On the Spectral Radius of Unicyclic Graphs with n Vertices and Edge Independence Number q *

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Abstract

We study the spectral radius of unicyclic graphs with n vertices and edge independence number q. In this paper, we show that of all unicyclic graphs with n vertices and edge independence number q, the maximal spectral radius is obtained uniquely at $\Delta_n(q)$, where $\Delta_n(q)$ is a graph on n vertices obtained from the cycle C_3 by attaching $n-2q+1$ pendant edges and $q-2$ paths of length 2 at one vertex.

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1. Introduction

The graphs in this paper are simple. Let $A(G)$ be a $(0,1)$-adjacency matrix of G. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. The characteristic polynomial of G is just $\det(\lambda I - A(G))$, and is denoted by $P(G;\lambda)$. The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of G, denoted by $\rho(G)$. If G is connected, then $A(G)$ is irreducible, and by the Perron-Frobenius theory of non-negative matrices $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. We shall refer to such an eigenvector as the Perron vector of G.

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The following problem concerning spectral radii was proposed by Brualdi and Solheid [2]: Given a set φ of graphs, find an upper bound for maximal spectral radii of graphs in φ and characterize the graphs in which the maximal spectral radii is attained. Recently, the problem has been studied extensively. The reader is referred to [1, 4, 9, 13] and the references therein.

Two distinct edges in a graph G are independent if they are not adjacent in G. A set of pairwise independent edges of G is called a matching in G, while a matching of maximum cardinality is a maximum matching in G denoted by $M(G)$ or M. The cardinality $|M|$ of a maximum matching M of G is commonly known as its edge independence number denoted by q. A matching $M(G)$ that satisfies $2q = n = |V(G)|$ is called a perfect matching. Unicyclic graphs are connected graphs in which the number of edges equals the number of vertices. Let $U(n)$ and $U^+(2k)$ denote the set of all unicyclic graphs on n vertices and the set of all unicyclic graphs with perfect matchings on $2k$ vertices, respectively. The eigenvalues of graphs in $U(n)$ have been studied by several authors (see [3, 6-8, 11, 12]). Very recently, Chang [4] gave two graphs which have the largest and the second largest spectral radius, respectively, among the graphs in $U^+(2k)$.

In this paper, we study the spectral radius of unicyclic graphs with n vertices and edge independence number q. We show that of all unicyclic graphs with n vertices and edge independence number q, the maximal spectral radius is obtained uniquely at $\Delta_n(q)$, where $\Delta_n(q)$ is a graph on n vertices obtained from the cycle C_3 by attaching $n-2q+1$ pendant edges and $q-2$ paths of length 2 at one vertex.

2. Preliminaries

Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. Let $G - xy$ denote the graph that arises from G by deleting the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. Let M be a maximum matching of a graph G. An edge $e = uv$ which belongs to M is called an M-saturated edge and both u and v are called M-saturated vertices. A pendant vertex of G is a vertex of degree 1. A pendant edge is an edge with which a pendant vertex is incident. We denote by $\mathcal{U}_n(q)$ the set of all unicyclic graphs with n vertices and edge independence number q. A unicyclic graph is either a cycle or a cycle with trees attached. For any $U \in \mathcal{U}_n(q)$, denote by $C_k = u_1u_2\cdots u_ku_1$ the unique cycle in U and denote U by $C_k(T_1, T_2, \cdots, T_k)$, where T_i is a tree attached to vertex u_i $(i = 1, 2, \cdots, k)$, T_i contains u_i as a vertex and $|V(T_i)| \geq 1$. If there exists a pendant vertex adjacent to some vertex, for example u_1, of the cycle C_k, we also denote U by $C_k(P_2 \cup T_1, T_2, \cdots, T_k)$, where path P_2 and tree T_1 are attached to u_1. Denote by $\Delta_n^+(q)$ the graph obtained from $C_3(P_2, P_2, P_2)$ by attaching $n-2q$ pendant edges and $q-3$ paths of length 2 at one vertex. It is clear that $\Delta_n^+(q) \in \mathcal{U}_n(q)$.

In order to complete proof of our main result, we need following lemmas. For $v \in V(G)$, $N(v)$ denotes the set of all neighbors of vertex v in G.

Lemma 1 [9, 13]. *Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \cdots, v_s \in N(v)\backslash N(u)(1 \le s \le d_v)$ and $x = (x_1, x_2, \cdots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i(1 \le i \le n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i(1 \le i \le s)$. If $x_u \ge x_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 1 was first given in [13] and cited in [9]. The proof can also be found in [9]. By Lemma 1, we obtain easily following Lemma 2-6 which may be regard as immediate consequences of Lemma 1. Proofs of the lemmas are similar, so we only give the proof of Lemma 2.

Lemma 2. *Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \cap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v, and adding a pendant edge to $u(= v)$. Then $\rho(G) < \rho(G^*)$.*

Proof. We use x_u and x_v to denote the components of the Perron vector of G corresponding to u and v. Suppose that $N(u) = \{v, v_1, \cdots, v_s\}$ and $N(v) = \{u, u_1, \cdots, u_t\}$. Since $e = uv$ is a non-pendant edge of G, it follows that $s, t \ge 1$. If $x_u \ge x_v$, let

$$G' = G - \{vu_1, \cdots, vu_t\} + \{uu_1, \cdots, uu_t\}.$$

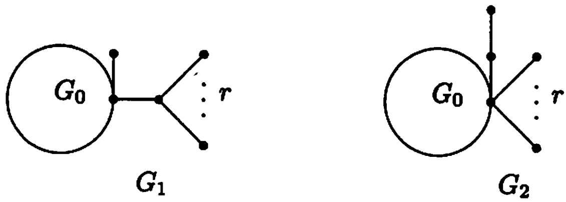
If $x_u < x_v$, let

$$G'' = G - \{uv_1, \cdots, uv_s\} + \{vv_1, \cdots, vu_s\}.$$

Obviously, $G^* = G' = G''$. By Lemma 1, we have $\rho(G) < \rho(G^*)$. This completes the proof.

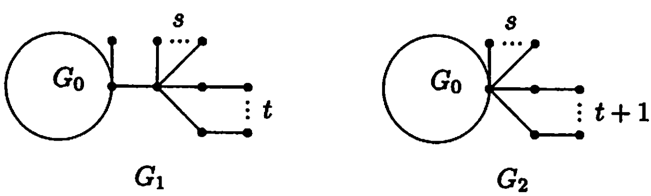
Lemma 3. *Let G, G', G'' be three connected graphs pairwise disjoint. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $\rho(G_1) < \rho(G_2)$ or $\rho(G_1) < \rho(G_3)$.*

Lemma 4. *Let G_1, G_2 be the graphs shown in Figs. 1, where G_0 is a connected graph, $r \ge 2$. Then $\rho(G_1) < \rho(G_2)$.*



Figs. 1

Lemma 5. *Let G_1, G_2 be the graphs shown in Figs. 2, where G_0 is a connected graph, $s \geq 2, t \geq 0$, or $s = 1, t \geq 1$. Then $\rho(G_1) < \rho(G_2)$.*



Figs. 2

Lemma 6. *Let $k > 3$, and let $C_{k-1}(P_2 \cup T \cup P_3, P_2, \cdots, P_2,))$ be a cycle $C = u_1u_2 \cdots u_{k-1}u_1$ of order $k-1$ with one pendant edge u_iv_i attached to vertex u_i $(i = 1, 2, \cdots, k-1)$ and with one tree T and a path of length 2 attached to u_1. Then*

$$\rho(C_k(P_2 \cup T, P_2, \cdots, P_2,)) < \rho(C_{k-1}(P_2 \cup T \cup P_3, P_2, \cdots, P_2,)).$$

Lemma 7[10]. *Let u be a vertex of G, and let $C(u)$ be the set of all cycles containing u. The characteristic polynomial of G satisfies*

$$P(G; \lambda) = \lambda P(G-u; \lambda) - \sum_{v \in N(u)} P(G-u-v; \lambda) - 2 \sum_{Z \in C(u)} P(G \backslash V(Z); \lambda).$$

Lemma 8[7, 8]. *Let v be a vertex in a non-trivial connected graph G and suppose that two paths of lengths k, m $(k \geq m \geq 1)$ are attached to G by their end vertices at v to form $G_{k,m}$. Then $\rho(G_{k,m}) > \rho(G_{k+1,m-1})$.*

Lemma 9. *Let $q \geq 3$ and $n \geq 8$, Then $\rho(\Delta_n(q)) > \rho(\Delta_n^+(q))$.*

Proof. Applying Lemma 7 to the highest degree vertex of $\Delta_n(q)$ and $\Delta_n^+(q)$, respectively, we have

$$P(\Delta_n(q); \lambda) = \lambda^{n-2q}(\lambda^2-1)^{q-2}[\lambda^4 - (n-q+2)\lambda^2 - 2\lambda + n - 2q + 1],$$

$$P\left(\triangle_n^+(q);\ \lambda\right) = \lambda^{n-2q}(\lambda^2-1)^{q-4}[\lambda^8-(n-q+4)\lambda^6-2\lambda^5+ \\ (4n-5q+3)\lambda^4+2\lambda^3-(4n-7q+4)\lambda^2+n-2q+1].$$

Since $q \geq 3$ and $n \geq 8$, it follows that the star $K_{1,4}$ is an induced subgraph of $\triangle_n^+(q)$, and so $\rho\left(\triangle_n^+(q)\right) \geq \rho(K_{1,4}) = 2$. Hence

$$P(\triangle_n(q);\ \lambda) - P\left(\triangle_n^+(q);\ \lambda\right) = \lambda^{n-2q+1}(\lambda^2-1)^{q-4}[(q-n+3)\lambda^2(\lambda-2)- \\ (n-2q)\lambda(\lambda-1)-(n-8)\lambda^2-2] < 0$$

for all $\lambda \geq \rho\left(\triangle_n^+(q)\right)$. This implies that

$$P\left(\triangle_n(q),\ \rho(\triangle_n^+(q))\right) < 0.$$

Thus

$$\rho(\triangle_n(q)) > \rho\left(\triangle_n^+(q)\right).$$

This completes the proof.

3. Main results

Theorem 1. *Let U be a unicyclic graph with n vertices and edge independence number q, $n \geq 8$. Then*

$$\rho(U) \leq \rho(\triangle_n(q)),$$

and the equality holds if and only if $U = \triangle_n(q)$, where $\rho(\triangle_n(q))$ is the largest root of the equation

$$\lambda^4-(n-q+2)\lambda^2-2\lambda+n-2q+1=0.$$

Proof. Let $\mathcal{U}_n(q)$ be the set of all unicyclic graphs with n vertices and edge independence number q. For any $U \in \mathcal{U}_n(q)$, assume that M is a maximum matching in U. Then $|M| = q$ and there are three cases for a non-pendant edge $e = uv$ in U: (1) $e = uv$ is an M-saturated edge; (2) $e = uv$ has exactly one M-saturated vertex; (3) $e = uv$ is not an M-saturated edge but both u and v are M-saturated vertices. If there exists a non-pendant edge $e = uv$ of case (1) or case (2) in U, applying the transformation described in Lemma 2, we can transform U into a graph $U^* \in \mathcal{U}_n(q)$ such that edge e is a pendant edge and $\rho(U) < \rho(U^*)$ unless the cycle in U is C_3 and $e = uv$ belongs to the cycle. Accordingly, if U is not the graph $U_0 \in \mathcal{U}_n(q)$ described in the following 4 cases, applying the transformation described in Lemma 2 repeatedly, we can transform U into a graph $U_0 \in \mathcal{U}_n(q)$ such that $\rho(U) < \rho(U_0)$.

Case 1. U_0 is a cycle $C_k = u_1u_2\cdots u_ku_1$ of order k with one pendant edge u_iv_i and one tree T_i attached to vertex u_i $(i = 1, 2, \cdots, k)$, where T_i contains u_i as a vertex and $|V(T_i)| \geq 1$. Each M-saturated edge in U_0 is pendant edge. Each vertex of U_0, which is not M-saturated, is a pendant vertex adjacent to some M-saturated vertex. The two vertices of each non-pendant edge are M-saturated. We denote the U_0 by $C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cup T_k)$.

Case 2. U_0 is a cycle $C_3 = u_1u_2u_3u_1$ of order 3 with one tree T_i attached to vertex u_i $(i = 1, 2, 3)$ and one pendant edge u_1v_1 attached to vertex u_1, where T_i contains u_i as a vertex and $|V(T_i)| \geq 1$. Edge u_2u_3 is an M-saturated edge, while all other M-saturated edges in U_0 are pendant edges. Each vertex of U_0, which is not M-saturated, is a pendant vertex adjacent to some M-saturated vertex. The two vertices of each non-pendant edge are M-saturated. We denote the U_0 by $C_3(P_2 \cup T_1, T_2, T_3)$.

Case 3. U_0 is a cycle $C_3 = u_1u_2u_3u_1$ of order 3 with one pendant edge u_iv_i and one tree T_i attached to vertex u_i $(i = 1, 2)$, where T_i contains u_i as a vertex and $|V(T_i)| \geq 1$. u_3 is not an M-saturated vertex. All other vertices of U_0, which are not M-saturated, are pendant vertices adjacent to some M-saturated vertices. Each M-saturated edges in U_0 is a pendant edge. The two vertices of each non-pendant edge except u_1u_3 and u_2u_3 are M-saturated. We denote the U_0 by $C_3(P_2 \cup T_1, P_2 \cup T_2, P_1)$.

Case 4. U_0 is a cycle $C_3 = u_1u_2u_3u_1$ of order 3 with one tree T_i attached to vertex u_i $(i = 1, 2)$, where T_i contains u_i as a vertex and $|V(T_i)| \geq 1$. Edge u_1u_2 is an M-saturated edge and u_3 is not an M-saturated vertex. All other M-saturated edges in U_0 are pendant edges. All other vertices of U_0, which are not M-saturated, are pendant vertices adjacent to some M-saturated vertices. The two vertices of each non-pendant edge except u_1u_3 and u_2u_3 are M-saturated. We denote the U_0 by $C_3(T_1, T_2, P_1)$.

For Case 1, if there exist two trees T_i, T_j such that $|V(T_i)| \geq 2$, $|V(T_j)| \geq 2$, let

$$N(u_i) \bigcap V(T_i) = \{ a_1, \cdots, a_s \}, \quad N(u_j) \bigcap V(T_j) = \{ b_1, \cdots, b_t \},$$

then $s \geq 1$, $t \geq 1$. Denote by U_i the graph

$$C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cup T_k) - \{ u_jb_1, \cdots, u_jb_t \} + \{ u_ib_1, \cdots, u_ib_t \},$$

and by U_j the graph

$$C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cup T_k) - \{ u_ia_1, \cdots, u_ia_s \} + \{ u_ja_1, \cdots, u_ja_s \}.$$

Then $U_i, U_j \in \mathcal{U}_n(q)$, and by Lemma 3 we have either

$$\rho(C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cap T_k)) < \rho(U_i)$$

or

$$\rho(C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cap T_k)) < \rho(U_j).$$

So if more than one T_i in $C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cup T_k)$ satisfies $|V(T_i)| \geq 2$, by Lemma 3 we can transform $C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cup T_k)$ to the graph $C_k(P_2 \cup T, P_2, \cdots, P_2)$ in $\mathcal{U}_n(q)$ such that

$$\rho(C_k(P_2 \cup T_1, P_2 \cup T_2, \cdots, P_2 \cup T_k)) < \rho(C_k(P_2 \cup T, P_2, \cdots, P_2)).$$

Beginning with the vertex of T furthest from u_1 and applying Lemma 4 and Lemma 5 repeatedly, we can transform the graph $C_k(P_2 \cup T, P_2, \cdots, P_2)$ into a graph $C_k((n-2q+1)P_2 \cup (q-k)P_3, P_2, \cdots, P_2)$ in $\mathcal{U}_n(q)$, which is formed by attaching $n-2q$ pendant edges and $q-k$ paths of length 2 to the vertex u_1 of $C_k(P_2, P_2, \cdots, P_2)$. Applying Lemma 6 $k-3$ times, we can transform $C_k((n-2q+1)P_2 \cup (q-k)P_3, P_2, \cdots, P_2)$ into the graph $\Delta_n^+(q)$. By Lemma 3-6, we have

$$\rho(C_k(P_2 \cup T, P_2, \cdots, P_2)) \leq \rho(\Delta_n^+(q)),$$

and the equality holds if and only if $C_k(P_2 \cup T, P_2, \cdots, P_2,) = \Delta_n^+(q)$.

For Case 2, if $|V(T_2)| \geq 2$ or $|V(T_3)| \geq 2$, applying Lemma 3 as above we can transform $C_3(P_2 \cup T_1, T_2, T_3)$ into $C_3(P_2 \cup T, P_1, P_1)$ in $\mathcal{U}_n(q)$ such that

$$\rho(C_3(P_2 \cup T_1, T_2, T_3)) < \rho(C_3(P_2 \cup T, P_1, P_1)).$$

Applying Lemma 4 and 5 repeatedly, we can transform $C_3(P_2 \cup T, P_1, P_1)$ into the graph $\Delta_n(q)$. By Lemma 3-5, we have

$$\rho(C_3(P_2 \cup T_1, T_2, T_3)) \leq \rho(\Delta_n(q)),$$

and the equality holds if and only if $C_3(P_2 \cup T_1, T_2, T_3) = \Delta_n(q)$.

For Case 3, applying Lemma 3 as above, we can transform $C_3(P_2 \cup T_1, P_2 \cup T_2, P_1)$ into a graph $C_3(P_2 \cup T, P_1, P_1)$ in $\mathcal{U}_n(q)$ such that

$$\rho(C_3(P_2 \cup T_1, P_2 \cup T_2, P_1)) < \rho(C_3(P_2 \cup T, P_1, P_1)),$$

where T is a tree. Applying Lemma 4 and 5 repeatedly, we can transform $C_3(P_2 \cup T, P_1, P_1)$ into the graph $\Delta_n(q)$. By Lemma 3-5, we have

$$\rho(C_3(P_2 \cup T_1, P_2 \cup T_2, P_1)) < \rho(\Delta_n(q)).$$

For Case 4, if $|V(T_1)| \geq 2$ and $|V(T_2)| \geq 2$, applying Lemma 3, we can transform $C_3(T_1, T_2, P_1)$ into a graph $C_3(T, P_1, P_1)$ in $\mathcal{U}_n(q)$ such that

$$\rho(C_3(T_1, T_2, P_1)) < \rho(C_3(T, P_1, P_1)).$$

Applying Lemma 2 to an edge with which u_1 is incident in T, we can transform the graph $C_3(T, P_1, P_1)$ into a graph $C_3(P_2 \cup T', P_1, P_1)$ in $\mathcal{U}_n(q)$, where T' is a tree. By Lemma 2, we have

$$\rho(C_3(T, P_1, P_1)) < \rho(C_3(P_2 \cup T', P_1, P_1).$$

Applying Lemma 4 and 5 repeatedly, we can transform $C_3(P_2 \cup T', P_1, P_1)$ into a graph $\Delta_n(q)$. By Lemma 3-5, we have

$$\rho(C_3(T_1, T_2, P_1)) < \rho(\Delta_n(q)).$$

Combining the above arguments and Lemma 9, we have

$$\rho(U) \leq \rho(U_0) \leq \rho(\Delta_n(q))$$

and the equality holds if and only if $U = \Delta_n(q)$. By the proof of Lemma 9, we have $\rho(\Delta_n(q))$ is the largest root of the equation

$$x^4 - (n - q + 2)x^2 - 2x + n - 2q + 1 = 0.$$

The proof is completed.

By Lemma 8, we can easily see that $\rho(\Delta_n(q)) > \rho(\Delta_n(q + 1))$ for $n \geq 2q + 2$. From this fact and Theorem 1, we have the following corollary.

Corollary 1. *Let U be a unicyclic graph on n vertices with edge independence number not less than q, $n \geq 8$. Then*

$$\rho(U) \leq \rho(\Delta_n(q)),$$

and the equality holds if and only if $U = \Delta_n(q)$.

As a particular case of Corollary 1, we obtain the upper bound for the spectral radii of unicyclic graphs on n vertices, which is the main result of [12] and can also be found in [3, 11].

Corollary 2. *Let U be a unicyclic graph on n vertices, and $n \geq 8$. Then*

$$\rho(U) \leq \rho(\Delta_n(2)),$$

and the equality holds if and only if $U = \Delta_n(2)$.

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