Average distance in bipartite tournaments

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Abstract

The average distance $\mu(D)$ of a strong digraph D is the average of the distances between all ordered pairs of distinct vertices of D. Plesník [3] proved that if D is a strong tournament of order n, then $\mu(D) \le \frac{n+4}{6} + \frac{1}{n}$. In this paper we show that, asymptotically, the same inequality holds for strong bipartite tournaments. We also give an improved upper bound on the average distance of a k-connected bipartite tournament.

Let $D = (V, A)$ be a strong digraph of order n. The *average distance* of D, $\mu(D)$, is the average of the distances between all ordered pairs of distinct vertices of D, i.e.,

$$\mu(D) = \frac{1}{n(n-1)} \sum_{(u,v)\in V\times V} d_D(u,v),$$

where $d_D(u,v)$ denotes the distance from u to v in D. The *total distance* of D is defined as $d(D) = \sum_{(u,v)\in V\times V} d_D(u,v)$. The *diameter* of D, diam(D), is the maximum of the distances between all ordered pairs of vertices of D. In this paper, we are concerned with the average distance of strong bipartite tournaments. The following bound on the average distance is due to Plesník.

Theorem 1 [3] *Let T be a strong tournament of order $n \ge 3$. Then*

$$\frac{3}{2} \le \mu(T) \le \frac{n+4}{6} + \frac{1}{n}.$$

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Moreover, $\mu(T) = \frac{3}{2}$ *if and only if* T *has diameter* 2, *and* $\mu(T) = \frac{n+4}{6} + \frac{1}{n}$ *if and only if* T *is the unique strong tournament of diameter* $n-1$.

Taking the degree of a vertex, after whose removal the tournament remains strong, into account, Moon [2] obtained a slight improvement of Theorem 1. In [1], the present authors showed that for k-connected tournaments, Plesník's upper bound can be improved significantly.

Theorem 2 [1] *Let* $k \geq 1$. *If* T *is a* k*-connected tournament of order* n, *then*

$$\mu(T) < \frac{n}{6k} + \frac{19}{6} + \frac{k}{n},$$

and this bound is, apart from an additive constant, best possible.

In this paper, we are concerned with the average distance of bipartite tournaments, i.e., orientations of complete bipartite graphs. We show that Theorems 1 and 2 essentially hold also for bipartite tournaments.

Let T be a digraph. The *out-distance* $d^+(v)$ and the *in-distance* $d^-(v)$ of a vertex v of T is defined as $d^+(v) = \sum_{w \in V(T)} d(v,w)$ and $d^-(v) = \sum_{w \in V(T)} d(w,v)$, respectively. The *distance* of v, $d(v)$, is the sum $d^+(v) + d^-(v)$. If i is an integer, then we define $D_i = D_i(T)$ to be the number of ordered pairs of vertices (x,y) with $d_T(x,y) = i$. If v is a vertex of D and $A \subset V(D)$, then $d(v,A)$ is defined as $\min_{a \in A} d(v,a)$. The *converse* $\overline{D}$ of a digraph D is the digraph obtained from D by reversing the directions of all arcs. Clearly, $d(D) = d(\overline{D})$ if D is strong.

Two bipartite tournaments will be of importance in our considerations. For $n \geq 4$ we define T_n to be the bipartite tournament with vertex set $\{v_1, v_2, \ldots, v_n\}$ and arc set $E(T_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i v_j | \, 1 \leq j \leq i-3 \leq n-3, \, i-j \text{ odd}\}$. It is easy to verify that T_n is the unique strong bipartite tournament of order n and diameter $n-1$. For $k \geq 1$, the bipartite tournament T_n^k is the bipartite tournament of order $n+k$ obtained from T_n by adding k copies of v_n, i.e., by adding k new vertices with the same in-neighbourhood and out-neighbourhood as v_n.

We adopt the following notation. Let v be a vertex of out-eccentricity e. For $i \geq 0$ let $V_i(v) = \{w \in V(T) \mid d(v,w) = i\}$, $V_{-i}(v) = \{w \in V(T) \mid d(w,v) = i\}$ and, for any integer i, $n_i(v) = |V_i(v)|$. If v is understood then we simply write V_i and n_i. Also let $v = a_0, a_1, a_2, \ldots, a_e$ be a shortest path from v to V_e.

Our first result is a lower bound on the average distance of a strong bipartite tournament.

Proposition 1 *Let* T *be a strong bipartite tournament. Then*

$$\mu(T) \geq 2.$$

Equality holds if and only if $\text{diam}(T) = 3$.

Proof. Let u, v be two vertices of T. If u and v are in the same partition set then $d(u,v) \geq 2$ and $d(v,u) \geq 2$. If u and v are not in the same partition set and, say, $uv \in E(T)$, then $d(u,v) = 1$ and $d(v,u) \geq 3$. In either case we have

$$d(u,v) + d(v,u) \geq 4.$$

Summation over all u, v and division by $n(n-1)$ now yields $\mu(T) \geq 2$. Now $\mu(T) = 2$ if and only if $d(u,v) + d(v,u) = 4$ for all $u, v \in V(T)$, which holds if and only if $\mathrm{diam}(T) \leq 3$. Since every bipartite tournament has diameter at least 3, we have $\mu(T) = 2$ if and only if $\mathrm{diam}(T) = 3$. □

We remark that by a result by Šoltés [4], who characterizes the complete bipartite graphs that admit a strong orientation of diameter 3, bipartite tournaments of order n and diameter 3 exist for all $n \geq 4$. Hence the bound in Proposition 1 is sharp for all $n \geq 4$.

A sharp upper bound on μ is significantly harder to prove. The basic idea - remove a vertex v and estimate $d(v)$ and $d(T-v)$ - is simple. However, the fact that the bipartite tournaments of order n that maximize $d(v)$ and $d(T)$ change shape at $n = 10$ and $n = 13$, respectively, makes some technical detail necessary. We first state preparatory Lemmas.

Lemma 1 *Let v, x be vertices of a bipartite tournament T with $d_T(v,x) \geq 4$. If in $T - x$ every vertex is reachable from v, then $T - x$ is strong. In particular, if $\mathrm{diam}(T) \geq 4$ and $d(v,x) = \mathrm{diam}(T)$, then $T - x$ is strong.*

Proof. It suffices to show that for every vertex w of $T - x$ there exists a (w,v)-path. Let $w \in V_i$. If $i \geq 3$, i odd, then the arc wv is present in $T - x$. If $i \geq 6$, i even, then w, a_3, v is a (w,v)-path. If $i \in \{1,2\}$, then let P be a shortest path in T from w to V_3. Since the internal vertices of P are all in $V_1 \cup V_2$, P is also a path in $T - x$, and thus appending v to P yields a (w,v)-path. Finally, if $i = 4$ then w has an out-neighbour $w' \in V_1$. Since there exists a (w',v) path, the existence of a (w,v)-path follows. This proves the first part.

The second statement follows from the first statement and the fact that in $T - x$ every vertex is reachable from v if $d(v,x) = diam(T)$. □

Lemma 2 *Let T be a strong bipartite tournament and $v \in V(T)$. Let $V_i = V_i(v)$ for $i = 1,2,3$. Then $V_3 \neq \emptyset$ and*

$$\sum_{w \in V_1 \cup V_2} \Big(d_T(v,w) + d_T(w,v) \Big) \leq \lfloor \frac{1}{2}(n_1 + n_2 + 3)^2 - 4 \rfloor. \qquad (1)$$

If $n_3 = 1$, then equality in (1) implies that (i) $V_0 \cup V_1 \cup V_2 \cup V_3$ induce a $T_{n_1+n_2+2}$ or (ii) $n_1 + n_2$ is even and $V_0 \cup V_1 \cup V_2 \cup V_3$ induce a $T^1_{n_1+n_2+1}$. In both cases v corresponds to v_1.

Proof. Let $w \in V_1 \cup V_2$. Since v is not adjacent from any vertex in $V_1 \cup V_2$, we have $V_3 \neq \emptyset$. A shortest path from w to V_3 contains only vertices in $V_1 \cup V_2$, except for the terminal vertex. Moreover, $d(w,v) = d(w,V_3)+1$. Let W_i be the set of vertices $w \in V_1 \cup V_2$ with $d(w,V_3) = i$ and let $m_i = |W_i|$. Since T is bipartite, we have $W_i \subseteq V_1$ if i is even, and $W_i \subseteq V_2$ if i is odd and thus $d(v,w)+d(w,v) = 2j+2$ if $w \in W_{2j-1} \cup W_{2j}$. Hence

$$\begin{aligned}\sum_{w\in V_1\cup V_2}\Big(d(v,w)+d(w,v)\Big) &= \sum_{j\geq 1}\Big(\sum_{w\in W_{2j-1}\cup W_{2j}}\Big)\Big(d(v,w)+d(w,v)\Big)\\ &= \sum_{j\geq 1}(2j+2)(m_{2j-1}+m_{2j}). \qquad (2)\end{aligned}$$

Clearly, $m_i \geq 0$ for $i \geq 0$. Also, if there is a vertex in $V_1 \cup V_2$ at distance i to V_3, then there exists a vertex at distance $i-1$. Hence, $m_i \geq 1$ implies $m_{i-1} \geq 1$ for $i \geq 1$. Subject to this condition and $\sum_{i\geq 1} m_i = n_1+n_2$, the term in (2) is maximized if and only if $m_i = 1$ for $i = 1,2,\ldots,n_1+n_2-2$ and either (i) $m_i = 1$ for $i = n_1+n_2-1, n_1+n_2$ or (ii) n_1+n_2 is even and $m_i = 2$ for $i = n_1+n_2-1$ and $m_i = 0$ for $i = n_1+n_2$, while the remaining m_i equal 0. Hence, with $n_1+n_2 =: N$,

$$\begin{aligned}\sum_{w\in V_1\cup V_2}\Big(d(v,w)+d(w,v)\Big) &\leq \begin{cases}\sum_{j=1}^{N/2} 2(2j+2), & N \text{ even},\\ \sum_{j=1}^{(N-1)/2} 2(2j+2)+(N+3) & N \text{ odd},\end{cases}\\ &= \lfloor \frac{1}{2}(n_1+n_2+3)^2 - 4 \rfloor,\end{aligned}$$

as desired.

Assume (1) holds with equality and $n_3 = 1$. Then $m_i = 1$ for $i = 1,2,\ldots,n_1+n_2-2$, $m_i = 0$ for $i > n_1+n_2$ and either (i) $m_i = 1$ for $i = n_1+n_2-1, n_1+n_2$ or (ii) n_1+n_2 is even and $m_{n_1+n_2-1} = 2$ and $m_{n_1+n_2} = 0$. Hence in the tournament induced by $V_0 \cup V_1 \cup V_2 \cup V_3$, we have $n_i(v) = 1$ for $i = -1,-2,,\ldots,-(n_1+n_2)$ and either (i) $n_i(v) = 1$ for $i = -(n_1+n_2), -(n_1+n_2)$ or (ii) n_1+n_2 is even and $n_{n_1+n_2-1}(v) = 2$ and $n_{n_1+n_2}(v) = 0$. Hence $T = \overline{T_{n_1+n_2+2}}$ or n_1+n_2 is even and $T = \overline{T^1_{n_1+n_2+1}}$. □

For a vertex $v \in V(T)$ define the *out-eccentricity* and *in-eccentricity* of v by $e^+(v) = \max_{w\in V(T)} d_T(v,w)$ and $e^-(v) = \max_{w\in V(T)} d_T(w,v)$, respectively.

Lemma 3 *Let T be a strong bipartite tournament of order n and $v \in V(T)$ with $5 \leq e^+(v) \leq 7$ or $5 \leq e^-(v) \leq 7$. Then*

$$d(v) \leq \begin{cases} 8n-24 & \text{if } 6 \leq n \leq 12,\\ \lfloor \frac{1}{2}n^2 - n + \frac{25}{2} \rfloor & \text{if } n \geq 13.\end{cases}$$

Proof. We only consider the case $5 \leq e^+(v) \leq 7$. The other case is proved analogously. We first consider the distances between v and the vertices in $\bigcup_{i\geq 3} V_i$. Let $e = e^+(v)$. Clearly,

$$d^+(v) = \sum_{i=1}^{e} i n_i. \tag{3}$$

To bound the in-distance of v, note that the same considerations as in the proof of Lemma 1 yield for $w \in V_i$,

$$d(w,v) \leq \begin{cases} 1 & \text{if } i \geq 3,\ i \text{ odd},\\ 2 & \text{if } i \geq 6,\ i \text{ even},\\ 2 & \text{if } i = 4 \text{ and } N^+(w) \cap V_5 \neq \emptyset,\\ 4 & \text{if } i = 4 \text{ and } N^+(w) \cap V_5 = \emptyset. \end{cases} \tag{4}$$

Since $V_5 \neq \emptyset$, the set $N^+(w)$ is not empty for some $w \in V_4$. In conjunction with Lemma 2 we obtain

$$\begin{aligned} d(v) &= \Big(\sum_{w\in V_1\cup V_2} + \sum_{w\in V_3\cup V_4\cup\ldots\cup V_7} \Big)\Big(d(v,w) + d(w,v)\Big) \\ &\leq \lfloor \frac{1}{2}(n_1+n_2+3)^2 - 4 \rfloor + 4n_3 + 8(n_4-1) + 6 + 6n_5 + 8n_6 + 8n_7. \end{aligned}$$

Let n_1+n_2 be fixed. Then $n_3+n_4+n_5+n_6+n_7 = n-n_1-n_2-1$, $n_3, n_4, n_5 \geq 1$, and $n_6, n_7 \geq 0$. Subject to these conditions the right hand side above is maximized if $n_6 = n_7 = 0$, $n_3 = n_5 = 1$ and thus $n_4 = n - n_1 - n_2 - 3$. Substituting these values yields, after simplification,

$$d(v) \leq \lfloor \frac{1}{2}(n_1+n_2)^2 - 5(n_1+n_2) + 8n - \frac{31}{2} \rfloor.$$

Now $2 \leq n_1 + n_2 \leq n-4$ since $e \geq 5$. A simple maximization of the right hand side shows that

$$d(v) \leq \begin{cases} 8n-24 & \text{if } 6 \leq n \leq 12,\\ \lfloor \frac{1}{2}n^2 - n + \frac{25}{2} \rfloor & \text{if } n \geq 13, \end{cases}$$

as desired. □

Lemma 4 *Let T be a strong bipartite tournament of order $n \geq 4$ and let $v \in V(T)$. Then*

$$d(v) \leq \begin{cases} 8n-20 & \text{if } 4 \leq n \leq 10,\\ \lceil \frac{1}{2}n^2 + n \rceil & \text{if } n \geq 11. \end{cases}$$

Equality holds for some vertex v of T if and only if
(i) $4 \leq n \leq 10$ and $T = T_5^{n-5}$ or
(ii) $n \geq 10$ and $T = T_n$ or $T = \overline{T_n}$ or
(iii) $n \geq 10$, n even and $T = T_{n-1}^1$ or $T = \overline{T_{n-1}^1}$.

Proof. Suppose there exists a counter example T of minimum order. We can assume that T and v are chosen such that $d(v)$ is maximum among all strong bipartite tournaments T of order n and all vertices v of T. It is easy to check that the Lemma holds for $n = 4, 5$. Let $e := e^+(v)$. By Lemma 3, we have $e \leq 4$ or $e \geq 8$.

CASE 1: $e \leq 4$.
Since T is strong and bipartite, V_3 is non-empty. By (4) and Lemma 2, we have

$$\begin{aligned} d(v) &= \Big(\sum_{w \in V_1 \cup V_2} + \sum_{w \in V_3} + \sum_{w \in V_4}\Big)\Big(d(v,w) + d(w,v)\Big) \\ &\leq \lfloor \frac{1}{2}(n_1 + n_2 + 3)^2 - 4 \rfloor + 4n_3 + 8n_4. \end{aligned}$$

Denote the last term by $f(n_1, n_2, n_3, n_4)$. We have $1 + n_1 + n_2 + n_3 + n_4 = n$ and $n_1, n_2, n_3 \geq 1$. Clearly, if f is maximized subject to these conditions then $n_3 = 1$, since otherwise decreasing n_3 and increasing n_4 by the same amount yields a larger value for f. Hence we can assume that $n_3 = 1$ and $n_1 + n_2 = n - 2 - n_4$. By elementary calculus, f is maximized iff $n_4 = n - 4$ and $n \leq 10$ or $n_4 = 0$ and $n \geq 10$. Substituting this yields, after simplification,

$$d(v) \leq \begin{cases} 8n - 20 & \text{if } n \leq 10, \\ \lceil \frac{1}{2}n^2 + n \rceil & \text{if } n \geq 11, \end{cases}$$

as desired.
We note that equality above implies $n_3 = 1$ and thus, by Lemma 2, that $n \leq 10$ and $T = T_5^{n-5}$ or $n \geq 10$ and $T = \overline{T_n}$.

CASE 2: $e \geq 8$.

We show that

$$n_4 = n_5 = \ldots = n_{e-1} = 1 \text{ and} \begin{cases} n_e = 1 & \text{or} \\ n_e = 2 & \text{and } e \text{ is even.} \end{cases} \tag{5}$$

Suppose not. Then $e \leq n - 2$. Let i be maximal with $n_i \geq 2$. Then there exists a vertex $x \in V_i$ satisfying the hypothesis of Lemma 1. Hence $T - x$ is strong. Since T is a minimal counter example, we have

$$d_{T-x}(v) \leq \lceil \frac{1}{2}(n-1)^2 + n - 1 \rceil = \lceil \frac{1}{2}n^2 - \frac{1}{2} \rceil.$$

If $i \geq 5$ then we have $d_T(v,x) + d_T(x,v) = i + 1$ if i is odd and $d_T(v,x) + d_T(x,v) = i + 2$ if i is even. Hence $d_T(v,x) + d_T(x,v) = 2\lceil (i+1)/2 \rceil$ for all i. Let T' be the bipartite tournament obtained from T be removing all arcs incident with x and adding an arc from a_e to x and arcs from x to all vertices in the same partition set as a_e, except a_e. Clearly, T' is strong and

$$\begin{aligned} d_{T'}(v) &= d_{T-x}(v) + d_{T'}(v,x) + d_{T'}(x,v) \\ &\geq d_{T-x}(v) + 2\lceil (e+2)/2 \rceil \\ &\geq d_{T-x}(v) + 2\lceil (i+1)/2 \rceil \\ &\geq d_T(v). \end{aligned}$$

Note that $2\lceil (e+2)/2\rceil > 2\lceil (i+1)/2\rceil$, unless $i = e$ and e is even. Note also that $n_e > 1$ implies $n_e = 2$, since otherwise repeated application of the above procedure strictly increases $d(v)$. Hence (5) follows, except possibly for n_4.

It remains to show that $n_4 = 1$. Suppose $n_4 \geq 2$. Then there exists a vertex $x \in V_4 - \{a_4\}$. By Lemma 1 and $n_5 = 1$, $T - x$ is strong. Let T' be the bipartite tournament obtained from T by removing all arcs incident with x and adding an arc from a_e to x and arcs from x to all vertices in the same partition set as a_e. By (4) we have

$$d_T(v) = d_{T-x}(v) + d_T(v,x) + d_T(x,v) \leq d_{T-x}(v) + 8,$$

$$d_{T'}(v) = d_{T'-x}(v) + d_{T'}(v,x) + d_{T'}(x,v) = d_{T-x}(v) + \begin{cases} e+2 & \text{if } e \text{ is even,} \\ e+3 & \text{if } e \text{ is odd.} \end{cases}$$

By $e \geq 8$, we have $d_{T'}(v) > d_T(v)$, a contradiction. Hence $n_4 = 1$. This completes the proof of (5).

Assume that $n_i = 1$ for $4 \leq i \leq e$ and that e is odd. (If e is even and $n_i = 1$ for $i = 4, 5, \ldots, e-1$ and $n_e \in \{1, 2\}$ the calculations are identical to the ones below.) Then $e = n - n_1 - n_2 - n_3 + 2$. Since T is bipartite, we have $d(v,w) + d(w,v) = 2i + 4$ for all $w \in V_{2i+2} \cup V_{2i+3}$, $i \geq 1$.

$$\begin{aligned} d(v) &= \Big(\sum_{w \in V_1 \cup V_2} + \sum_{w \in V_3} + \sum_{i=1}^{(e-3)/2} \sum_{w \in V_{2i+2} \cup V_{2i+3}} \Big) \Big(d(v,w) + d(w,v) \Big) \\ &= \Big(\sum_{w \in V_1 \cup V_2} d(v,w) + d(w,v) \Big) + 4n_3 + \sum_{i=1}^{(e-3)/2} 2(2i+4) \\ &= \Big(\sum_{w \in V_1 \cup V_2} d(v,w) + d(w,v) \Big) + 4n_3 + \frac{1}{2}(e+7)(e-3). \end{aligned}$$

Hence, by $e = n - n_1 - n_2 - n_3 + 2$ and Lemma 2,

$$d(v) \leq \lfloor \frac{1}{2}(n_1+n_2+3)^2 - 4 \rfloor + 4n_3 + \frac{1}{2}(n-n_1-n_2-n_3+9)(n-n_1-n_2-n_3-1).$$

Since $n_3 \geq 1$ and the last expression is decreasing in n_3, we obtain, after some simplification,

$$d(v) \leq \lfloor (n_1+n_2)^2 - (n_1+n_2)n + \frac{1}{2}n^2 + 3n - \frac{7}{2} \rfloor.$$

Now $2 \leq n_1 + n_2 \leq n-4$. Elementary calculus shows that the right hand side of the upper bound on $d(v)$ is maximized if and only if $n_1 + n_2 = 2$. Substituting this yields

$$d(v) \leq \lceil \frac{1}{2}n^2 + n \rceil,$$

as desired.

Now assume that the bound holds with equality. Then $n_1 = n_2 = 1$. In conjunction with $n_3 = 1$ and (5), it follows that either $T = T_n$, or n is even and $T = T^1_{n-1}$. □

Lemma 5 *Let T be a strong bipartite tournament of order $n \geq 4$ and diameter at most 4. Then*

$$d(T) \leq 4n^2 - 16n + 24.$$

Equality holds if and only if $T = T_5^{n-5}$.

Proof. Since T is bipartite, the diameter of T is at least 3.

CASE 1: $\mathrm{diam}(T) = 3$.

Let $u, w \in V(T)$. If u and w are in the same partition set, then $d(u,w) = d(w,u) = 2$. If u and w are in distinct partition sets, then either $d(u,w) = 1$ and $d(w,u) = 3$, or $d(u,w) = 3$ and $d(w,u) = 1$. In either case we have $d(u,w) + d(w,u) = 4$ and thus

$$d(T) = 4\binom{n}{2} = 2n(n-1) \leq 4n^2 - 16n + 24.$$

Equality holds if and only if $n = 4$, which implies $T = T_4$, since T_4 is the only strong bipartite tournament of order 4.

CASE 2: $\mathrm{diam}(T) = 4$.

Let A and B be the partition sets of T and let a and b be their respective cardinalities. Since two vertices u and w are in distinct partition sets if and only if $d(u,w) = 1$ and $d(w,u) = 3$ (or vice versa), we have

$$D_1 = D_3 = ab \text{ and } D_2 + D_4 = n(n-1) - 2ab.$$

Therefore,

$$d(T) = \sum_{i=1}^{4} iD_i = 1ab + 2D_2 + 3ab + 4(n(n-1) - 2ab - D_2) = 4n^2 - 4n - 4ab - 2D_2. \tag{6}$$

Let v be a vertex of out-eccentricity 4. Without loss of generality we can assume that $v \in A$. We prove that

$$D_2 \geq n - 2 + a. \tag{7}$$

Let $V_i = V_i(v)$ for $i \geq 0$. Then $A = V_0 \cup V_2 \cup V_4$ and $B = V_1 \cup V_3$. For $i \geq 2$ each vertex $x \in V_i$ has a vertex $y \in V_{i-2}$ with $d(y, x) = 2$. Also $d(x, y) = 2$ for all $x \in V_i$ and all $y \in V_{i-2}$ for $i \geq 2$. Finally, there is a vertex $x \in V_2$ which is adjacent to some vertex in V_3, hence $d(x, v) = 2$ for this $x \in V_2$. Since all these pairs of vertices are distinct, we have

$$D_2 \geq n_2 + n_3 + n_4 + n_1 n_3 + n_2 n_4 + 1 = n + n_1(n_3 - 1) + n_2 n_4.$$

Now $n_2, n_4 \geq 1$ and $n_2 + n_4 = a - 1$. Subject to these conditions, $n_2 n_4$ is minimized if $\{n_2, n_4\} = \{1, a-2\}$. Hence $n_2 n_4 \geq a-2$. Since $n_1(n_3-1) \geq 0$, (7) follows.
In conjunction with (6) we obtain

$$d(T) \leq 4n(n-1) - 4ab - 2(n - 2 + a) = 4n^2 - 6n + 4 - 2a(2b + 1). \tag{8}$$

Now $a \geq 3$, $b \geq 2$ and $a + b = n$. Subject to these conditions, $a(2b + 1)$ is minimized if $a = n - 2$ and $b = 2$. Hence $2a(2b + 1) \geq 10n - 20$ and thus

$$d(T) \leq 4n^2 - 16n + 24, \tag{9}$$

as desired.

Now let T be a bipartite tournament of diameter at most 4 with $d(T) = 4n^2 - 16n + 24$. Then (8) holds with equality, hence $b = 2$, $a = n - 2$ and $\{n_2, n_4\} = \{1, n - 4\}$. Now $b = 2$ implies $n_1 = n_3 = 1$. If $n_2 = 1$ and $n_4 = n - 4$ then $T = T_5^{n-5}$. If $n_2 = n - 4$ and $n_4 = 1$, then let $V_4 = \{w\}$. It is easy to check that $n_1(w) = n_2(w) = n_3(w) = 1$ and $n_4(w) = n - 4$, and thus $T = T_5^{n-5}$. □

Theorem 3 *Let T be a strong bipartite tournament of order $n \geq 4$.*
(i) If $4 \leq n \leq 12$ then

$$\mu(T) \leq \frac{4n^2 - 16n + 24}{n(n-1)},$$

with equality if and only if $T = T_5^{n-5}$.
(ii) If $n \geq 13$, then

$$\mu(T) \leq \begin{cases} \frac{n}{6} + \frac{11}{12} + \frac{7n-8}{4n(n-1)} & \textit{if } n \textit{ is even,} \\ \frac{n}{6} + \frac{11}{12} + \frac{28n-7}{16n(n-1)} & \textit{if } n \textit{ is odd.} \end{cases}$$

Equality holds if and only if $T = T_n$.

Proof. We prove the equivalent upper bounds on $d(T)$, obtained from the right hand sides above by multiplying by $n(n-1)$.
The proof is by induction on n. Clearly, T_4 is the only strong bipartite tournament of order 4, hence the statement holds for $n = 4$.

CASE 1: $n \leq 10$.

If $\mathrm{diam}(T) \leq 4$, then the statement follows from Lemma 5, so we can assume that $\mathrm{diam}(T) \geq 5$. By Lemma 1, T contains a vertex x whose removal leaves $T - x$ strong. By our induction hypothesis and Lemma 4

$$d(T) \leq d(T-x) + d_T(x) \leq d(T_5^{n-6}) + 8n - 20 = d(T_5^{n-5}).$$

Equality implies that $d_T(v) = 8n-20$ and $T - x = T_5^{n-6}$, hence $T = T_5^{n-5}$.

CASE 2: $n \in \{11, 12, 13\}$.

If $\mathrm{diam}(T) \leq 4$, then the statement follows from Lemma 5.
CASE 2A: $5 \leq \mathrm{diam}(T) \leq 7$.
Let $v, x \in V(T)$ with $d(v,x) = \mathrm{diam}(T)$. By Lemma 1, $T - x$ is strong. Hence

$$d(T) \leq d(T-x) + d_T(x).$$

Applying the induction hypothesis to $T - x$ and Lemma 3 to $d_T(x)$ yields $d(T) \leq 264 + 64 < 332 = d(T_5^6)$ for $n = 11$, $d(T) \leq 332 + 72 \leq 408 = d(T_5^7)$ for $n = 12$ and $d(T) \leq 408 + 84 < 492 = d(T_{13})$ for $n = 13$.
CASE 2B: $8 \leq \mathrm{diam}(T) \leq n-1$.
Let $\mathrm{diam}(T) = n - k$. Then a shortest path between two diametral vertices induces a strong subtournament $\hat{T}$ isomorphic to T_{n-k+1}. Let $u_1, u_2, \ldots, u_{k-1}$ be the vertices of T not in $\hat{T}$. We show that, if $\mathrm{diam}(T) > 5$, then, possibly after renumbering of $u_1, u_2, \ldots, u_{k-1}$,

$$T - u_1, T - \{u_1, u_2\}, \ldots, T - \{u_1, \ldots, u_{k-1}\} \text{ are strong.} \tag{10}$$

Suppose not. Clearly, $T - \{u_1, \ldots, u_{k-1}\} = \hat{T}$ is strong. Renumber $u_1, \ldots, u_{k-1}$ such that $T - \{u_1, \ldots, u_{k-1}\}$, $T - \{u_1, \ldots, u_{k-2}\}$, $\ldots$, $T - \{u_1, \ldots, u_j\}$ are strong, but there is no $u_i \in \{u_1, \ldots, u_j\}$ such that $T - (\{u_1, \ldots, u_j\} - \{u_i\})$ is strong. Then each $u_i \in \{u_1, \ldots, u_j\}$ is either adjacent to all vertices in a partition set of $T - \{u_1, \ldots, u_j\}$ or adjacent from all vertices in a partition set of $T - \{u_1, \ldots, u_j\}$. Since T is strong, there exist vertices $u_i, u_i' \in \{u_1, \ldots, u_j\}$ such that u_i is adjacent to all vertices in a partition set of $T - \{u_1, \ldots, u_j\}$ and u_i' is adjacent from all vertices in a partition set of $T - \{u_1, \ldots, u_j\}$. Choosing u_i and u_i' at minimum distance in T yields that also $u_i'u_i \in E(T)$. Hence, in T, there is a path of length at most 5, containing $u_i'u_i$, between any two vertices of $T - \{u_1, \ldots, u_j\}$, a contradiction to $\mathrm{diam}(T) > 5$. This proves (10).

Let $T(0) := T$ and $T(i) = T - \{u_1, u_2, \ldots, u_i\}$ for $i = 1, 2, \ldots, k-1$. Then

$$d(T) \leq d(T(k-1)) + d_{T(k-2)}(u_{k-1}) + d_{T(k-3)}(u_{k-2}) + \ldots + d_T(u_1).$$

Since $\mathrm{diam}(T) \geq 8$, each $T(i)$ has at least 10 vertices. By Lemma 4, we have $d_{T(i)}(u_{i+1}) \leq \lceil \frac{1}{2}(n-i)^2 + (n-i) \rceil$. Hence, since $T(k-1) = T_{n-k+1}$,

$$d(T) \leq d(T_{n-k+1}) + \sum_{i=0}^{k-2} d_{T(i)}(u_{i+1}) \leq d(T_{n-k+1}) + \sum_{i=0}^{k-2} \lceil \frac{1}{2}(n-i)^2 + (n-i) \rceil.$$

It is easy to prove by induction on k that the last term equals $d(T_n)$. Hence $d(T) \leq d(T_n)$. Now $d(T_n) < d(T_5^{n-5})$ for $n = 11, 12$. For $n = 13$, equality above implies by Lemma 4 that either $T = T_{13}$ (if $\mathrm{diam}(T) = 12$) or $d_{T(0)}(u_1) = \lceil \frac{1}{2}(13)^2 + 13 \rceil$ and thus, by Lemma 4, either $T = T_{13}$ or $T = T_{12}^1$. Since $d(T_{12}^1) < d(T_{13})$, equality holds only for T_{13}.

CASE 3: $n \geq 14$.

If $\mathrm{diam}(T) \leq 4$ then $d(T) \leq 4n^2 - 16n + 24 < d(T_n)$, so we can assume that T has diameter at least 5. By Lemma 1, there exists a vertex x of T such that $T - x$ is strong. Applying our induction hypothesis and Lemma 4 yields

$$d(T) \leq d(T - x) + d_T(x) \leq d(T_{n-1}) + \lceil \frac{1}{2}n^2 - n \rceil = d(T_n),$$

as desired.
Equality implies that $d_T(x) = \lceil \frac{1}{2}n^2 - n \rceil$ and thus, by Lemma 4, that $T = T_n$ or $T = T_{n-1}^1$. Since $d(T_{n-1}^1) < d(T_n)$, equality implies $T = T_n$. □

We conclude this paper with the observation that Theorem 2 holds in a slightly weaker form also for bipartite k-connected tournaments. The proof is omitted since it is almost identical to the proof of Theorem 2 in [1].

Theorem 4 [1] *Let $k \geq 1$. If T is a k-connected bipartite tournament of order n, then*

$$\mu(T) < \frac{n}{6k} + \frac{25}{6} + \frac{k}{n},$$

and this bound is, apart from an additive constant, best possible. □

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