{2(t+1)}}{F_{t+2}^2}-\frac{(\frac{1-\sqrt5}{2})^{2(t+2)}}{F_{t+3}^2}\right];$$

and

$$\sum_{n=1}^{t}\frac{(-1)^n}{F_n^3F_{n+1}^3}\sum_{i=0}^{2}\left[\left(\frac{3+\sqrt5}{2}\right)(-1)^{n-1}-\left(\frac{3-\sqrt5}{2}\right)^n\right]^{2-i}$$
$$\times\left[(-1)^n-\left(\frac{3-\sqrt5}{2}\right)^n\right]^i$$
$$=\frac{5(1-\sqrt5)}{2}\left[1-\frac{(\frac{1-\sqrt5}{2})^{3t}}{F_{t+1}^3}\right];$$

which are equivalent to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}} = 2 - \sqrt{5};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n[2(-1)^{n-1} + (\frac{1-\sqrt{5}}{2})^{2n+3}]}{F_n^2 F_{n+3}^2} = \frac{60 - 29\sqrt{5}}{16};$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n^3 F_{n+1}^3} \sum_{i=0}^{2} & \left[\left(\frac{3+\sqrt{5}}{2}\right)(-1)^{n-1} - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]^{2-i} \\ & \times \left[(-1)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]^i \\ = & \ \frac{5(1-\sqrt{5})}{2}.\end{aligned}$$

3.2 Lucas Numbers

In this case, $W_n(2,1;1,-1) = L_n$, the Lucas number. Then according to above theorems and corollaries we obtain

$$\begin{aligned}\sum_{n=k}^{t} \frac{(-1)^n}{L_n^l L_{n+m}^l} \sum_{i=0}^{l-1} & [(L_{k+1} - L_k\beta)(-1)^{n-k}\alpha^m - (L_{k+1} - L_k\alpha) \\ & \times \beta^{2(n-k)+m}]^{l-1-i} [(L_{k+1} - L_k\beta)(-1)^{n-k}\beta^m \\ & -(L_{k+1} - L_k\alpha)\beta^{2(n-k)+m}]^i \\ = & \ \frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(L_{k+1} - L_k\beta)} \sum_{i=0}^{m-1} \left[\frac{\beta^{li}}{L_{k+i}^l} - \frac{\beta^{l(t+i+1-k)}}{L_{t+i+1}^l}\right],\end{aligned}$$

and

$$\begin{aligned}\sum_{n=k}^{\infty} \frac{(-1)^n}{L_n^l L_{n+m}^l} \sum_{i=0}^{l-1} & [(L_{k+1} - L_k\beta)(-1)^{n-k}\alpha^m - (L_{k+1} - L_k\alpha) \\ & \times \beta^{2(n-k)+m}]^{l-1-i} [(L_{k+1} - L_k\beta)(-1)^{n-k}\beta^m\end{aligned}$$

$$-(L_{k+1} - L_k\alpha)\beta^{2(n-k)+m}\Big]^i$$

$$= \frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m-\beta^m)(L_{k+1}-L_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{L_{k+i}^l}.$$

In particular,

$$\sum_{n=1}^{t} \frac{(-1)^n}{L_nL_{n+2}} = \frac{\sqrt{5}-5}{10}\left[\frac{7-\sqrt{5}}{6} - \frac{(\frac{1-\sqrt{5}}{2})^t}{L_{t+1}} - \frac{(\frac{1-\sqrt{5}}{2})^{t+1}}{L_{t+2}}\right];$$

$$\sum_{n=1}^{t} \frac{(-1)^n\,[2(-1)^{n-1} - (\frac{1-\sqrt{5}}{2})^{2n+3}]}{L_n^2L_{n+3}^2}$$

$$= \frac{5-3\sqrt{5}}{40}\left[\frac{399-43\sqrt{5}}{288} - \frac{(\frac{1-\sqrt{5}}{2})^{2t}}{L_{t+1}^2} - \frac{(\frac{1-\sqrt{5}}{2})^{2(t+1)}}{L_{t+2}^2} - \frac{(\frac{1-\sqrt{5}}{2})^{2(t+2)}}{L_{t+3}^2}\right];$$

and

$$\sum_{n=1}^{t} \frac{(-1)^n}{L_n^3L_{n+1}^3} \sum_{i=0}^{2}\left[\frac{3+\sqrt{5}}{2}(-1)^{n-1} + \left(\frac{3-\sqrt{5}}{2}\right)^n\right]^{2-i}$$

$$\times\left[(-1)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n\right]^i$$

$$= \frac{\sqrt{5}-5}{10}\left[1 - \frac{(\frac{1-\sqrt{5}}{2})^{3t}}{L_{t+1}^3}\right];$$

which are equivalent to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_nL_{n+2}} = \frac{3\sqrt{5}-10}{15};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n[2(-1)^{n-1} - (\frac{1-\sqrt{5}}{2})^{2n+3}]}{L_n^2L_{n+3}^2} = \frac{660-353\sqrt{5}}{2880};$$

and

$$\sum_{n=1}^{t} \frac{(-1)^n}{L_n^3L_{n+1}^3} \sum_{i=0}^{2}\left[\frac{3+\sqrt{5}}{2}(-1)^{n-1} + \left(\frac{3-\sqrt{5}}{2}\right)^n\right]^{2-i}$$

$$\times\left[(-1)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n\right]^i$$

$$= \frac{\sqrt{5}-5}{10}.$$

References

[1] Richard Andre-Jeannin, Summation of Reciprocalsin Certain Second-order Recurring Sequences, The Fibonacci Quarterly 35.1(1997): 68-74.

[2] Br. A. Brousseau, Summation of infinite Fibonacci series, The Fibonacci Quarterly, 7.2(1969): 143-168.

[3] I. J. Good, A symmetry property of alternating sums of products of reciprocals, The Fibonacci Quarterly, 32.3(1994): 284-287.

[4] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, The Fibonacci Quarterly, 3.2(1965): 161-176.

[5] T. Mansour, A formula for the generating functions of powers of Horadam's sequence, Australasian Journal of Combinatorics 30 (2004), 207-212.

[6] T. Mansour, Squaring the terms of an l-th order linear recurrence, Australasian Journal of Combinatorics 31 (2005), 15-20.