Some Results on Fractional Edge coloring of Graphs*

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Abstract

A *fractional edge coloring* of graph G is an assignment of a nonnegative weight ω_M to each matching M of G such that for each edge e we have $\sum_{M \ni e} \omega_M \geq 1$. The fractional edge coloring chromatic number of a graph G, denoted by $\chi'_f(G)$, is the minimum value of $\sum_M \omega_M$(where the minimum is over all fractional edge colorings ω). It is known that for any simple graph G with maximum degree Δ, $\Delta \leq \chi'_f(G) \leq \Delta + 1$. And $\chi'_f(G) = \Delta + 1$ if and only if G is K_{2n+1}. In this paper, we give some sufficient conditions for a graph G to have $\chi'_f(G) = \Delta$. Furthermore we show that the results in this paper is the best possible.

1 Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. The graphs in this paper are simple, that is, they have no loops or multiple edges. We use $V(G)$, $E(G)$, $|V(G)|$, $\Delta(G)$ and $\delta(G)$ to denote, respectively, the vertex set, edge set, order,

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maximum degree and minimum degree of a graph G. Let $N_G(v)$ denote the neighborhood of v and let $d_G(v) = |N_G(v)|$ be the degree of v in G. For a subset $S \subseteq V(G)$, we write ∂S to stand for those edges that have exactly one end in S. Let G_Δ denote the subgraph of G induced by the vertices of degree $\Delta(G)$. An edge k-coloring of a graph G is a mapping ϕ from $E(G)$ to the set of colors $\{1, 2, \cdots, k\}$ such that no two incident edges receive the same color. The edge coloring chromatic number $\chi'(G)$ of G is the smallest integer k such that G admits an edge k-coloring. A well-known theorem of Vizing [6] states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

G is said to be Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 if $\chi'(G) = \Delta(G)+1$. The problem of deciding whether a given graph is Class 1 or Class 2 is known as the *classification problem* and Holyer [2] showed that this problem is NP−complete.

A graph G is *overfull* if $|E(G)| \geq \lfloor \frac{|V(G)|}{2} \rfloor \Delta(G) + 1$. We say that a subgraph H of G is an *overfull subgraph* if H is overfull and $\Delta(H) = \Delta(G)$ holds. A sufficient condition for a graph to be Class 2 is that G is overfull, or more generally, that G has an overfull subgraph H. This is easy to see, since the edges of H colored with the same color form a matching, and at most $\lfloor |V(G)|/2 \rfloor$ edges of H can receive the same color. There are many edge coloring problems on overfull subgraph still unsolved [3].

A fractional edge coloring of graph G is an assignment of a nonnegative weight ω_M to each matching M of G, such that for each edge e we have $\sum_{M \ni e} \omega_M \geq 1$. The fractional edge coloring chromatic number of a graph G, denoted by $\chi'_f(G)$, is the minimum value of $\sum_M \omega_M$(where the minimum is over all fractional edge colorings ω). Three other equivalent definitions and some interesting results on $\chi'_f(G)$ can be found in [5]. In which, The main results are the following Theorems.

Theorem A[5] *For any loopless multigraph G, there is*

$$\chi'_f(G) = \max\{\Delta(G), \max_H \frac{2|E(H)|}{|V(H)| - 1}\},$$

Where H is an induced subgraph of G with $|V(H)| \geq 3$ and $|V(H)|$ odd.

Theorem B[5] *Let G be a simple graph. Then $\Delta(G) \leq \chi'_f(G) \leq \chi'(G) \leq \Delta(G)+1$.*

From above Theorems we can see that $\chi'_f(G)$ may be $\Delta(G)$, or $\Delta(G)+1$, or between $\Delta(G)$ and $\Delta(G)+1$. Now it is reasonable to ask which graphs have $\chi'_f(G) = \Delta(G)$ and which have $\chi'_f(G) = \Delta(G)+1$? The following two results also can be found in [5].

Lemma 1.1[5] *Let G be an r-regular graph. Then $\chi'_f(G) = r$ if and only if G is a graph such that for any subset $X \subseteq V(G)$ with $|X|$ odd and $|\partial X| \geq r$.*

Lemma 1.2[5] *Let G be a connected graph. Then $\chi'_f(G) = \Delta(G)+1$ if and only if G is complete graph $K_{2n+1} (n \geq 1)$.*

In this paper we give some sufficient conditions for a graph G to have $\chi'_f(G) = \Delta(G)$. Our main results are Theorem 1 and Theorem 2.

Theorem 1 *Let G be a graph with maximum degree $\Delta(G) = 3$. Then $\chi'_f(G) > \Delta(G)$ if and only if G contains an induced subgraph H which has exactly one vertex of degree 2 and the remaining vertices of H have degree 3 in H.*

Theorem 2 *Let G be a graph with maximum degree $\Delta(G)$. If $\Delta(G) > \frac{2}{3}(|V(G)|-3)$ and $\delta(G_\Delta) \leq 1$, then $\chi'_f(G) = \Delta(G)$.*

Theorem 2 suggests that if $\Delta(G)$ is sufficiently large compared with $|V(G)|$, then $\delta(G_\Delta) \leq 1$ is a sufficient condition for G to have $\chi'_f(G) = \Delta(G)$. Furthermore, we will show that the condition $\Delta(G) > \frac{2}{3}(|V(G)|-3)$ is sharp in section 3.

2 Some Useful Lemmas

In this section, we state some basic results on overfull subgraphs, which will be used in the sequel.

Let the *deficiency* def(G) of a graph G be defined by

$$\text{def}(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v)).$$

The following three Lemmas were proved in [4].

Lemma2.1[4] *A graph G is overfull if and only if $|V(G)|$ is odd and*

$$\mathrm{def}(G) \le \Delta(G) - 2.$$

Lemma2.2[4] *Let G be a graph. Then H is an overfull subgraph of G if and only if H is an induced subgraph of G satisfying*

$$|E(H)| \ge \lfloor \frac{|V(H)|}{2} \rfloor \Delta(G) + 1.$$

Lemma2.3[4] *Let G be an overfull graph. Then each vertex of G is adjacent to at least two vertices of maximum degree.*

Now we give a new lemma.

Lemma 2.4 *Let G be a graph. Then G contains an induced overfull subgraph if and only if $\chi'_f(G) > \Delta(G)$.*

Proof. Let H_0 be the induced overfull subgraph in G. By Lemma 2.1 and 2.2, $|V(H_0)|$ is odd and

$$|E(H_0)| \ge \lfloor \frac{|V(H_0)|}{2} \rfloor \Delta(G) + 1,$$

That is

$$\frac{2|E(H_0)|}{|V(H_0)| - 1} > \Delta(G),$$

By Theorem A,

$$\chi'_f(G) = \max_H \frac{2|E(H)|}{|V(H)| - 1} \ge \frac{2|E(H_0)|}{|V(H_0)| - 1} > \Delta(G).$$

This proves the Necessity.

Conversely, since $\chi'_f(G) > \Delta(G)$, by Theorem A, there must be an induced subgraph H such that $|V(H)| (\ge 3)$ is odd and

$$\chi'_f(G) = \frac{2|E(H)|}{|V(H)| - 1} > \Delta(G).$$

That is

$$2|E(H)| > (|V(H)| - 1)\Delta(G).$$

Since two sides of the above inequality are even, we have

$$2|E(H)| \ge (|V(H)| - 1)\Delta(G) + 2,$$

that is

$$|E(H)| \geq \frac{|V(H)|-1}{2}\Delta(G)+1.$$

Since $|V(H)|$ is odd, it follows that

$$|E(H)| \geq \lfloor\frac{|V(H)|}{2}\rfloor\Delta(G)+1.$$

By Lemma 2.2, H is overfull in G. The proof is completed.

3 Our Main Results

In this section we prove Theorem 1 and Theorem 2.

Theorem 1 *Let G be a graph with maximum degree $\Delta(G) = 3$. Then $\chi'_f(G) > \Delta(G)$ if and only if G contains an induced subgraph H which has exactly one vertex of degree 2 and the remaining vertices of H have degree 3 in H.*

Proof. Sufficiency. Since the induced subgraph H of G has exactly one vertex of degree 2 and other vertices are of maximum degree in H, by Lemma 2.1, H is the induced overfull subgraph of G, and by Lemma 2.4, $\chi'_f(G) > \Delta(G)$.

Necessity. Since $\chi'_f(G) > \Delta(G)$, by Lemma 2.4, G must contain an induced overfull subgraph H, and

$$2|E(H)| \geq (|V(H)|-1)\Delta(G)+2.$$

On the other hand, we have known that

$$2|E(H)| \leq |V(H)|\Delta(G)-1,$$

and $\Delta(G) = 3$. So it follows that

$$2|E(H)| = |V(H)|\Delta(G)-1.$$

Hence, H must have exactly one vertex degree 2 and the remaining vertices of H have degree 3. The proof of Theorem 1 is completed.

From Theorem 1, we obtain the following corollary which is an exercise of [5].

Corollary1[5] *Let G be a simple 2-edge-connected 3-regular planar graph. Then $\chi'_f(G) = 3$.*

Theorem 2 *Let G be a graph with maximum degree $\Delta(G)$. If $\Delta(G) > \frac{2}{3}(|V(G)| - 3)$ and $\delta(G_\Delta) \leq 1$. Then $\chi'_f(G) = \Delta(G)$.*

Proof. Suppose that $\chi'_f(G) > \Delta(G)$. We shall obtain a contradiction. Let vertex $x \in V(G_\Delta)$ and $d_{G_\Delta}(x) \leq 1$. Since $\chi'_f(G) > \Delta(G)$, by Lemma 2.3 and 2.4, G must contain an induced overfull subgraph H and $x \notin V(H)$. Then at most one vertex in $N_G(x) \bigcap V(H)$ has maximum degree and for any other vertex $y \in N_G(x) \bigcap V(H)$, there must be $d_H(y) \leq \Delta(G) - 2$. Since subgraph H is overfull, by Lemma 2.1 and 2.2,

$$\mathrm{def}(H) \leq \Delta(H) - 2 = \Delta(G) - 2.$$

We consider two case.

Case 1. $\Delta(G)$ is even. Then there are at most $\frac{1}{2}(\Delta(G) - 2)$ edges joining x to $V(H)$. Since $d_G(x) = \Delta(G)$, there are at least $\frac{1}{2}\Delta(G) + 1$ vertices of $V(G) \setminus V(H)$ which are adjacent to x. Hence

$$\begin{aligned} |V(G)| &\geq 1 + |V(H)| + \frac{1}{2}\Delta(G) + 1 \\ &\geq 1 + \Delta(H) + 1 + \frac{1}{2}\Delta(G) + 1 \\ &= \Delta(G) + \frac{1}{2}\Delta(G) + 3 \\ &= \frac{3}{2}\Delta(G) + 3. \end{aligned}$$

This contradicts that $\Delta(G) > \frac{2}{3}(|V(G)| - 3)$.

Case 2. $\Delta(G)$ is odd. Similarly, there are at most $\frac{1}{2}(\Delta(G) - 1)$ edges joining x to $V(H)$. Then there are at least $\frac{1}{2}(\Delta(G) + 1)$ vertices of $V(G) \setminus V(H)$ which are adjacent to x. Hence

$$|V(G)| \geq 1 + |V(H)| + \frac{1}{2}(\Delta(G) + 1).$$

Since H is overfull, $|V(H)|$ is odd and $|V(H)| \geq \Delta(H) + 2$. Therefore

$$\begin{aligned} |V(G)| &\geq 1 + \Delta(H) + 2 + \frac{1}{2}(\Delta(G) + 1) \\ &= \Delta(G) + \frac{1}{2}\Delta(G) + \frac{7}{2} \\ &> \frac{3}{2}\Delta(G) + 3, \end{aligned}$$

a contradiction too. Theorem 2 is proved.

Remark. Theorem 2 is the best possible in the following sense. The condition that $\Delta(G) > \frac{2}{3}(|V(G)| - 3)$ can not be replaced by $\Delta(G) = \frac{2}{3}(|V(G)| - 3)$. For example, Let H_1 be a graph obtained from K_9 by deleting 3 edges which constructs a triangle. Then H_1 has 3 vertices of degree 6 and 6 vertices of degree 8. Let $H_2 = K_6$ be a complete graph with 6 vertices and $V(H_1) \bigcap V(H_2) = \phi$. Construct graph G by joining a vertex x of H_2 to each vertex of degree 6 in H_1. Then $\Delta(G) = \Delta(H_1) = 8$ and x is a vertex with maximum degree in G. Thus x is an isolated vertex of G_Δ. Clearly $\delta(G_\Delta) \leq 1$ and $\Delta(G) = \frac{2}{3}(|V(G)| - 3)$. It is easy to see that $|E(H_1)| \geq \lfloor \frac{|V(H_1)|}{2} \rfloor \Delta(G) + 1$, that is, H_1 is overfull in G. By lemma 2.4, $\chi'_f(G) > \Delta(G)$. The result in Theorem 2 is not true. So we have known that the condition that $\Delta(G) > \frac{2}{3}(|V(G)| - 3)$ is sharp.

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