Counting pan-fan maps on nonorientable surfaces

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Abstract In this paper, we obtain a general enumerating functional equation about rooted pan-fan maps on nonorientable surfaces. Basen on this equation, an explicit expression of rooted pan-fan maps on the Klein bottle is given. Meanwhile, some simple explicit expressions with up to two parameters: the valency of root face and the size for rooted one-vertexed maps on surfaces (Klein bottle, Tours, N_3) are provided.
Keywords: rooted map, enumerating function, enumeration, Lagrangian inversion, pan-fan maps

1 Introduction

A map is *rooted* if a vertex and an edge together with a direction along the edge and a face on one side of the edge are all distinguished. An edge belonging to only one face is called *double* (or singular by some authors), all others are called *single.* Two rooted maps on a surface Σ are considered the same if there is a self-homeomorphism of Σ which induces an isomorphism between them preserving the rooting. The surfaces considered here are compact 2-manifolds. Surfaces with *p handles*, i.e., O_p (*q crosscaps*, i.e., N_q) are called *orientable* (*nonorientable*) of genus p (q). Concerning other definitions or denotations of a map on a surface, reader can refer to the literature [6] or [7]. An *one-vertexed map* is a map with only one vertex. Apparently, the dual maps of all one-vertexed maps on the surface of genus g are the *g-essential* maps which was mentioned in lit.[5]. A rooted *pan-fan map* (including loop and mult-edge) on a surface is one such that it is a plane tree by deleting the root-vertex with its incident edges.

We continue to use the notations of lit.[6]. For any rooted pan-fan map M (the number of vertices $v(M) \geq 3$), $v_r(M) = (r, \mathcal{J}r, \mathcal{J}^2r, \cdots, \mathcal{J}^{k-1}r)$ (k is the valency of the root-vertex) and Kr denote the edge incident with

*Supported by NNSFC under Grant No. 60373030 and Grant No. 10571031

r. Suppose $K\mathcal{J}^i r = Kx$ ($i = 0, ...k-1$) is the first edge which connect v_r to $v_1 \neq v_r$, $v_1 = (\delta x, \mathcal{J}\delta x, \mathcal{J}^2\delta x, \cdots, \mathcal{J}^{k_1-1}\delta x)$ (k_1 is the valency of v_1, $\delta = \alpha\beta$). Similarly, since $v(M) \geq 3$, the first edge $K\mathcal{J}^j\delta x$ ($j = 1, ...k_1 - 1$) which connects v_1 to $v_2 \neq v_1 \neq v_r$ must exist. So the plane tree $M' = M - \{v_r(M)\}$ can be rooted as follows: $\tilde{r}(M') = \mathcal{J}^j\delta x$.

The enumeration of rooted planar maps was first noticed by W.T.Tutte[9] in 1963. Since then, much work has been done by W.T.Tutte himself, Arquès, Brown, Mullin, Harary, E.A.Bender, Y.P.Liu *et al.* in a series of papers. They investigated nearly all sets of rooted planar maps for enumerating them. Especially, in the early 80's, the appearance of enumerating equations for planar maps simplified the processes and the results of enumerating planar maps greatly. However, any kind of exact enumerations of rooted maps on nonplanar surface is quite difficult. Arquès, Brown[3], Walsh[11], Lehman, Bender[2] *et al.* and Gao[4] *et al.* did some influential work in this field. Among them, Bender *et al.* and Gao studied several classes of rooted maps on general surfaces and got asymptotic evaluations of nonplanar maps mainly by an asymptotic method. Now many people apply himself to research maps on nonplanar surfaces such as Ren Han[8] (4-regular maps *etc.*) and Rongxia Hao *et al.*. In literature [12], the authors studied a new kind of rooted maps: pan-fan maps (derived from circuit boundary maps which was brought forward on the basis of Halin maps and pan-halin maps), and gave explicit expressions of planar pan-fan maps with different parameters. In literature [13], the authors investigated nonplanar surface and got explicit expressions for two kinds of maps on the projective plane: rooted one-vertexed maps and pan-fan maps.

In this article, the authors study rooted pan-fan maps further. First of all, a general equation about rooted pan-fan maps on nonorientable surfaces is given. As an example, the number of pan-fan maps with the size n on the Klein bottle is derived. For obtaining this number, by a series of operations, some simpler (than lit[13]) explicit expressions of one-vertexed maps with more parameters (the valency of root face m and the size n) than lit[1] which enumerated by numbers of vertices and faces (i.e. the size) on Klein bottle are obtained. Finally, the explicit enumerative expressions of one-vertexed maps on the Tours and N_3 with more parameters are also given.

For convenience, we introduce an operation as follows. For two maps M_1 and M_2 with their respective roots $r_i = r(M_i), i = 1, 2$, the map $M = M_1 \cup M_2$ provided $M_1 \cap M_2 = v$ such that $v = v_{r_1} = v_{r_2}$ is defined to have that its root, root-vertex and root-edge are as the same as those of M_1 while the root face is the composition of $f_r(M_i), i = 1, 2$. This operation is called 1-*addition* and write

$$M = M_1 \dot{+} M_2$$

Further, for any two sets of rooted maps $\mathcal{M}_1$ and $\mathcal{M}_2$, the set of maps

$$\mathcal{M}_1 \odot \mathcal{M}_2 = \{M_1 \dot{+} M_2 | \forall M_i \in \mathcal{M}_i, i = 1, 2\}$$

is said to be 1-*production* of $\mathcal{M}_1$ and $\mathcal{M}_2$ as shown in [6]. Especially we write

$$\mathcal{M} \odot \mathcal{M} = \mathcal{M}^{\odot^2}$$
$$\mathcal{M}^{\odot^k} = \mathcal{M}^{\odot^{k-1}} \odot \mathcal{M}$$

Let $\mathcal{H}_q(\mathcal{S}_p)$ and $\mathcal{F}_q(\mathcal{L}_p)$ be the sets of all rooted pan-fan maps and one-vertexed maps on $N_q(O_p)$ respectively. Suppose their enumerating functions are respectively:

$$h_q(x, y) = \sum_{m,n \geq 1} H^q_{m,n} x^m y^n, \quad s_p(x, y) = \sum_{m,n \geq 0} S^p_{m,n} x^m y^n;$$

$$f_q(x, y) = \sum_{m,n \geq 1} F^q_{m,n} x^m y^n, \quad l_p(x, y) = \sum_{m,n \geq 0} L^p_{m,n} x^m y^n,$$

where $p \geq 0, q \geq 1$, $m(M), n(M)$ are the valency of the root face and the size respectively.

Let $\mathcal{H}^i_q(\mathcal{S}^i_p)$ and $\mathcal{F}^i_q(\mathcal{L}^i_p)$ $(p \geq 0, q \geq 1)$ be the sets of all rooted pan-fan maps and one-vertexed maps with i $(i \geq 1)$ distinguished non-rooted faces on $N_q(O_p)$ respectively. Their enumerating functions are respectively:

Denote $\underline{z} = (z_1, z_2, \cdots, z_i)$, $\underline{k} = (k_1, k_2, \cdots, k_i)$. Then

$$h^i_q(x, y, \underline{z}) = \sum_{m,n,\underline{k} \geq 1} H^q_{m,n,\underline{k}} x^m y^n \underline{z}^{\underline{k}}, \quad s^i_p(x, y, \underline{z}) = \sum_{m,n,\underline{k} \geq 1} S^p_{m,n,\underline{k}} x^m y^n \underline{z}^{\underline{k}};$$

$$f^i_q(x, y, \underline{z}) = \sum_{m,n,\underline{k} \geq 1} F^q_{m,n,\underline{k}} x^m y^n \underline{z}^{\underline{k}}, \quad l^i_p(x, y, \underline{z}) = \sum_{m,n,\underline{k} \geq 1} L^p_{m,n,\underline{k}} x^m y^n \underline{z}^{\underline{k}},$$

where $H^q_{m,n,\underline{k}}$ $(S^p_{m,n,\underline{k}})$ and $F^q_{m,n,\underline{k}}$ $(L^p_{m,n,\underline{k}})$ count, respectively, rooted pan-fan maps and one-vertexed maps in $\mathcal{H}^i_q(\mathcal{S}^i_p)$ and $\mathcal{F}^i_q(\mathcal{L}^i_p)$ with the valency of root-face m, the size n and the valency of j-th distinguished non-rooted face k_j $(k_j \geq 1, j = 1, \cdots, i)$. What's more, $h^i_q = f^i_q = 0, q \leq 0, s^i_p = l^i_p = 0, p < 0$.

Theorem 1[10] The enumerating function of planted trees $\mathcal{T}_1$ has the following form:

$$t_{\mathcal{T}_1} = t_1 = \sum_{n \geq 1} \frac{(2n-2)!}{n!(n-1)!} y^n \tag{1.1}$$

For the enumerating function $l_0(x, y)$, the following equation is well-known.

$$(1 - x + x^2 y) l_0 = 1 - x + xy l^*_0 \tag{1.2}$$

where $l_0^* = l_0(1, y)$.

Parametric expressions can be extracted as follows:

$$\begin{cases} \eta = y(\eta+1)^2, l_0^* = \eta + 1, x = \beta(\eta+1); \\ l_0 = \dfrac{1}{1-\beta\eta}, \dfrac{\partial(xl_0)}{\partial x} = \dfrac{1}{(1-\beta\eta)^2}. \end{cases} \tag{1.3}$$

Theorem 2[12] The enumerating function s_0 satisfies the functional equation as follows:

$$(1 - x + x^2 y)s_0 = xys_0^* + (1-x)t_1 l_0 + (1-x) \tag{1.4}$$

where $s_0^* = s_0(1, y)$.

From (1.4), the following parametric equations can be found.

$$\begin{cases} \eta = y(\eta+1)^2, s_0^* = \eta + 1 + \dfrac{(\eta+1)(1-\sqrt{1-4\eta})}{2(1-\eta)}, x = \beta(\eta+1); \\ s_0 = \dfrac{[1-(\eta+1)\beta](1-\sqrt{1-4\beta^2\eta})}{2(1-\beta)(1-\beta\eta)^2} + \dfrac{\beta\eta(1-\sqrt{1-4\eta})}{2(1-\beta)(1-\eta)(1-\beta\eta)} \\ +\dfrac{1}{1-\beta\eta}. \end{cases} \tag{1.5}$$

Let

$$A(p, q, \lambda) = \sum_{i \geq 0}^{n-m-p} \frac{(i+q)!\lambda(i)}{q!i!} \binom{2n-m-1}{n-m-p-i} \tag{1.6}$$

By employing Lagrangian inversion with two parameter for (1.5), the number of rooted pan-fan maps on the sphere is: $S_{0,0}^0 = 1$

$S_{m,n}^0 = \dfrac{(2n-2)!}{n!(n-1)!}$ (when m=2n); $S_{m,n}^0 = \dfrac{2(2n-4)!}{(n-1)!(n-2)!}$ (when m=2n-1);

$$S_{m,n}^0 = \frac{m(2n-m-1)!}{(n-m)!n!} + \sum_{k\geq 0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(2k)![A(1-k,0,1) - A(k-m+2,0,1)]}{k!(k+1)!}$$

$$+ \sum_{k\geq 0}^{n-2} \frac{(2k)!A(k-m+2,0,1)}{k!(k+1)!} - \sum_{k\geq 0}^{n-m-2} \frac{(2k)!A(k+2,0,1)}{k!(k+1)!}$$

$$+ \sum_{k\geq 0}^{\lfloor \frac{m-2}{2} \rfloor} \frac{(m-2k-1)(2k)!(2n-m-1)!}{k!(k+1)!(n-m+k+1)!(n-k-2)!} \tag{1.7}$$

where the initial values are listed in the following table:

$S^0_{m,n}$	$n=0$	1	2	3	4	5	6
$m=0$	1	0	0	0	0	0	0
1	0	1	2	7	28	119	526
2	0	1	2	7	28	119	*
3	0	0	2	6	23	96	*
4	0	0	1	4	16	68	*
5	0	0	0	2	10	45	*
6	0	0	0	2	5	27	*
7	0	0	0	0	4	17	*
8	0	0	0	0	5	11	*
9	0	0	0	0	0	10	*
10	0	0	0	0	0	14	*
11	0	0	0	0	0	0	*
12	...	...	...	...	...	...	*

Theorem 3[13] The enumerating function f_1 satisfies the following equation:

$$(1-x+x^2y)f_1 = xy\left[x(1-x)\frac{\partial(xl_0)}{\partial x} + f_1^*\right] \tag{1.8}$$

where $f_1^* = f_1(1, y)$.

Parametric expressions is as follows:

$$\begin{cases} \eta = y(\eta+1)^2, f_1^* = \dfrac{\eta(\eta+1)}{(1-\eta)^2}, x = \beta(\eta+1); \\ f_1 = \dfrac{\beta\eta^2}{(1-\eta)^2(1-\beta\eta)^2} + \dfrac{\beta^2\eta}{(1-\eta)(1-\beta\eta)^3}; \\ \dfrac{\partial f_1}{\partial x} = \dfrac{\eta^2(1+\beta\eta)}{(1-\eta)^2(1-\beta\eta)^3(1+\eta)} + \dfrac{\beta\eta(2+\beta\eta)}{(1-\eta)(1-\beta\eta)^4(1+\eta)}. \end{cases} \tag{1.9}$$

Also, by employing Lagrangian inversion with two parameter for (1.9), the following two results, which are simpler than [13], are obtained.

Theorem 4 The enumerating functions $f_1(x,y)$ has the following explicit expression:

$$f_1(x,y) = \sum_{m,n\geq 1} F^1_{m,n} x^m y^n,$$

where $F^1_{2,1} = 1$; $F^1_{3,2} = 3$;

$$F^1_{m,n} = m\sum_{k\geq 0}^{n-m-1}\binom{2n-m-1}{k} + \frac{m(m-1)}{2}\binom{2n-m-1}{n-2}. \tag{1.10}$$

Corollary 1 The number of one-vertexed maps on the projective plane with the size n is:

$$2^{2n-1} - \frac{(2n)!}{2(n!)^2} \tag{1.11}$$

In this article, some main results are listed as follows:
Let

$$\gamma_q = \begin{cases} 1, & q \text{ is odd number;} \\ 0, & q \text{ is even number.} \end{cases}$$

Theorem A the enumerating function $h_q(x,y)(q \geq 1)$ about the rooted pan-fan maps on nonorientable surface satisfies the following general equation:

$$(1-x+x^2y)h_q(x,y) = xyh_q^* + (1-x)t_1 f_q + (1-x)x^2y[\frac{\partial(xh_{q-1})}{\partial x} + \gamma_q \frac{\partial(xs_{\frac{q-1}{2}})}{\partial x}]$$

$$+(1-x)x^3y[2\frac{\partial h^1_{q-2}}{\partial z_1}(x,y,x) + \gamma_{q+1}\frac{\partial s^1_{\frac{q}{2}-1}}{\partial z_1}(x,y,x)] \tag{1.12}$$

where $h_q^* = h_q(1,y)$.

Theorem B enumerating function of $f_2(x,y)$ has the following expression:

$$f_2(x,y) = \sum_{m,n\geq 1} F^2_{m,n} x^m y^n,$$

where $F^2_{3,2} = 0$; $F^2_{4,2} = 4$; $F^2_{5,3} = 20$;

$$F^2_{m,n} = \sum_{i\geq 0}^{n-m} \frac{m!(i+1)}{(m-3)!}\binom{2n-m-1}{n+i-1} + \sum_{i\geq 0}^{n-m+2} \frac{m!}{6(m-4)!}\binom{2n-m-1}{n+i-3}$$

$$+\sum_{i\geq 0}^{n-m-2} \frac{m\lambda_i(m,n)}{3i!} + \sum_{i\geq 0}^{n-m-1} \frac{m(m-1)\lambda_{i-1}(m,n)}{2i!} \tag{1.13}$$

where

$$\lambda_i(m,n) = (i+3)!\left[\binom{2n-m}{n+i+2} + \binom{2n-m-1}{n+i+1}\right]$$

Theorem C The enumerating function of $f_3(x,y)$ has the following expression:

$$f_3(x,y) = \sum_{m,n\geq 1} F^3_{m,n} x^m y^n,$$

where $F^3_{6,3} = 41$; $F^3_{7,4} = 287$;

$$F^3_{m,n} = 12mA(5,6,1) + \frac{m}{6}A(4,5,\lambda_1) + \frac{m}{30}A(3,4,\lambda_2) + \frac{m(m-1)}{10}A(2,4,\lambda_3)$$

$$+\frac{m!}{4!(m-3)!}A(1,3,\lambda_4)+\frac{62m!}{4!(m-4)!}A(0,3,1)+\frac{85m!}{5!(m-5)!}A(-1,2,1)$$
$$+\frac{41m!}{6!(m-6)!}A(-3,1,1) \qquad (1.14)$$

where

$$\begin{cases}\lambda_1(i)=75i+36m+414;\\ \lambda_2(i)=41i^2+40m^2+225mi+1005m+226i+185;\\ \lambda_3(i)=41i+115m-25;\\ \lambda_4(i)=51i+52m+48.\end{cases}$$

2 General functional equation

For $\mathcal{H}_q(q\geq 1)$, it can be separated into four categories: $\mathcal{H}_{q_{\rm I}}$, $\mathcal{H}_{q_{\rm II}}$, $\mathcal{H}_{q_{\rm III}}$ and $\mathcal{H}_{q_{\rm IV}}$, such that

- $M\in\mathcal{H}_{q_{\rm I}}, e_r(M)$ is a single edge.
- $M\in\mathcal{H}_{q_{\rm II}}, e_r(M)$ is a double edge and $M-e_r(M)$ is not a map.
- $M\in\mathcal{H}_{q_{\rm III}}, e_r(M)$ is a double edge and $M-e_r(M)$ is still nonorientable.
- $M\in\mathcal{H}_{q_{\rm IV}}, e_r(M)$ is a double edge and $M-e_r(M)$ becomes orientable.

Lemma 1 Let $\mathcal{H}_{q_{(\rm I)}}=\{M-e_r(M)\mid M\in\mathcal{H}_{q_{\rm I}}\}$, then $\mathcal{H}_{q_{(\rm I)}}=\mathcal{H}_q$.
Proof It is easy to see that $\mathcal{H}_{q_{(\rm I)}}\subseteq\mathcal{H}_q$. On the other hand, for any $M\in\mathcal{H}_q$, a map $M'\in\mathcal{H}_{q_{\rm I}}$ can be obtained by adding an edge R' connecting the root vertex $v_r(M)$ and any vertex on the root face boundary, such that $M=M'-R'$, here, $R'=e_r(M')$. By all appearances, there are $m(M)+1$ ways to get M'. This means that $M\in\mathcal{H}_{q_{(\rm I)}}$, thus, $\mathcal{H}_{q_{(\rm I)}}\supseteq\mathcal{H}_q$. Then the lemma is true. □
Lemma 2 For $\mathcal{H}_{q_{\rm II}}$, $\mathcal{H}_{q_{\rm II}}=\mathcal{T}_1\odot\mathcal{F}_q$.
Proof It can easily get this result by the definitions of $\mathcal{H}_{q_{\rm II}}$ and $\mathcal{F}_q$. □

For a nonorientable map $M\in\mathcal{H}_{q_{\rm III}}$, from the Euler characteristic of M, we have $v(M)-\varepsilon(M)+\phi(M)=2-q$ $(q\geq 1)$. It is easy to see after deleting a double edge of M ($v(M)$ is invariable)the genus of the nonorientable surface will be reduced two at most. Therefore, $\mathcal{H}_{q_{\rm III}}$ can be further divided into two parts: $\mathcal{H}_{q_{\rm III_1}}=\{M\in\mathcal{H}_{q_{\rm III}}\mid M-e_r(M)\in\mathcal{H}_{q-1}\}$ and $\mathcal{H}_{q_{\rm III_2}}=\{M\in\mathcal{H}_{q_{\rm III}}\mid M-e_r(M)\in\mathcal{H}^1_{q-2}\}$
Lemma 3 Let $\mathcal{H}_{q_{\rm III_{(1)}}}=\{M-e_r(M)|M\in\mathcal{H}_{q_{\rm III_1}}\}$, then $\mathcal{H}_{q_{\rm III_{(1)}}}=\mathcal{H}_{q-1}$.
Proof Using the same method as Lemma 1, it is easy to check that, for any $M\in\mathcal{H}_{q-1}$, $M'\in\mathcal{H}_{q_{\rm III_1}}$ can be obtained by adding a new double edge

$e_r(M')$ ($(m(M)+1)$ ways) in the root face of M. Since $M = M' - e_r(M')$, then $\mathcal{H}_{q\text{III}_{(1)}} \supseteq \mathcal{H}_{q-1}$. This prove the lemma. □

Lemma 4 Let $\mathcal{H}_{q\text{III}_{(2)}} = \{M - e_r(M) | M \in \mathcal{H}_{q\text{III}_2}\}$, then $\mathcal{H}_{q\text{III}_{(2)}} = \mathcal{H}^1_{q-2}$.

Proof For any map $M \in \mathcal{H}^1_{q-2}$, since $v(M) - \varepsilon(M) + \phi(M) = 2 - (q-2)$, we can get $M' \in \mathcal{H}_{q\text{III}_2}$ by adding a new double edge R' ($2k_1(M)$ ways) from the tail of the root edge of M to each of vertices on the boundary of the distinguished non-rooted face as the root-edge of M' so that the distinguished non-rooted face and the root face can be turn into a new face whose valency is $m(M)+k_1(M)+2$. And $v(M')-\varepsilon(M')+\phi(M') = 2-q$. Since $M = M' - R'$, $M' \in \mathcal{H}_{q\text{III}_2}$ it is evident that $M \in \mathcal{H}_{q\text{III}_{(2)}}$ and $\mathcal{H}_{q\text{III}_{(2)}} \supseteq \mathcal{H}^1_{q-2}$. Therefore, $\mathcal{H}_{q\text{III}_{(2)}} = \mathcal{H}^1_{q-2}$. □

Concerning $\mathcal{H}_{q\text{IV}}$, it includes two circumstances: $\mathcal{H}_{q\text{IV}_1} = \{M \in \mathcal{H}_{q\text{IV}} \mid q$ is odd number$\}$ and $\mathcal{H}_{q\text{IV}_2} = \{M \in \mathcal{H}_{q\text{IV}} \mid q$ is even number$\}$.

Lemma 5 Let $\mathcal{H}_{q\text{IV}_{(1)}} = \{M - e_r(M) | M \in \mathcal{H}_{q\text{IV}_1}\}$, then $\mathcal{H}_{q\text{IV}_{(1)}} = \mathcal{S}_{\frac{q-1}{2}}$.

Proof According to the Euler formula and the defination of $\mathcal{H}_{q\text{IV}_1}$, for any map $M \in \mathcal{H}_{q\text{IV}_1}$, the number of vertices and faces of $M' = M - e_r(M)$ is equal to M. Since $v(M)-\varepsilon(M)+\phi(M) = 2-q$ and $v(M')-\varepsilon(M')+\phi(M') = 2-2p$, we can get $p = \frac{q-1}{2}$ easily. Conversely, referring to Lemma 3 and using the same calculational method of the above metioned, $\mathcal{H}_{q\text{IV}_{(1)}} \supseteq \mathcal{S}_{\frac{q-1}{2}}$ holds, just taking notice that the new added edge must get across a cross cap. So This lemma follows. □

Lemma 6 Let $\mathcal{H}_{q\text{IV}_{(2)}} = \{M - e_r(M) | M \in \mathcal{H}_{q\text{IV}_2}\}$, then $\mathcal{H}_{q\text{IV}_{(2)}} = \mathcal{S}^1_{\frac{q}{2}-1}$.

Proof The Lemma follows from Lemmas 4 and 5. □

Let $h_{q\text{I}}, h_{q\text{II}}, h_{q\text{III}}$ and $h_{q\text{IV}}$ be, respectively, the enumerating function of $\mathcal{H}_{q\text{I}}$, $\mathcal{H}_{q\text{II}}$, $\mathcal{H}_{q\text{III}}$ and $\mathcal{H}_{q\text{IV}}$; $h_{q\text{III}_t}$ and $h_{q\text{IV}_t}$ be the enumerating function of $\mathcal{H}_{q\text{III}_t}$ and $\mathcal{H}_{q\text{IV}_t}$ (t=1,2) respectively, then as a direct consequence of Lemmas 1-6, the following formulas hold.

$$h_{q\text{I}} = y \sum_{M \in \mathcal{H}_{q\text{I}}} (x + x^2 + \cdots + x^{m+1}) y^n = \frac{xy}{1-x}[h^*_q - x h_q];$$

where $h^*_q = h_q(1, y)$. And,

$$h_{q\text{II}} = \sum_{M \in \mathcal{H}_{q\text{II}}} x^{m(M)} y^{n(M)} = t_1 f_q;$$

$$h_{q\text{III}_1} = y \sum_{M \in \mathcal{H}_{q-1}} (m+1) x^{m+2} y^n = x^2 y \frac{\partial (x h_{q-1})}{\partial x};$$

$$h_{q\text{III}_2} = y \sum_{M \in \mathcal{H}^1_{q-2}} 2k_1 x^{m+k_1+2} y^n = 2x^3 y \frac{\partial h^1_{q-2}}{\partial z_1}(x, y, x);$$

$$h_{q\text{IV}_1} = y \sum_{M \in S_{\frac{q-1}{2}}} (m+1) x^{m+2} y^n = x^2 y \frac{\partial (x s_{\frac{q-1}{2}})}{\partial x};$$

$$h_{q\text{IV}_2} = y \sum_{M \in S^1_{\frac{q}{2}-1}} k_1 x^{m+k_1+2} y^n = x^3 y \frac{\partial s^1_{\frac{q}{2}-1}}{\partial z_1}(x,y,x).$$

Since $h_q = h_{q\text{I}} + h_{q\text{II}} + h_{q\text{III}} + h_{q\text{IV}}$, then the Theorem A is proved by simplification.

For the set $\mathcal{H}^1_{q-2}$, it includes the following two cases when a map $M \in \mathcal{H}^1_{q-2}$ and $e_r(M)$ is a single edge. If $e_r(M)$ is a double edge, the classifications are the same as that of $\mathcal{H}_q$.

- $M \in \mathcal{H}^1_{q-2}, e_r(M)$ is single and the distinguished non-rooted face is adjacent to the root edge.
- $M \in \mathcal{H}^1_{q-2}, e_r(M)$ is single and the distinguished non-rooted face is not adjacent to the root edge.

Therefore, like the above discussion, we can get an equation satisfied by the enumerating function h^1_{q-2}. In a general way, let $\tilde{\underline{z}} = (z_1, \cdots, z_i, z_{i+1})$, $\tilde{\underline{k}} = (k_1, \cdots, k_i, k_{i+1})$, $\dot{\underline{z}} = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_i)$ and $\dot{\underline{k}} = (k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_i)$,

$$h_q^{i+1}(x,y,\tilde{\underline{z}}) = \sum_{M \in \mathcal{H}_q^{i+1}} x^{m(M)} y^{n(M)} \tilde{\underline{z}}^{\tilde{\underline{k}}(M)} = \sum_{m,n,\tilde{\underline{k}} \geq 1} H^q_{m,n,\tilde{\underline{k}}} x^m y^n \tilde{\underline{z}}^{\tilde{\underline{k}}}$$

$$h_q^{i-1}(x,y,\dot{\underline{z}}) = \sum_{M \in \mathcal{H}_q^{i-1}} x^{m(M)} y^{n(M)} \dot{\underline{z}}^{\dot{\underline{k}}(M)} = \sum_{m,n,\dot{\underline{k}} \geq 1} H^q_{m,n,\dot{\underline{k}}} x^m y^n \dot{\underline{z}}^{\dot{\underline{k}}}$$

the following theorem can be obtained.

Theorem 5 The enumerating function $h^i_q(x) = h^i_q(x,y,\underline{z})(q \geq 1)$, satisfies the following equation:

$$(1-x+x^2y)h^i_q(x) = xyh_q^{i^*} + (1-x)t_1 f^i_q(x) + (1-x)\sum_{j=1}^{i} xyz_j \delta_{z_j,x}(uh_q^{i-1}(u))$$

$$+(1-x)x^2y[(xh^i_{q-1}(x))' + \gamma_q(xs^i_{\frac{q-1}{2}}(x))']$$

$$+(1-x)x^3y[2\frac{\partial h^{i+1}_{q-2}}{\partial z_{i+1}}(x,x) + \gamma_{q+1}\frac{\partial s^{i+1}_{\frac{q}{2}-1}}{\partial z_{i+1}}(x,x)] \qquad (2.1)$$

where $h_q^{i^*} = h^i_q(1)$, $\delta_{z_j,x}(uh_q^{i-1}(u)) = \dfrac{z_j h_q^{i-1}(z_j,y,\dot{\underline{z}}) - xh_q^{i-1}(x,y,\dot{\underline{z}})}{z_j - x}$.

As for the circumstance of orientable surface, $\mathcal{S}_p$ can be studied similar to $\mathcal{H}_q$. Especially, it is simpler. Here, we no longer discuss it in detail.

3 Pan-fan maps on the given surface

3.1 Projective plane.

Theorem 6[13] The enumerating function $h_1(x, y)$ satisfies the following equation:

$$(1 - x + x^2 y)h_1 = xyh_1^* + (1 - x)t_1 f_1 + (1 - x)x^2 y \frac{\partial(xs_0)}{\partial x} \tag{3.1}$$

where $h_1^* = h_1(1, y)$.

Proof According to Eq.(1.12), let $q = 1$, this theorem holds. □

Let $\sigma_{i,j}(n) = \dfrac{(i - 2j + 2n)!(i + 3)}{(n - j - 1)!(i - j + n + 2)!}$.

$$G_{i,j}(*) = \sum_{j > n_1 \geq 1} \frac{(2n_1 - 2)!}{(n_1 - 1)!n_1!} F^*_{m_2,n_2} \quad and \quad \begin{cases} n_1 + n_2 = j; \\ m_2 + 2n_1 = i. \end{cases} \tag{3.2}$$

By employing Lagrangian inversion (see [13]), we obtain that

$$h_1^* = \sum_{n \geq 1} H_n^1 y^n$$

where

$$H_n^1 = \sigma_{0,0}(n) + \sum_{\substack{j=n \\ i \leq 2j}} G_{i,j}(1) + \sum_{\substack{j \leq n-1 \\ i \leq 2j}} \left[G_{i,j}(1)\sigma_{i-2,j-1}(n) + (i+1)\sigma_{i,j}(n)S^0_{i,j}\right] \tag{3.3}$$

The number of rooted pan-fan maps on the projective plane with the valency of root face m and the size n is:

$$\partial_{x,y}^{m,n} h_1 = H^1_{m,n} = C_{m,n} + \sum_{\substack{1 \leq l+j \leq m \\ 0 \leq j \leq l}} (-1)^j \binom{l}{j} C_{m-l-j,n-j} \tag{3.4}$$

where

$$\begin{cases} C_{1,n} = G_{1,n}(1) + H^1_{n-1}, & n \geq 1; \\ C_{m,n} = G_{m,n}(1) - G_{m-1,n}(1) & \\ \qquad + (m-1)S^0_{m-2,n-1} - (m-2)S^0_{m-3,n-1}, & m \geq 2, n \geq 1. \end{cases}$$

$H^1_{m,n}$	$m=1$	2	3	4	5	6	7	8	9	10	11	12
n=1	0	1	0	0	0	0	0	0	0	0	0	...
2	1	1	3	4	0	0	0	0	0	0	0	...
3	9	9	13	14	15	6	0	0	0	0	0	...
4	66	66	76	74	66	48	21	16	0	0	0	...
5	433	*	*	*	*	*	*	*	*	*	0	...

3.2 Klein bottle.

Theorem 7 The enumerating function $h_2(x, y)$ satisfies the following equation:

$$(1-x+x^2y)h_2 = xyh_2^* + (1-x)t_1f_2 + (1-x)x^2y\frac{\partial(xh_1)}{\partial x}$$
$$+(1-x)x^3y\frac{\partial s_0^1}{\partial z_1}(x, y, x) \quad (3.5)$$

where $h_2^* = h_2(1, y)$

Proof For Eq.(1.12), let $q = 2$, it is easy to see this formula is valid. □

As for s_0^1, according to the discussion of section 2 and noticing the differences in nonorientable and orientable, the following theorem can be obtained.

Theorem 8 The function $s_0^1(x, y, z_1)$ satisfies the following equation:

$$(1-x+x^2y)s_0^1 = xys_0^{1^*} + (1-x)t_1l_0^1 + (1-x)xyz_1\delta_{u=(z_1,x)}(us_0(u, y)) \quad (3.6)$$

where $s_0^{1^*} = s_0^1(1, y, z_1)$, $\delta_{u=(z_1,x)}(us_0(u, y)) = \dfrac{z_1s_0(z_1, y) - xs_0(x, y)}{z_1 - x}$

In order to get the explicit expression of h_2, now we must study f_2 and $l_0^1(x, y, z_1)$. Observing the difference, which is mainly in the set of $\mathcal{H}_2^{II}$, between rooted pan-fan maps and one-vertexed maps, similar to Theorem 7 and Theorem 8, we can get the following two equations.

Theorem 9 The enumerating function f_2 satisfies the following equation:

$$(1-x+x^2y)f_2 = xyf_2^* + (1-x)x^2y\frac{\partial(xf_1)}{\partial x} + (1-x)x^3y\frac{\partial l_0^1}{\partial z_1}(x, y, x) \quad (3.7)$$

where $f_2^* = f_2(1, y)$

Theorem 10 The function l_0^1 satisfies the following equation:

$$(1-x+x^2y)l_0^1 = xyl_0^{1^*} + (1-x)xyz_1\delta_{u=(z_1,x)}(ul_0(u, y)) \quad (3.8)$$

where $\delta_{u=(z_1,x)}(ul_0(u, y)) = \dfrac{z_1l_0(z_1, y) - xl_0(x, y)}{z_1 - x}$

For Eq.(3.8) we can derive the parametric expressions of $l_0^{1^*}(y, z_1)$ and $l_0^1(x, y, z_1)$ are respectively:

$$\begin{cases} \eta = y(\eta+1)^2, z_1 = \zeta(\eta+1), l_0^{1^*} = \dfrac{\zeta\eta(\eta+1)}{(1-\eta)(1-\zeta\eta)}; \\ x = \beta(\eta+1), l_0^1 = \dfrac{\beta\zeta\eta}{(1-\zeta\eta)(1-\eta)(1-\beta\eta)^2}; \\ \dfrac{\partial l_0^1}{\partial z_1} = \dfrac{\beta\eta}{(1-\eta^2)(1-\zeta\eta)^2(1-\beta\eta)^2}. \end{cases} \tag{3.9}$$

Using Lagrangian inversion with three parameters, the following result can be gained.

Theorem 11 The number of one-vertexed maps on the sphere with the valency of root face m, the size n and the valency of distinguished non-rooted face k_1 is:

$$L^0_{m,n,k_1} = \frac{m(2n-m-k_1-1)!}{(n-2)!(n-m-k_1+1)!},$$

where $L^0_{1,1,1} = 1, L^0_{1,2,2} = 1, L^0_{2,2,1} = 2$

For Eq.(3.7), let $x = \xi$ be the characteristic solution of the equation (3.7) and $\eta = \xi - 1$, according to (1.9) and (3.9), then the following formula can be obtained.

$$\begin{cases} \eta = y(\eta+1)^2, f_2^* = \dfrac{2\eta^2(1+\eta)(2+\eta)}{(1-\eta)^5}, x = \beta(\eta+1); \\ f_2 = \dfrac{2\beta\eta^3(\eta+2)(1+\beta-2\beta\eta)}{(1-\eta)^5(1-\beta\eta)^3} + \dfrac{2\beta^3\eta^2[3\eta(1-\beta\eta)+2\beta(1-\eta)]}{(1-\eta)^3(1-\beta\eta)^5}; \\ \dfrac{\partial f_2}{\partial x} = \dfrac{2\eta^3(\eta+2)(1+\beta\eta)}{(1-\eta)^5(1-\beta\eta)^3(1+\eta)} + \dfrac{2\eta^3\beta(\eta+2)(2+\beta\eta)}{(1-\eta)^4(1-\beta\eta)^4(1+\eta)} \\ \qquad + \dfrac{6\eta^3\beta^2(3+\beta\eta)}{(1-\eta)^3(1-\beta\eta)^5(1+\eta)} + \dfrac{4\eta^2\beta^3(4+\beta\eta)}{(1-\eta)^2(1-\beta\eta)^6(1+\eta)}. \end{cases} \tag{3.10}$$

Now, by using Lagrangian inversion, the theorem B can be proved.

Corollary 2 The number of rooted one-vertexed maps on the Klein bottle with the size $n-1$ is:

$$\sum_{i\geq 0}^{n-3} \frac{(2n-2)!(3n+i+1)(i+3)!}{3i!(n+i+2)!(n-i-3)!}$$

Proof Let A and B be two subsets of $\mathcal{F}_2$. The set A includes all that maps which the size is $n-1$. The set B includes all that maps which the valency of root face is 1 and the size is n. For any $M \in A$, we can get a map $M' \in B$ by adding a single edge on M. It is easy to see there exists an one-one mapping between the two sets. So let $m = 1$ for (1.13), it is easy to check this corollary holds. □

$F^2_{m,n}$	$m=1$	2	3	4	5	6	7	8	9
n=2	0	0	0	4	0	0	0	0	...
3	4	4	6	8	20	0	0	0	...
4	42	42	48	52	60	60	0	0	...
5	304	304	306	296	280	240	140	0	...
6	1870	*	*	*	*	*	*	*	...

Next, solving the Eq.(3.5), let $A(x, y) = 1 - x + x^2 y$ and $x = \theta$ be the characteristic solution of the equation (3.5), then the following formula can be obtained.

$$\begin{cases} A(\theta, y) = 1 - \theta + \theta^2 y = 0; \\ \theta y h_2^* = (\theta - 1) t_1 f_2 \mid_{x=\theta} + (\theta - 1)\theta^2 y \dfrac{\partial (x h_1)}{\partial x} \mid_{x=\theta} \\ \quad + (\theta - 1)\theta^3 y \dfrac{\partial s_0^1}{\partial z_1}(\theta, y, \theta); \end{cases} \tag{3.11}$$

From Eq.(3.6) and $A(\theta, y) = 0$, let $A^{(k)}$ denote the kth partial with respect to x of $A(x, y)$ at θ and likewise for other functions, we have

$$s_0^1(\theta, y, z_1) = \frac{\theta y z_1 C^{(1)}}{A^{(1)}} + \frac{[(1 - x) t_1 l_0^1]^{(1)}}{A^{(1)}}$$

where $C(x, y, z_1) = \dfrac{1 - x}{z_1 - x}(z_1 s_0(z_1, y) - x s_0(x, y))$ and $C^{(1)} = \dfrac{\partial C}{\partial x} \mid_{x=\theta}$. Then it can be check that

$$A^{(1)} \frac{\partial s_0^1}{\partial z_1}(\theta, y, \theta) = \frac{\theta y}{6}(x^2(1 - x) s_0)^{(3)}$$

$$+ \sum_{m,n,k_1 \geq 1} k_1 G_{m,n,k_1}[m - (m + 1)\theta]\theta^{m + k_1 - 2} y^n \tag{3.12}$$

where

$$G_{m,n,k_1} = \sum_{n > n_1 \geq 1} \frac{(2n_1 - 2)!}{(n_1 - 1)! n_1!} L^0_{m_2, n_2, k_1} \quad \text{and} \quad \begin{cases} n_1 + n_2 = n; \\ m_2 + 2n_1 = m. \end{cases}$$

In formula (3.11), substituting $\dfrac{\partial s_0^1}{\partial z_1}(\theta, y, \theta)$ by formula (3.12) and introducing a parameter $\eta = \theta - 1$, from Eq.(3.5) and Eq.(3.6), by employing

Lagrangian inversion, we obtain:

Theorem 12 The number of rooted pan-fan maps on the Klein bottle is as follows:

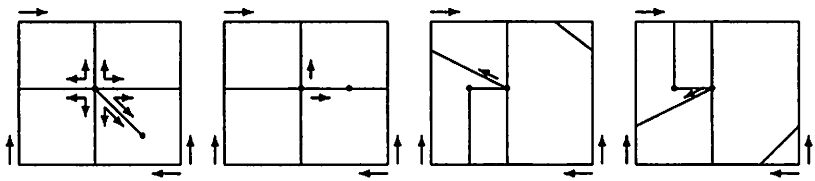
$$\partial_y^n h_2^* = \sum_{i,j\geq 0}\left[(i+1)\sigma_{i,j}(n)H_{i,j}^1 + \frac{1}{6n}C(i)B(i+1,j,n)(S_{i,j}^0 - S_{i+1,j}^0)\right]$$
$$+\frac{1}{n}\sum_{i,j,k_1\geq 1} G_{i,j,k_1}k_1[C(i+1,j,k_1,n) - C(i,j,k_1,n)]$$
$$+\sum_{i,j\geq 1}\sigma_{i-2,j-1}(n)G_{i,j}(2) \tag{3.13}$$

where $H_{i,j}^1 = (3.4)$, $S_{i,j}^0 = (1.7)$, $C(i) = (i+3)(i+2)(i+1)$.

$$\begin{cases} C(i,j,k_1,n) = \dfrac{i(j+1)(i-2j+k_1+2n+1)!}{(n-j-1)!(i-j+n+k_1+2)!} \\ \qquad + \displaystyle\sum_{l\geq 0}^{n-j-2}\frac{i(i-2j+k_1+2n)!D_l(i,j,n,k_1)}{l!(i-2j+k_1+2n-l)!}; \\ D_l(i,j,k_1,n) = \dfrac{(i-2j+k_1+2n-2l+1)(i-2j+k_1+n+1)}{i-2j+k_1+2n-l+1}; \\ B(i+1,j,n) = \displaystyle\sum_{l\geq 0}^{n-j-3}\frac{(i-j+n+4)(i-2j+2n+1)!D_l(i,j,1,n)}{(i-2j+n+2)l!(i-2j+2n-l+1)!} \\ \qquad + \dfrac{(j+2)(i-2j+2n+1)!}{(n-j-2)!(i-j+n+3)!}. \end{cases}$$

Let $H_{\cdot,n}^2(x) = \sum_{m=1}^{\infty} H_{m,n}^2 x^m$. According to Eq.(3.5), Eq.(3.6) and formula (3.13), it is easy to see:

- $H_{\cdot,2}^2(x) = 4x^4$;
- $H_{\cdot,3}^2(x) = 4x + 4x^2 + 6x^3 + 8x^4 + 20x^5 + 31x^6$.



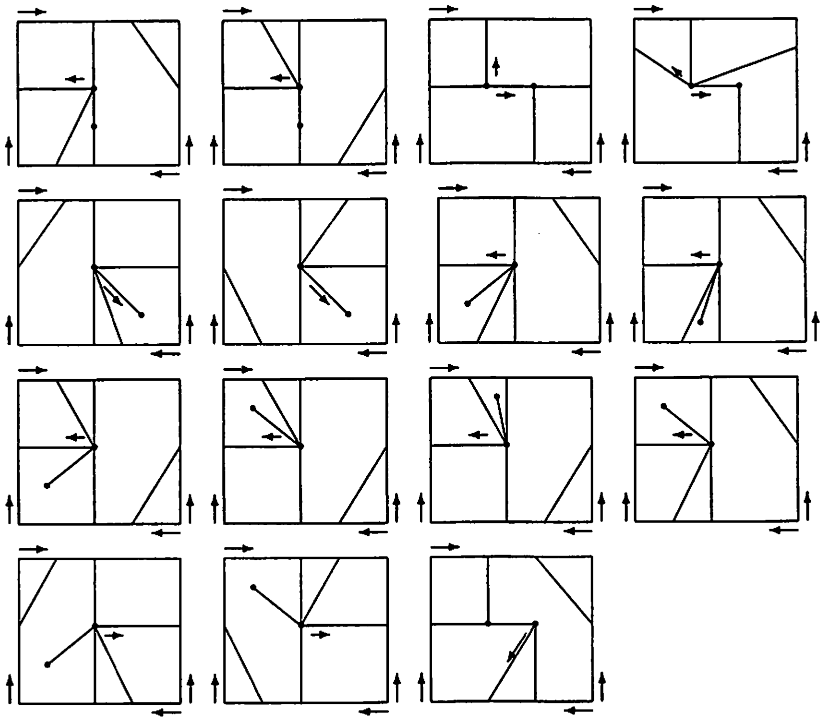


Fig 1.
(31 rooted pan-fan maps with the valency of root face 6 and the size 3)

4 One-vertexed maps on the Torus and N_3

After a similar procedure used in the proof of Theorem 5, the following theorems can be derived.

Theorem 13 The enumerating function f_3 satisfies the following equation:

$$(1-x+x^2y)f_3 = xyf_3^* + (1-x)x^2y[\frac{\partial(xf_2)}{\partial x} + \frac{\partial(xl_1)}{\partial x}] + 2(1-x)x^3y\frac{\partial f_1^1}{\partial z_1}(x,y,x) \tag{4.1}$$

where $f_3^* = f_3(1,y)$

Theorem 14 The functions f_1^1 satisfy the following equation:

$$(1-x+x^2y)f_1^1 = xyf_1^{1^*} + (1-x)xyz_1\delta_{u=(z_1,x)}(uf_1(u,y)) + (1-x)x^2y\frac{\partial(xl_0^1)}{\partial x} \tag{4.2}$$

where $\delta_{u=(z_1,x)}(uf_1(u,y)) = \dfrac{z_1f_1(z_1,y) - xf_1(x,y)}{z_1 - x}$

According to formula (1.9) and (3.9), it can be checked Eq.(4.2) satisfied the following parametric expressions:

$$
\begin{cases}
\eta = y(\eta+1)^2, z_1 = \zeta(\eta+1), x = \beta(\eta+1); \\
f_1^{1^*} = \frac{\eta^2\zeta(\eta+1)[(\eta+1)(1-\zeta\eta)(1+\zeta-2\zeta\eta)+\zeta^2(1-\eta)^2]}{(1-\eta)^4(1-\zeta\eta)^3} \\
\quad + \frac{2\eta^2\zeta(\eta+1)}{(1-\zeta\eta)(1-\eta)^4}; \\
f_1^1 = \frac{\beta\eta^3[(4\zeta-4\zeta\eta+\eta-\zeta\eta^2)(1-\beta\eta)+(3-7\beta\eta+4\beta)(1-\zeta\eta)]}{(\eta+1)(1-\eta)^4(1-\beta\eta)^3(1-\zeta\eta)^3} \\
\quad + \frac{\beta\eta^2[\zeta(3\zeta-5\beta\eta\zeta+2\beta)(1-\beta\eta)+3\beta^2(1-\zeta\eta)^2]}{(\eta+1)(1-\eta)^2(1-\beta\eta)^4(1-\zeta\eta)^4}.
\end{cases}
\tag{4.3}
$$

As for l_1, it can be studied by the same method. It is easy to check that the enumerating function l_1 satisfies the following equation.

Theorem 15 The enumerating function l_1 satisfies:

$$
(1-x+x^2y)l_1 = xyl_1^* + (1-x)x^3y\frac{\partial l_0^1}{\partial z_1}(x,y,x) \tag{4.4}
$$

where $l_1^* = l_1(1,y)$

We can also get the parametric expressions as follows:

$$
\begin{cases}
\eta = y(\eta+1)^2, l_1^* = \frac{\eta^2(\eta+1)}{(1-\eta)^5}, x = \beta(\eta+1); \\
l_1 = \frac{\beta\eta^3}{(1-\eta)^5(1-\beta)(1-\beta\eta)} + \frac{\beta^4\eta^2[1-\beta(\eta+1)]}{(1-\eta)(1-\beta)(1-\beta\eta)^5}; \\
\frac{\partial l_1}{\partial x} = \frac{\eta^3(1+\beta\eta)}{(1-\eta)^5(1-\beta\eta)^3(1+\eta)} + \frac{\eta^3\beta(2+\beta\eta)}{(1-\eta)^4(1-\beta\eta)^4(1+\eta)} \\
\quad + \frac{\eta^3\beta^2(3+\beta\eta)}{(1-\eta)^3(1-\beta\eta)^5(1+\eta)} + \frac{\eta^3\beta^3(4+\beta\eta)}{(1-\eta)^2(1-\beta\eta)^6(1+\eta)}.
\end{cases}
\tag{4.5}
$$

Using Lagrangian inversion, the following theorem can be gained.

Theorem 16 The number of one-vertexed maps on the Torus with the size $n-1$ is:

$$
\sum_{i\geq 0}^{n-3} \frac{(2n-2)!(i+3)!}{6i!(n+i+1)!(n-i-3)!}
$$

Due to (3.10), (4.3) and (4.5), it is easy to see that Eq.(4.1) satisfies the following parametric expressions.

$$\begin{cases} \eta = y(\eta+1)^2, f_3^* = \dfrac{\eta(\eta+1)(12\eta^4+75\eta^3+41\eta^2)}{(1-\eta)^8}, x = \beta(\eta+1); \\ f_3 = \dfrac{\beta\eta^4(12\eta^2+75\eta+41)(1+\beta-2\beta\eta)}{(1-\eta)^8(1-\beta\eta)^3} + \dfrac{\beta^3\eta^4(8\eta^2+69\eta+51)}{(1-\eta)^6(1-\beta\eta)^4} \\ \quad + \dfrac{\beta^4\eta^4(52\eta+62)}{(1-\eta)^5(1-\beta\eta)^5} + \dfrac{\beta^5\eta^3(85\eta-85\beta\eta^2+41\beta-41\beta\eta)}{(1-\eta)^4(1-\beta\eta)^7} \end{cases}$$

Therefore, by employing Lagrangian inversion, it can be checked that the theorem C is correct. What's more, the following corollary comes into existence.

Corollary 3 The number of one-vertexed maps on N_3 with the size $n-1$ is:

$$\frac{(n+1)!(82n^2-294n+110)}{6!(n-4)!} + \sum_{i\geq 0}^{n-6} \frac{(i+6)!(2n-2)!\lambda_i(n)}{6!i!(n+i+4)!(n-i-4)!}.$$

where $\lambda_i(n) = 128n^2 - 22i^2 + 58ni + 179n - 205i - 468$.

$F^3_{m,n}$	$m=1$	2	3	4	5	6	7	8	9	10
n=3	0	0	0	0	0	41	0	0	0	...
4	41	41	51	62	85	123	287	0	0	...
5	690	690	753	814	905	1002	1148	1148	0	...
6	7150	*	*	*	*	*	*	*	*	...

Remark: From the above table, it is easy to see that the sum of any row is the same as the result of lit[1] which gave the number of rooted maps with 1 vertices and $f = n-2$ faces on a non-orientable surface of type 3/2.

From Theorem A, we can discuss the rooted pan-fan maps on N_3 basen on one-vertexed maps. Since its equation has known, it is just a problem of calculation. So here we do not dissertate it in detail.

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