# On a well-spread halving of regular multigraphs

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**Abstract**

Let $G$ be a connected multigraph with an even number of edges and suppose that the degree of each vertex of $G$ is even. Let $\mu(uv, G)$ denote the multiplicity of edge $(u, v)$ in $G$. It is well known that we can obtain a *halving* of $G$ into two *halves* $G_1$ and $G_2$, i.e. that $G$ can be decomposed into multigraphs $G_1$ and $G_2$, where for each vertex $v$, $\deg(v, G_1) = \deg(v, G_2) = \frac{1}{2}\deg(v, G)$. It is also easy to see that if the edges with odd multiplicity in $G$ induce no components with an odd number of edges then we can obtain such a halving of $G$ into two halves $G_1$ and $G_2$ that is *well-spread*, i.e. for each edge $(u, v)$ of $G$, $|\mu(uv, G_1) - \mu(uv, G_2)| \leq 1$. We show that if $G$ is a $\Delta$-regular multigraph with an even number of vertices and with $\Delta$ being even, then even if the edges with odd multiplicity in $G$ induce components with an odd number of edges, we can still obtain a well-spread halving of $G$ provided that we allow the addition/removal of a Hamilton cycle to/from $G$. We give an application of this result to obtaining sports schedules such that multiple encounters between teams are well-spread throughout the season.

# 1 Introduction

Let $G$ be a multigraph with vertex set $V(G)$ and edge set $E(G)$. We denote the degree of vertex $v \in V(G)$ by $\deg(v, G)$ and the maximum degree of $G$ by $\Delta(G)$. The *multiplicity* of an edge $(u, v) \in E(G)$ is defined to be the number of edges joining $u$ and $v$ and is denoted by $\mu(uv, G)$ and the maximum edge-multiplicity of $G$ is denoted by $\mu(G)$ respectively. $G$ is said to be *simple* if $\mu(G) = 1$. The *simple graph underlying* $G$ is defined to be the simple graph obtained from $G$ by replacing all multiple edges in $G$ by single edges and is denoted by $\mathrm{simp}(G)$.

We denote by $G_{\mathrm{odd}}$ the multigraph induced by the edges of odd multiplicity in $G$. We will say that multigraph $G$ is an *even degree* multigraph if $\deg(v, G)$ is even for all $v \in V(G)$. It is well known that every even degree multigraph $G$ that is connected has an Euler tour. We refer the reader to ([2,4]) for all terminology and notation that is not defined in this paper.

Let $G$ be an even degree multigraph. A decomposition of $G$ into multigraphs $G_1$ and $G_2$ is said to be a *halving* of $G$ into two *halves* $G_1$ and $G_2$, if for each vertex $v$, $\deg(v, G_1) = \deg(v, G_2) = \frac{1}{2}\deg(v, G)$. It is an easy exercise to show that an even degree multigraph has a halving if and only if every component of $G$ has an even number of edges. One need only place alternate edges in an Euler tour of each component of $G$ into halves $G_1$ and $G_2$ respectively. Many results are known about halvings of multigraphs. See for instance [1] where it was shown that a multigraph can be decomposed into any given number of spanning subgraphs, each with almost the same degree sequence and almost the same number of edges. However, none of these results pay any attention to how the multiple edges of the multigraph split among the spanning subgraphs in the decomposition. A halving of an even degree multigraph $G$ into halves $G_1$ and $G_2$ is said to be *well-spread* if in addition to the condition that $\deg(v, G_1) = \deg(v, G_2) = \frac{1}{2}\deg(v, G)$ for all $v \in V(G)$, we also have that for each edge $(u, v) \in E(G)$, $|\mu(uv, G_1) - \mu(uv, G_2)| \leq 1$. The following theorem gives a simple necessary and sufficient condition for the existence of a well-spread halving of an even degree multigraph.

**Theorem 1** *Let $G$ be an even degree multigraph and let $G_{\mathrm{odd}}$ be the multigraph induced in $G$ by the edges of odd multiplicity in $G$. There exists a well-spread halving of $G$ if and only if simp($G_{\mathrm{odd}}$) has no components with an odd number of edges.*

**Proof.** In a well-spread halving of $G$ into halves $G_1$ and $G_2$, for each edge $(u, v) \in E(G)$, at least $\lfloor \frac{\mu(uv,G)}{2} \rfloor$ edges between $u$ and $v$ must be included in each of $G_1$ and $G_2$. The graph obtained from $G$ by removing exactly $2\lfloor \frac{\mu(uv,G)}{2} \rfloor$ edges between $u$ and $v$ for each edge $(u, v) \in E(G)$ is precisely the simple graph simp($G_{\text{odd}}$). Now since $G$ is an even degree multigraph, so is simp($G_{\text{odd}}$), and we have that there exists a well-spread halving if and only if simp($G_{\text{odd}}$) has no components with an odd number of edges. ■

If $G$ is an even degree multigraph and if simp($G_{\text{odd}}$) has some component with an odd number of edges then by Theorem 1, there does not exist a well-spread halving of $G$. However, it was shown in [5] that if $G$ is a $\Delta$-regular multigraph on $n$ vertices with $n$ and $\Delta$ being even and $\mu(G) = 2$ then generally there exists a well-spread halving of $G$ if we allow the addition/removal of a Hamilton cycle to/from $G$. We denote by $K_n^{(r)}$ the complete multigraph on $n$ vertices with $r$ parallel edges between each pair of vertices.

**Theorem 2 (Plantholt and Tipnis [5])** *Let $G$ be a $\Delta$-regular multigraph on $n$ vertices with $n$ and $\Delta$ being even and with maximum multiplicity $\mu(G) = 2$.*

*(i) If $\Delta \geq n + 2$, then $G$ contains a Hamilton cycle $H$ such that $G - E(H) = H_1 \bigcup H_2$, where $H_1$ and $H_2$ are edge-disjoint $(\frac{\Delta}{2} - 1)$- regular simple graphs.*

*(ii) If $\Delta \leq n - 4$, then the complement of $G$ relative to $K_n^{(2)}$ contains a Hamilton cycle $H$ such that $G \bigcup E(H) = H_1 \bigcup H_2$, where $H_1$ and $H_2$ are edge-disjoint $(\frac{\Delta}{2} + 1)$- regular simple graphs.*

We note that since $\mu(G) = 2$ the decomposition of $G - E(H)$ (respectively $G \bigcup E(H)$) in (i) (respectively (ii)) of Theorem 2 gives a well-spread halving of $G - E(H)$ (respectively $G \bigcup E(H)$) into halves $H_1$ and $H_2$. In this paper we extend Theorem 2 to a similar theorem for regular, even degree multigraphs $G$ with maximum multiplicity $\mu(G) \leq \rho$, where $\rho$ is any even integer.

**Theorem 3** *Let $G$ be a $\Delta$-regular multigraph of even order $n$ and maximum multiplicity $\mu(G) \leq \rho$ with $\rho$ and $\Delta$ being even.*

*(i) If $\Delta \geq \rho(\frac{n}{2} + 1)$, then $G$ contains a Hamilton cycle $H$ such that there exists a well-spread halving of $G - E(H)$.*

*(ii) If $\Delta \leq \rho(\frac{n}{2} - 2)$, then the complement of $G$ relative to $K_n^{(\rho)}$ contains a Hamilton cycle $H$ such that there exists a well-spread halving of $G \bigcup E(H)$.*

Our motivation behind studying well-spread halvings of multigraphs is illustrated by the following example. Suppose that we have 8 teams that are to play a total of 22 games each during the season and suppose that any pair of teams play each other at most 4 times during the season. The number of games to be played between each pair of teams is given by the matrix $M(G)$ in Figure 1. We would like to schedule these games such that multiple encounters between teams are 'well-spread' during the season. The situation is modeled by a 22-regular multigraph $G$ on 8 vertices with incidence matrix $M(G)$. Notice that simp($G_{\text{odd}}$) has two components with an odd number of edges and hence by Theorem 1 there does not exist a well-spread halving of $G$. However, we can apply Theorem 3 to find a Hamilton cycle $H$ such that $G - E(H)$ has a well-spread halving into halves $G_1$ and $G_2$. This divides the games into three parts: those corresponding to edges in $G_1$, in $H$, and in $G_2$. The incidence matrices $M(G_1)$, $M(G_2)$ and $M(H)$ of $G_1$, $G_2$ and $H$ respectively are given in Figure 1. Notice that each of $G_1$ and $G_2$ is 10-regular and the edges of odd multiplicity in $G_1$ and $G_2$ do not induce any components with an odd number of edges. Hence Theorem 1 implies that $G_1$ has a well-spread halving into halves $P_1$ and $P_2$ and $G_2$ has a well-spread halving into halves $Q_1$ and $Q_2$. Thus, we can achieve a scheduling of the games into five parts with multiple games between teams well-spread during the season. The five parts are those corresponding to edges in $P_1$, $P_2$, $H$, $Q_1$ and $Q_2$. Note that in general, the halves $G_1$ and $G_2$ given by Theorem 3 in turn may not have a well-spread halving and that even when Theorem 3 can be applied to $G_1$ and $G_2$, iterated application of Theorem 3 will result in the addition and deletion of several Hamilton cycles.

The National Hockey League (NHL) schedule for the 2003-2004 season is an example of a schedule for which a variety of multiplicities occur in the multigraph corresponding to the schedule. The set of edge multiplicities in the multigraph for this NHL schedule is {1,2,4,6}. In the context of scheduling games between teams, we note that since $n$ is even in Theorem 3, a Hamilton cycle in $G$ consists of two perfect matchings. Part (i) of Theorem 3 thus divides the schedule into two well-spread halves and games scheduled in two other time periods. Part (ii) of Theorem 3 can be viewed as dividing the schedule into two well-spread halves with the Hamilton cycle being added corresponding to two 'bye' time periods. The 1993 National Football League schedule was an example of a situation where each team had two 'bye' weeks over an 18-week schedule.

# 2 Proof of Theorem 3

We will need the following Theorem 4 and Lemma 1 in order to prove Theorem 3. Theorem 4 is a result by Chvátal [3] that gives a sufficient condition for the existence of a Hamilton cycle in a simple graph $G$ and Lemma 1

$$M(G)=\begin{pmatrix}0&1&3&2&4&4&4&4\\1&0&1&4&4&4&4&4\\3&1&0&4&4&2&4&4\\2&4&4&0&3&4&2&3\\4&4&4&3&0&3&2&2\\4&4&2&4&3&0&3&2\\4&4&4&2&2&3&0&3\\4&4&4&3&2&2&3&0\end{pmatrix},M(H)=\begin{pmatrix}0&0&0&0&0&0&1&1\\0&0&1&0&0&0&0&1\\0&1&0&1&0&0&0&0\\0&0&1&0&0&1&0&0\\0&0&0&0&0&1&1&0\\0&0&0&1&1&0&0&0\\1&0&0&0&1&0&0&0\\1&1&0&0&0&0&0&0\end{pmatrix}$$

$$M(G_1)=\begin{pmatrix}0&0&2&1&2&2&2&1\\0&0&0&2&2&2&2&2\\2&0&0&1&2&1&2&2\\1&2&1&0&2&2&1&1\\2&2&2&2&0&1&0&1\\2&2&1&2&1&0&1&1\\2&2&2&1&0&1&0&2\\1&2&2&1&1&1&2&0\end{pmatrix},M(G_2)=\begin{pmatrix}0&1&1&1&2&2&1&2\\1&0&0&2&2&2&2&1\\1&0&0&2&2&1&2&2\\1&2&2&0&1&1&1&2\\2&2&2&1&0&1&1&1\\2&2&1&1&1&0&2&1\\1&2&2&1&1&2&0&1\\2&1&2&2&1&1&1&0\end{pmatrix}$$

Figure 1: A well-spread sports schedule.

guarantees a spanning tree $T$ in a connected, even degree multigraph $G$ such that the degrees of the vertices in $T$ are small relative to the degrees of the vertices in $G$. Although Lemma 1 appeared in [5], we include its proof here for the sake of completeness.

**Theorem 4 (Chvátal [3])** *Let $G$ be a graph on $n$ vertices with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If $r < \frac{n}{2}$ implies that $d_r > r$ or $d_{n-r} \geq n-r$, then $G$ contains a hamilton cycle.*

**Lemma 1 (Plantholt and Tipnis [5])** *Every connected, even degree multigraph $G$ contains a spanning tree $T$ such that $1 \leq deg(v,T) \leq 1+\frac{1}{2}\deg(v,G)$ for each $v \in V(G)$.*

**Proof.** Since $G$ is an even degree multigraph and is connected, $G$ contains an Euler tour. We consider the edges in an Euler tour in $G$ for inclusion in $T$ in the order that they are encountered in the Euler tour and we include an edge $e$ in $T$ if and only if $e$ is incident with a vertex that is previously unvisited in the Euler tour. The result now follows because in any Euler tour in $G$, each vertex $v \in V(G)$ is revisited at least $(\frac{1}{2}\deg(v,G)-1)$ times after the first time that the Euler tour visits vertex $v$. ■

We are now ready to prove Theorem 3 in the Introduction.

**Proof.** We prove only part (i) of the theorem and point out that part (ii) follows from part (i) by considering the complement of $G$ with respect to $K_n^{(\rho)}$. Let $G$ be a $\Delta$-regular multigraph of even order $n$ and maximum multiplicity $\mu(G) \leq \rho$ with $\rho$ and $\Delta$ being even, and suppose that $\Delta \geq \rho(\frac{n}{2}+1)$. We will show that $G$ contains a Hamilton cycle $H$ such that there exists a well-spread halving of $G - E(H)$ into halves denoted by $G_1$ and $G_2$. For each edge $(u,v) \in E(G)$, we begin by including $\lfloor\frac{(\mu(uv,G)-1)}{2}\rfloor$ parallel edges between vertices $u$ and $v$ in each of $G_1$ and $G_2$. Denote by $S$ the spanning multigraph of $G$ that remains after removing these two sets of $\lfloor\frac{(\mu(uv,G)-1)}{2}\rfloor$ parallel edges from $G$ for each edge $(u,v) \in E(G)$. Clearly, $\mu(S) \leq 2$ and we have that

$$\mu(uv,S) = \begin{cases} 0 & \text{if } \mu(uv,G) = 0 \\ 1 & \text{if } \mu(uv,G) \text{ is odd} \\ 2 & \text{if } \mu(uv,G) \text{ is even and non-zero.} \end{cases}$$

Note that $\mathrm{simp}(S) = \mathrm{simp}(G)$ and that since $G$ is an even degree multigraph and in constructing $S$ from $G$ we removed an even number of edges incident on each vertex $v \in V(G)$, we have that $S$ is an even degree multigraph. To complete the proof we now show that $S$ contains a Hamilton cycle $H$ such that there exists a well-spread halving of $S - E(H)$ into (simple) halves $H_1$ and $H_2$.

Note that $\mathrm{simp}(G_{\mathrm{odd}}) = S_{\mathrm{odd}}$, and $S_{\mathrm{odd}}$ is an even degree multigraph because $G$ is an even degree multigraph. Hence, by Lemma 1 there exists a spanning forest $F$ in $S_{\mathrm{odd}}$ such that there is a maximal spanning tree in $F$ for each non-trivial component of $S_{\mathrm{odd}}$, and $1 \leq \deg(v,F) \leq 1 + \frac{1}{2}\deg(v, S_{\mathrm{odd}})$ for each $v \in V(S_{\mathrm{odd}})$. We claim that $S - E(F)$ contains a Hamilton cycle.

**Claim:** $S - E(F)$ contains a Hamilton cycle.
**Proof of Claim:** We will prove that $S^* = \mathrm{simp}(S - E(F))$ contains a Hamilton cycle. Note that since $\Delta(G) \geq \rho(\frac{n}{2}+1)$, we have that $\deg(v, \mathrm{simp}(G)) \geq \frac{n}{2} + 1$ for each $v \in V(G)$ and that we have strict inequality in this relationship if vertex $v$ has at least one edge of multiplicity less than $\rho$ incident on it in $G$. Hence, $\deg(v, \mathrm{simp}(S)) \geq \frac{n}{2} + 1$ for each $v \in V(G)$ and we have strict inequality in this relationship if vertex $v$ has at least one edge of multiplicity less than $\rho$ incident on it in $G$. Hence for each $v \in V(F)$ we have that $\deg(v, \mathrm{simp}(S)) \geq \frac{n}{2} + 2$.

Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of $S^*$ and suppose for contradiction that $S^*$ does not contain a Hamilton cycle. Then by Theorem 4 there exists a positive integer $r < \frac{n}{2}$ such that $d_r \leq r$. Let $V_r \subseteq V(F)$ denote the set of these $r$ vertices that have degree at most $r$ in $S^*$. For each $v \in V_r$ we have that

$$\begin{aligned}\deg(v,S^*) &= \deg(v,\mathrm{simp}(S)) - \deg(v,F)\\ &\geq \deg(v,\mathrm{simp}(S)) - (1+\frac{1}{2}\deg(v,S_{\mathrm{odd}}))\\ &\geq \deg(v,\mathrm{simp}(S)) - (1+\frac{1}{2}\deg(v,\mathrm{simp}(S))\\ &\geq \frac{1}{2}(\frac{n}{2}+2)-1=\frac{n}{4}.\end{aligned}$$

Hence we have that $\frac{n}{4} \leq d_r \leq r < \frac{n}{2}$. Let $r = \frac{n}{2} - k$ where $k$ is some integer satisfying $1 \leq k \leq \frac{n}{4}$ and let $|V(F)| = p$. Let $s = \sum_{v\in V(G)} \deg(v,\mathrm{simp}(S))$ and $s^* = \sum_{v\in V(G)} \deg(v,S^*)$. If $v \notin V(F)$ then $\deg(v,\mathrm{simp}(S)) = \deg(v,S^*)$. Thus, $\sum_{v\in V_G} \deg(v,\mathrm{simp}(S)) - \sum_{v\in V_G} \deg(v,S^*) = 2|E(F)| \leq 2(p-1)$. For each $v \in V(F)$, $\deg(v,\mathrm{simp}(S)) - \deg(v,S^*) \geq 1$ and for each $v \in V_r$, $\deg(v,\mathrm{simp}(S)) - \deg(v,S^*) \geq (\frac{n}{2}+2) - r = k+2$. Hence we have that $s - s^* \geq p + r(k+1) = p + (\frac{n}{2} - k)(k+1)$, i.e. $s - s^* \geq f(k)$, where $f(k)$ is the quadratic function given by $f(k) = p + (\frac{n}{2} - k)(k+1)$ for $1 \leq k \leq \frac{n}{4}$. To summarize, we have that $2(p-1) \geq s - s^* \geq f(k)$, for $1 \leq k \leq \frac{n}{4}$. Note that $f(k)$ achieves its maximum value at $k = \frac{n}{4} - \frac{1}{2}$. We address the possible values of $k$ in the following three cases and in each case obtain a contradiction to the assumption that $S^*$ does not contain a Hamilton cycle.

(i) $2 \leq k \leq \frac{n}{4} - \frac{1}{2}$: $f(k)$ is an increasing function on the interval $[1, \frac{n}{4} - \frac{1}{2})$ and hence for each $k$ satisfying $2 \leq k \leq \frac{n}{4} - \frac{1}{2}$ we have that $f(k) > f(1) = p + (n-2) \geq 2(p-1)$, a contradiction.

(ii) $k = 1$: In this case, $2(p-1) \geq s - s^* \geq f(1) = p + (n-2) \geq 2(p-1)$ and hence we must have $p = n$ and $s - s^* = f(1) = 2(n-1)$. Thus, since for each vertex $v$, $\deg(v,\mathrm{simp}(S)) \geq \frac{n}{2}$ and $\deg(v,S^*) \leq \frac{n}{2}+1$, it is easy to verify that this implies that in $S^*$ we must have precisely $(\frac{n}{2}-1)$ vertices of degree $(\frac{n}{2}-1)$ each and precisely $(\frac{n}{2}+1)$ vertices of degree at least $(\frac{n}{2}+1)$ each. So, $i < \frac{n}{2}$ implies that $d_{n-i} \geq (n-i)$ and Theorem 4 implies that $S^*$ contains a Hamilton cycle, a contradiction.

(iii) $k = \frac{n}{4}$: In this case, $f(k) = p + \frac{n}{4}(\frac{n}{4}+1)$. Now, for each $n > 8$, it is true that $\frac{n}{4}(\frac{n}{4}+1) > (n-2)$ and hence for each $n > 8$, $f(k) > p + (n-2) \geq 2(p-1)$, a contradiction. Since $k$ is an integer, this implies that we must have $n = 4$ or $n = 8$. If $n = 4$ then $k = 1$ which in turn is impossible by case (ii) above, and so we must have that $n = 8$ and $k = 2$. Now, for $v \in V(G)$, if $\deg(v,S^*) = 2$ then $\deg(v,\mathrm{simp}(S)) \geq 6$, and hence it follows that $S^*$ has exactly two vertices $u$ and $v$ such that $\deg(u,S^*) = \deg(v,S^*) = 2$, and that $F$ is the unique tree on 8 vertices that has exactly two vertices of degree

4 each, and exactly 6 vertices of degree one each. Thus it is clear that $S^*$ has six vertices, each of whose degree is at least 5, and two vertices $u$ and $v$ of degree two each, with $u$ and $v$ having no common neighbors. Given this form for $S^*$ it is easy to check that $S^*$ must have a Hamilton cycle, a contradiction.

This ends the proof of the claim.

Let $H$ be a Hamilton cycle in $S - E(F)$ and denote by $X$ the spanning subgraph of $S$ consisting of the singleton edges of $S - E(H)$. $X$ is connected because any edge of $H$ that joins two vertices that are in the same connected component of $G_{\text{odd}}$ has multiplicity two in $S$. In addition, $X$ is an even degree simple graph and has an even number of edges because $G$ is an even degree multigraph, $|E(G)|$ is even and $X$ is obtained from $G$ by deleting edges incident at each vertex in pairs. A well-spread halving of $S - E(H)$ into (simple) halves $H_1$ and $H_2$ is now easily obtained by including one edge from each pair of two parallel edges in $S - E(H)$ in each of $H_1$ and $H_2$ and by including alternate edges in an Euler tour of $X$ in each of $H_1$ and $H_2$. To summarize, we have constructed a well-spread halving of $G - E(H)$ into halves $G_1$ and $G_2$, where $H$ is a Hamilton cycle in $S - E(F)$, and for $i = 1, 2$, $G_i$ consists of the edges in $H_i$ together with the $\lfloor\frac{(\mu(uv,G)-1)}{2}\rfloor$ parallel edges of $G$ included in $G_i$ for each pair of vertices $u, v \in V(G)$ at the beginning of the proof. ■

# 3 Conclusions

Part (i) of Theorem 3 in this paper gives a decomposition of a regular multigraph $G$ of even order and degree that is high relative to the order and maximum multiplicity of $G$ into a Hamilton cycle and two halves with the multiple edges of $G$ being well-spread between the two halves. If $G$ is a regular multigraph of even order and degree that is low relative to the order and maximum multiplicity of $G$, part (ii) of Theorem 3 gives a decomposition of $G \bigcup E(H)$, where $H$ is a particular Hamilton cycle that is not in $G$, into two halves with the multiple edges of $G$ being well-spread between the two halves. We point out two problems that require further work. First, the case when $G$ is a $\Delta$-regular multigraph of even order $n$ and maximum multiplicity $\mu(G) \leq \rho$ with $\rho(\frac{n}{2} - 2) < \Delta < \rho(\frac{n}{2} + 1)$ is not addressed by Theorem 3. We expect that the use of a theorem that gives a sufficient condition that is weaker than that of Theorem 4 for the existence of a Hamilton cycle might be able to address this gap between parts (i) and (ii) of Theorem 3. Secondly, if $\rho$ is odd we can surely obtain a decomposition as in part (i) of Theorem 3 if we assume that $\Delta \geq (\rho + 1)(\frac{n}{2} + 1)$ and similarly a decomposition as in part (ii) of Theorem 3 if we assume that

$\Delta \leq (\rho+1)(\frac{n}{2}+1)$. However, this is too restrictive and we conjecture that Theorem 3 is also true if $\rho$ is odd.

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