Four Colour Classes in Projective Planes with Non-Fano Quads

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Abstract

By introducing 4 colour classes in projective planes with non-Fano quads, discussion of the planes of small order is simplified.

1 Introduction

This is a continuation of the discussion in [7]. For simplicity, we repeat some of the arguments there. We take the non-Fano quad based on A, B, C, D, in the form

ABE4 CDE1 ACF5 BDF2 ADG6 BCG3	blue
EF63 FG41 GE52	green

If this quad occurs in a plane with n^2+n+1 points, there are $n-3$ additional points on each of the 9 lines of the non-Fano quad. The $6(n-3)$ points on the first 6 lines are coloured blue; the $3(n-3)$ points determined by the 3 diagonal lines are coloured green; the 6 numbered points are coloured yellow. This gives a total of $9(n-3)+13 = 9n-14$ points. So there are $n^2+n+1-(9n-14) = (n-3)(n-5)$ additional points. We colour these "foreign" points red.

Each of A, B, C, D, occurs with all the red points, all the green points, half of the blue points, and 3 yellow points (A with 1,2,3; B with 1,5,6; C with 2,4,6; D with 3,4,5). This gives a total of $(n-3)(n-5)+6(n-3)+3 = n(n-2)$ points with each of A, B, C, D.

Each of E, F, G, occurs with one-third of the green points, all the red points, two-thirds of the blue points, for a total of $5(n-3)+(n-3)(n-5) = n(n-3)$ points.

Additionally, there are $(n^2+n+1)-9-4(n-2)-3(n-3) = (n-3)^2$ lines. These "foreign" lines contain $6(n-3)$ yellow points, $6(n-3)(n-4)$ blue points, $3(n-3)(n-5)$ green points, $(n-3)(n-5)(n-6)$ red points, for a total of $(n-3)[6+3(n-5)+6(n-4)+(n-5)(n-6)] = (n-3)^2(n+1)$ points.

Of course, some of these sets may be empty for values of n less than 7.

We might also remark that when we assign 41 to the diagonal line FG, we could have denoted the lines ABE and CDE by ABE1, CDE4. If we had made that allotment, the triples occuring with A, B, C, D, would (in a different order) be 4,5,6; 2,3,4; 1,3,5; 1,2,6. A change of allotment of 2 and 5 as well as 1 and 4 would preserve the triples 1,2,3; 1,5,6; 2,4,6; 3,4,5; although in a different order.

2 The Pattern Theorem

Altogether, there are $3(n-3)+6 = 3(n-1)$ green and yellow points. We will indicate the points which are either green or yellow as the green-yellow points. So the total number of pairs involving green-yellow points is $(3n-3)(3n-4)/2 = (9n^2-21n+12)/2$.

Now there are 3 sets of $n-1$ green-yellow points in the lines EG, EF, FG. Each of A, B, C, D, must occur with $3+3(n-3) = 3(n-2)$ green-yellow points. Note that each of A, B, C, D, must occur $n-2$ more times. The $(n-3)^2$ foreign lines have $3(n-3)^2$ green-yellow points. So the total number of green-yellow pairs is at least

$$\begin{aligned} & \frac{3(n-1)(n-2)}{2} + 4(3)(n-2) + (n-3)^2 3 \\ = & \frac{9n^2-21n+12}{2}. \end{aligned}$$

This is because the number of pairs in a collection of sets is minimal when the sets all have the same cardinality, or when the cardinalities are as nearly equal as possible (see, for example, [8]). But this is the proper total, and so we have established.

Lemma 1. *The green-yellow points occurring with A, B, C, D, and in the* $(n-3)^2$ *additional lines occur in triples ((n − 2) triples with each of A, B, C, D, and* $(n-3)^2$ *triples in the additional lines).*

Now we consider the yellow-blue pairs. The total number of yellow-blue points is $6 + 6(n-3) = 6n - 12$, and so the total number of blue-yellow pairs is $(6n-13)(3n-6) = 18n^2 - 75n + 78$.

There are 6 sets of $n-2$ blue-yellow points in the 6 lines ABE to BCG. Each of A, B, C, D, occurs with $3 + 3(n-3) = 3(n-2)$ blue-yellow points. Note that each of E, F, G, mus still occur with $4(n-3)$ blue points, and the lines EF, FG, GE contain 3 yellow pairs. Each of E, F, G, contains $4(n-3)$ blue points, and the lines EF, FG, GE, contain 3 yellow points. The $(n-3)^2$ foreign lines contain $6(n-3) + 6(n-3)(n-4) = 6(n-3)^2$ blue-yellow points. So the total number of blue-yellow pairs is at least

$$\begin{aligned} & 3(n-2)(n-3) + 4(n-2)3 + 18(n-3) + 3 + (n-3)^2 15 \\ = \; & 18n^2 - 75n + 78. \end{aligned}$$

Again, this is the proper total and so we have established.

Lemma 2. *The yellow and blue points occurring with A, B, C, D, occur in triples and the yellow and blue points occurring in the* $(n-3)^2$ *additional lines occur in sextuples.*

Now let us look at the A, B, C, D, lines and the foreign lines. The yellow-green points occur in the sets of the same cardinality, and so therefore do the red-blue points. The yellow-blue points occur in the sets of the same cardinality, and so also do the red-green points. So if we remove the yellow points from the yellow-green sets, the red points can be moved over to exactly fill the gaps produced and give sets of equal cardinality. We will say that the red and yellow points have the *same pattern* in a set of lines, if the difference of the number of red and yellow points in each line of the set is a constant. Thus we have established the

Pattern Theorem. *The distribution patterns of the red and yellow points in A, B, C, D, lines is exactly the same. Similarly, the distribution pattern of the red and yellow points in the foreign lines is exactly the same.*

This does not mean that there are equal numbers of red and yellow points. For example, if $n = 7$, there are 8 red points, 6 yellow points. With A, we might have $A123xxx$, $Axxxxxx$ (4 times) as the situation with yellow points. Actually, this can not occur, but, if it did, then the red points would have to follow the same pattern, namely,

$$Ar_1xxxxx, Ar_2xxxxx, Ar_3xxxxx, Ar_4xxxxx, Ar_5r_6r_7r_8xx.$$

3 Remarks on Small Planes

If $n = 3$, then there are no red points and the plane is completed with A123, B156, C246, D345.

If $n = 4$, there are no non-Fano quads, since one would need to have 9 additional blue and green points, and the total number of points is only 21.

If $n = 5$ and there is a Fano quad, then there are 21 blue and green points. This requires 3 red points (impossible, since they would occur with $A, \dots, G$, and thus have frequency 7 each). Hence, if $n = 5$, there are 12 blue points, 6 green points, no red points. The four foreign lines contain no green points, 12 yellow points, 12 blue points. Since there are no red points, the four foreign lines can be written as 123, 156, 246, 345, by the Pattern Theorem.

Also, we can assign the blue and green points to the 9 lines of the non-Fano quad as b_1b_2; b_3b_4; b_5b_6; b_7b_8; b_9b_{10}; $b_{11}b_{12}$; g_1g_2; g_3g_4; g_5g_6; to produce

$ABE4b_1b_2$	$EF63g_1g_2$
$CDE1b_3b_4$	$FG41g_3g_4$
$ACF5b_5b_6$	$GE52g_5g_6$
$BDF2b_7b_8$	
$ADG6b_9b_{10}$	
$BCG3b_{11}b_{12}$	

There is no loss of generality in writing $123b_1b_5b_9$. Then 156 requires $b_2b_7b_{11}$. This forces $246b_6b_3b_{12}$ and $345b_4b_8b_{10}$.

It is then easy to complete the remaining 18 lines. The A, B, C, D, lines each contain 3 yellow points, 6 blue points, 6 green points; the E, F, G, lines each contain 2 green points, 8 blue points. We obtain perforce

$A1b_8b_{12}$	$B1b_6b_{10}$	$C2b_2b_{10}$	$D3b_2b_6$
$A2b_4b_{11}$	$B5b_3b_9$	$C4b_7b_9$	$D4b_5b_{11}$
$A3b_3b_7$	$B6b_4b_5$	$C6b_1b_8$	$D5b_1b_{12}$

Eg_3	Fg_5	Gg_1
Eg_4	Fg_6	Gg_2

Now the g positions are the four unused 1-factors of $g_1, \dots, g_6$. There are $G = g_2g_3, g_4g_5, g_6g_1$; $G_3 = g_1g_4, g_2g_6, g_3g_5$; $G_4 = g_2g_5, g_1g_3, g_4g_6$; $G_5 =$

g_3g_6, g_2g_4, g_1g_5. Since these 1-factors are unique up to isomorphism, we can assign G_2 to A and obtain

$$A1b_8b_{12}g_6g_1,\quad A2b_4b_{11}g_2g_3,\quad A3b_3b_7g_4g_5.$$

This forces G_4 to occur with B as

$$B1b_6b_{10}g_2g_5,\quad B5b_3b_9g_1g_3,\quad B6b_4b_5g_4g_6.$$

It also forces G_3 to occur with C as

$$C2b_2b_{10}g_1g_4,\quad C4b_7b_9g_2g_6,\quad C6b_1b_8g_3g_5.$$

Finally, G_5 must occur with D as

$$D3b_2b_6g_3g_6,\quad D4b_5b_{11}g_1g_5,\quad D5b_1b_{12}g_2g_4.$$

The remaining 6 blocks require blue points and are easily obtained as

$$\begin{array}{lll} Eg_4b_6b_8b_9b_{11} & Fg_5b_2b_4b_9b_{12} & Gg_1b_1b_4b_6b_7 \\ Eg_3b_5b_7b_{10}b_{12} & Fg_6b_1b_3b_{10}b_{11} & Gg_2b_2b_3b_5b_8 \end{array}$$

This use of the Pattern Theorem constructs the 31-point geometry uniquely (up to isomorphism).

4 The Plane with $n = 6$

The Pattern Theorem gives a shorter proof of the impossibility of this plane than was given in [7]. If there is a non-Fano quad, then there are 3 red points which we may call P, Q, R. By the Pattern Theorem (no yellow points in the E, F, G, lines), the red points must occur disjointly in the E, F, G, lines. So the 3 red pairs must occur in the A, B, C, D, lines.

If PQR appears, it must appear with 3 yellow points, say A123PQR. But then P, Q, and R occur separately with B, C, D. So P, Q, R, must each occur with 6 and this is impossible (3 disjoint letters and only 2 symbols 6).

If PQ, PR, QR, occur separately, we may suppose PQ occurs with 12 and thus R occurs with 3. So PR and QR must occur with two of 45, 46, 56. Whichever choice is taken, a repeat with R occurs. So this distribution is likewise impossible.

Thus a non-Fano quad can not occur in a 43-point geometry. But then all quads are Fano and [1] guarantees that one must have a field plane. Since there is no field with 6 elements, we see that the case $n = 6$ is impossible.

5 Remarks on the Plane with $n = 7$

In this case, there are 8 red points and they occur once each among the 16 foreign lines. There are 24 yellow points among the 16 foreign lines. This is the last occasion in which the number of yellow points exceeds the number of red points among the foreign lines. It follows that the yellow points must fill one column in the additional lines leaving 8 yellow points to have the same pattern as the 8 red points. So the 8 red points can only occur with frequencies 0, 1, or 2 in the additional lines (corresponding to yellow points of frequncies 1, 2, 3).

We first note that, if there is a triple of yellow points such as $A123xxxxx$, then it must correspond to lines Ar_1, Ar_2, Ar_3, Ar_4, $Ar_5r_6r_7r_8$. But then $r_5r_6r_7r_8$, must occur in separate additional lines and each must correspond to a number pair from 4, 5, 6. Since there are only 3 such number pairs, this is impossible. So no yellow triple occurs in the A, B, C, D, lines.

Let us now investigate whether we can have a yellow pair, say A12. Then we can write Ar_1, Ar_2, Ar_3, $A3r_4r_5$, $A12r_6r_7r_8$. The points r_6, r_7, r_8, must each correspond to at least 2 yellow points. If r_6345 occurs, then r_7 and r_8 must both occur with D3, and this is impossible. So we need r_634 and r_735. The choice r_845 is impossible since r_6, r_7, r_8, would then have to occur with 6 in B6 and C6 (this forces a repeat). So we must have r_8456 or r_846 (r_856 behaves equivalently).

If r_8456 occurs, then another r is required. Now r_6 and r_7 are impossible and r_1 and r_2 occur with 13 and 23 respectively. If r_3 occurs with r_8456, we have a contradiction, since r_3 and r_8 both must occur with D3. Hence our only possibility is r_8456r_4 (or, equivalently r_5). This forces $B1r_4$ and $C2r_4$; thus r_5 must occur with 1 and 2 in the foreign lines. This is impossible since r_5 occurs only once.

We now have r_846 as well as Dr_3r_83. This forces $B5r_8$, which in turn requires $D5r_6$. Now B6 must occur with r_6 or r_7; so 56 must appear in the foreign lines. If we have r_356, then we are forced to take $1r_3$ and $24r_3$. Then r_1 must appear with 2, 4, 5, 6; this requires $C24r_3r_1$, $B6r_6r_1$, $D5r_6r_1$ (a contradiction). So we must have r_456. Then r_4 occurs with B1 and C2; hence r_5 must appear with 1 and 2 in the foreign lines (impossible).

We have thus established that no triples nor pairs occur in the A, B, C, D, lines. But if there are x triples and y pairs in the additional lines, these correspond to x pairs and y singletons of the r_i in the additional lines; thus $2x + y = 8$. The total number of pairs is thus $3x + y = 12$, whence we have $x = 4$. Thus the red points occur in 4 pairs.

We have thus proved the

Conic Theorem. *The 8 red points form a system of points of which no three are collinear.*

The uniqueness of the plane of order 7 was first proved by Pierce [5] and Hall [2], [3]. Recently, Kocay [4] has given a different proof using an ingenious "Sum of Squares Theorem". An argument can also be given, analogous to those in Sections 4 and 5 for $n = 6$ and $n = 7$, by using the Pattern Theorem.

Since the plane of order 7 is unique, it must be the field plane. But Segre [6] proved that all sets of $n + 1$ points, no 3 collinear, in field planes of odd order are conics. Hence, such a system in PG(2,7) is a conic, and we have thus shown that a non-Fano quad in PG(2,7) determines a conic made up of the 8 red points determined by the initial quad ABCD.

6 Concluding Remarks

A number of interesting questions can now be asked concerning the plane of order 7. In particular, suppose that $A, B, C, D, P_1, P_2, P_3, P_4$, is a conic through A, B, C, D (there are 5 such conics). What is the relationship among the 5 red conics determined?

Also there are 70 possible quads determined by a specific conic A, B, C, D, P_1, P_2, P_3, P_4. How many foreign conics are determined by the 70 possible quads from A, B, C, D, P_1, P_2, P_3, P_4.

Another interesting question is whether 2 distinct quads can determine the same foreign conic. And, if so, how many quads can determine a fixed foreign conic?

If one applies this pattern approach with $n = 8$, it should be possible to rule out the possibility of a non-Fano quad. Then, from [1], the 73-point geometry would have to be a field plane, and this would provide a non-computer proof of its uniqueness.

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