GRACEFUL LABELING OF A FAMILY OF QUASISTARS WITH PATHS IN ARITHMETIC PROGRESSION

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The problem of graceful labeling of a particular class of trees called quasistars is considered. Such a quasistar is a tree Q with k distinct paths with lengths $1, d+1, 2d+1, ..., (k-1)d+1$ joined at a unique vertex θ. The k paths of Q intersect only in the vertex θ.
Thus, Q has $1+[1+(d+1)+(2d+1)+...+(k-1)d+1)] = 1+k+\frac{k(k-1)d}{2}$ vertices. The k paths of Q have lengths in arithmetic progression with common difference d. It is shown that Q has a graceful labeling for all k $\leq$ 6 and all values of d.
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Introduction:
Labeling of a graph is an assignment of a unique label to each of the vertices of a graph. Let G be a graph with n vertices. Let $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of G and E(G) the edge set of G. A bijective mapping $f: V(G) \rightarrow \{1, 2, ..., n\}$ is said to be a (vertex) labeling of graph G . A labeling $f: V(G) \rightarrow \{0, 1, 2, ..., n-1\}$ is called graceful and G is called a gracefully labeled graph if the induced edge labels $\{| f(v_i) - f(v_j) | : (v_i, v_j) \in E(G)\}$ are all distinct and take all values in the set $\{1, 2, \cdots, n-1\}$. The term graceful labeling was first introduced by S.W.Golomb. A tree T is a simple connected graph (without loops or multiple edges) which has no cycles. The Graceful Tree Conjecture (Kotzig-Rosa) states that every tree is graceful. This conjecture is widely believed to be true and some classes of trees are shown to have a graceful labeling. Rosa[3] proved that all paths and caterpillars are graceful. A k-quasistar T is a tree with k paths (of various lengths) rooted at a unique common vertex and there are no other vertices in T. Such a quasistar is called a balanced quasistar if the lengths of any two paths differ by at most one. A recent result of R.Badrinath and P.Panigrahi [2] proves that a balanced quasistar has a graceful labeling. We give a graceful labeling for a k-path quasi-star in which the lengths of the k paths increase in arithmetic progression by a fixed difference d $\in$ N.

Fix positive integers k and d. By a k-paths quasistar we mean a tree $T_{k,d} = T$ with the following properties:

1. T has k paths P_i meeting in a common vertex θ.
   $\cap_{i=1}^{k} V(P_i) = \{\theta\}$.

2. Length of path P_i of T denoted by $\ell(P_i)$ is $\ell(P_i) = 1 + (i-1)d$ where $i = 1, 2, \cdots, k$. $d \in$ N, d fixed.

The aim of this paper is to prove :
Theorem: Let $T_{k,d}$ denote a quasistar with k paths commonly attached to a vertex θ such that the paths have lengths $1 + (i-1)d$ where $i = 1, 2, \ldots, k$ and $k \leq 6$. Then $T_{k,d}$ has a graceful labeling.
Proof: We give a graceful labeling for the cases where $k \leq 6$. The number of vertices of tree $T_{k,d}$ is given by n = $1 + 1 + (1+d) + (1+2d) + +(1+(k-1)$d. Let $N = n - 1 = k + \frac{(k-1)kd}{2}$.
We obtain a layout of the tree by rooting at the vertex θ and paths flow vertically from the root θ with paths arranged in increasing size from left to right with increment d. The labeling of tree T can be described by constructing a matrix so that the k paths are written as k columns of the matrix. We construct matrix $L_{(k-1)d \times k} = L$ such that it contains distinct values from the set $\{1, 2, 3, ..., n\}$. The values in the matrix L will be the labels given to the vertices of the tree $T_{k,d}$. The common vertex θ is assigned the value N. The first row of the matrix will have the value N and the subsequent rows will have the remaining values.
The matrix L can be described in the following manner.
L = $[\ L_r\ ]$ where $r = 1, 2, 3, \ldots, k$ and L_r is the submatrix of L having the first $r-1$ columns blank. Thus L_1 consists of the k vertices at distance 1 from the root θ (i.e. neighbours of θ) and L_2 consists of the vertices on the paths $P_2, \ldots, P_k$ (in that order) that are distance $\leq d+1$ from the root θ. (The path P_1 has no such vertices and so on. For example in table 1.1 we have three columns that respectively correspond to three paths length 1, 1+d and 2+d.

Further R refers to the row of θ.
<u>The Labeling:</u> We consider the case $k = 3$.
Tree $T_{k,d}$ is a quasistar with 3 paths of length 1, 1+d and 1+2d.
$N = 3 + 3d$. The case d = 1 requires separate consideration.

$$L = \begin{pmatrix} 6 & 6 & 6 \\ 0 & 1 & 2 \\ & 3 & 5 \\ & & 4 \end{pmatrix}$$

For the case d=1 the edge differences are as follows:

$$D = \begin{pmatrix} 6 & 5 & 4 \\ & 2 & 3 \\ & & 1 \end{pmatrix}$$

Thus for d=1 tree $T_{3,1}$ has a graceful labeling. Now let d $\geq$ 2.
Table 1.1 gives a labeling for the tree $T_{3,d}$ which will be shown to be graceful.

R	$3d+3$	$3d+3$	$3d+3$	
R_0	0	1	2	
R_{2j-1}		$3d-3j+3$	$3d-3j+5$	$j = 1,2,3,\cdots,\lceil\frac{d}{2}\rceil$
R_{2j}		$3j$	$3j+1$	$j = 1\ 2,3,\cdots,\lfloor\frac{d}{2}\rfloor$
R_{2j-1}			$3d-3j+5$	$j = \lfloor\frac{d+3}{2}\rfloor,\cdots,d$
R_{2j}			$3j+1$	$j = \lceil\frac{d+1}{2}\rceil,\cdots\ d$

Table 1.1

$$L_{2d\times 3} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 3d+3 & 3d+3 & 3d+3 \\ 0 & 1 & 2 \end{pmatrix}_{2\times 3}$$

$$L_2 = \left(\begin{array}{ccc|l} \diamond & 3d-3j+3 & 3d-3j+5 & j = 1,2,3,\cdots,\lceil\frac{d}{2}\rceil \\ \diamond & & & \\ \diamond & 3j & 3j+1 & j = 1,2,3,\cdots,\lfloor\frac{d}{2}\rfloor \\ \diamond & & & \end{array}\right)_{d\times 3}$$

$$L_3 = \left(\begin{array}{ccc|l} \diamond & \diamond & 3d-3j+5 & j = \lfloor\frac{d+3}{2}\rfloor,\cdots,d \\ \diamond & \diamond & & \\ \diamond & \diamond & 3j+1 & j = \lceil\frac{d+1}{2}\rceil,\cdots,d \end{array}\right)_{d\times 3}$$

In the matrix L_1 the first row corresponds to the label given to the common vertex θ and the second row has distinct values. The values in odd rows of the matrix L_2 are of the form $3d-3j+p$ where $p = 3,5$ and $j = 1,2,3,\cdots,\lceil\frac{d}{2}\rceil$. The values are obviously distinct. In L_2 the entries in the even rows are of the form $3j+q$ where $q = 0,1$ and $j = 1,2,3,\cdots,\lfloor\frac{d}{2}\rfloor$.
We now prove that the values in columns are distinct.
If $3d-3j$+p= $3j+q$ then the congruence $3d-6j+p-q \equiv 0\ mod(3)$ is not true for $d > 2$.
Also values in L_1 and L_2 are distinct as $3+3d \neq 3d-3j+3$ for any $j = 1,2,3,\cdots,\lceil\frac{d}{2}\rceil$.

Further the values in odd rows of L_2 are of the form 3d-3j+p where $p = 3, 5$ and $j = 1, 2, 3, \cdots, \lceil \frac{d}{2} \rceil$ and the values in the odd rows of L_3 are $3d - 3j' + 3$ for any $j' = \lfloor \frac{d+3}{2} \rfloor, \cdots, d$.
$3d - 3j + p \neq 3d - 3j' + 3$ for any j, j'.

Similarly the values in the even rows of L_2 and L_3 are $3j + 1$ where $j = 1, 2, 3, \cdots, \lfloor \frac{d}{2} \rfloor$ and $3j' + 1$ where $j' = \lceil \frac{d+1}{2} \rceil, \cdots, d$ respectively and clearly $3j + 1 \neq 3j' + 1$ for any j, j'.
Thus the values in L_1, L_2 and L_3 are distinct.
Each value in the matrix L is associated with a vertex in tree $T_{k,d}$.
We now show that the differences obtained from the labeling given above is graceful.
We construct matrix $D_{(k-1)d \times k} = D$ which gives edge differences for the labeling of the tree described above. The common vertex θ is assigned the value N. $| R - R_0 |=| \; 3+3d \;\; 2+3d \;\; 1+3d \; |$ $| R_1 - R_0 | = | \; 3d-1 \;\; 3d \; |$

¿From L_2 we calculate the absolute difference :

$$| R_{2j} - R_{2j-1} | = | \; 3d - 6j + 3 \quad 3d - 6j + 4 \; |$$
$$| R_{2j+1} - R_{2j} | = | \; 3d - 6j \quad 3d - 6j + 1 \; |$$

¿From L_3 we calculate the absolute difference :

$$| R_{2j} - R_{2j-1} | = | \; 3d - 6j + 4 \; |$$
$$| R_{2j+1} - R_{2j} | = | \; 3d - 6j + 1 \; |$$

The matrix D can be described in the following manner. D= $\left[\; D_r \; \right]$ where r=1,2, 3 and D_r has first r-1 entries blank.
For the case k=3 the matrix $D = [D_r]$ where r=1,2,3. Specifically D is:

D_1	$R - R_0$	$3d+3$	$3d+2$	$3d+1$	
	$R_1 - R_0$	$\diamond$	$3d-1$	$3d$	
D_2	$R_{2j} - R_{2j-1}$	$\diamond$	$3d-6j+3$	$3d-6j+4$	$j = 1, 2, 3, \cdots, \lfloor \frac{d}{2} \rfloor$
	$R_{2j+1} - R_{2j}$	$\diamond$	$3d-6j$	$3d-6j+1$	$j = 1, 2, 3, \cdots, \lfloor \frac{d-1}{2} \rfloor$
D_3	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$3d-6j+4$	$j = \lceil \frac{d+1}{2} \rceil, \cdots, d$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$3d-6j-1$	$j = \lceil \frac{d}{2} \rceil, \cdots, d-1$

$_{1 \times 3}$

We now show that the values in D_1, D_2 and D_3 are distinct. The values in D_1 are 3d+3, 3d+2, 3d+1, 3d, 3d-1 which are obviously distinct. The values in D_2 are 3d-6j+p where p=3, 4 and $j = 1, 2, \cdots, \lfloor \frac{d}{2} \rfloor$ and $3d-6j+p'$ where p'=0, 1 and $j = 1, 2, \cdots, \lfloor \frac{d-1}{2} \rfloor$. The values in D_3 are $3d-6j'$+4 where $j' = \lceil \frac{d+1}{2} \rceil, \cdots$,d and 3d-6j'+1 where $j' = \lceil \frac{d}{2} \rceil, \cdots, d-1$. 3d-6j+p $\neq$ 3d-6j'+4 for any j, j'. Also 3d-6j+p $\neq$ 3d-6j'+1 for any j, j'. The values 3d-6j+p' $\neq$ $3d-6j$+1 for any j, j'. Comparing the values in D_1, D_2 and D_3 the values are distinct as j takes different values in each D_r where r=1,2,3. Therefore matrix D has distinct values. In conclusion, the tree $T_{3,d}$ has a graceful labeling, for every $d \in$ N.

Case 2: $k = 4$.

Tree $T_{4,d}$ is a quasistar with 4 paths of length 1, 1+d, 1+2d and 1+3d. Also, $N = 4 + 6$d.

Table 2.1 gives a labeling (which will be shown to be graceful). R_{2j} and R_{2j-1} are respectively the even rows and odd rows of the label matrix L.

Table 2.1

R	$6d+4$	$6d+4$	$6d+4$	$6d+4$	
R_0	0	1	2	3	
R_{2j-1}		$6d-3j+2$	$6d-3j+4$	$6d-3j+6$	$j = 1, 2, 3, \cdots, \lceil \frac{d}{2} \rceil$
R_{2j}		$3j+1$	$3j+2$	$3j+3$	$j = 1, 2, 3, \cdots, \lfloor \frac{d}{2} \rfloor$
R_{2j-1}			$6d-3j+4$	$6d-3j+6$	$j = \lfloor \frac{d+3}{2} \rfloor, \cdots, d$
R_{2j}			$3j+2$	$3j+3$	$j = \lceil \frac{d+1}{2} \rceil, \cdots, d-1$
R_{2d}			$6d+2$	$3d+3$	
R_{2j-1}				$6d-3j+4$	$j = d+1, d+2, \cdots, \lfloor \frac{3d+1}{2} \rfloor$
R_{2j}				$3j-1$	$j = d+1, d+2, \cdots, \lceil \frac{3d-1}{2} \rceil$

$$L_{2d\times 3} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 4+6d & 4+6d & 4+6d & 4+6d \\ 0 & 1 & 2 & 3 \end{pmatrix}_{2\times 4}$$

$$L_2 = \left(\begin{array}{cccc|l} \diamond & 6d-3j+2 & 6d-3j+4 & 6d-3j+6 & j=1,2,3,\cdots,\lceil \frac{d}{2}\rceil \\ \diamond & 3j+1 & 3j+2 & 3j+3 & j=1,2,3,\cdots,\lfloor \frac{d}{2}\rfloor \end{array}\right)_{d\times 4}$$

$$L_3 = \left(\begin{array}{cccc|l} \diamond & \diamond & 6d-3j+4 & 6d-3j+6 & j=\lfloor \frac{d+3}{2}\rfloor,\cdots,d \\ \diamond & \diamond & 3j+2 & 3j+3 & j=\lceil \frac{d+1}{2}\rceil,\cdots,d \\ \diamond & \diamond & 6d+2 & 3d+3 & \end{array}\right)_{d\times 4}$$

$$L_4 = \left(\begin{array}{cccc|l} \diamond & \diamond & \diamond & 6d-3j+4 & j=d+1,d+2,\cdots,\lfloor \frac{3d+1}{2}\rfloor \\ \diamond & \diamond & \diamond & 3j-1 & j=d+1,d+2,\cdots,\lceil \frac{3d-1}{2}\rceil \end{array}\right)_{d\times 4}$$

In the matrix L_1 the first row corresponds to the label given to the common vertex θ and the second row has distinct values. The values in L_1 are distinct. The values in the odd rows of L_2 are $6d-3j+p$ where $p = 2,4,6$ and $j = 1,2,3,\cdots,\lceil \frac{d}{2}\rceil$ which are obviously distinct. We also observe that L_2 has distinct values in the even rows which are 3j+q where q=1,2,3 and $j = 1,2,3,\cdots,\lfloor \frac{d}{2}\rfloor$.

We now prove that the values in L_1, L_2, L_3 and L_4 are distinct.

Considering the values in L_2:

If $6d-3j+p = 3j+q$ then the congruence $6d-6j+p-q \equiv 0 \bmod(6)$ is not true for any $d \in \mathbb{N}$.

The values in the odd rows of L_3 are $6d-3j+p$ where $p = 4,6$ and $j = \lfloor \frac{d+3}{2}\rfloor,\cdots,d$ which are distinct.

We also observe that L_3 has distinct values in the even rows which are 3j+q where q=2,3 and $j = \lceil \frac{d+1}{2}\rceil,\cdots,d-1$. Also $6d+2$ and $3d+3$ is distinct from any value in L_3.

The values in odd rows of L_2 are of the form 6d-3j+p where p=2,4,6 and j=$1,2,3,\cdots,\lceil \frac{d}{2}\rceil$ and in L_3 the values in the odd rows are 6d-3j′+p′ where p′=4,6 and j′=$\lfloor \frac{d+3}{2}\rfloor,\cdots,d$.

The values 6d-3j+p $\neq$ 6d-3j′+p′ for any j, j′. Similarly the values in the even rows of L_2 are 3j+q where q=1,2,3 and j= $1,2,3,\cdots,\lfloor \frac{d}{2}\rfloor$ and the values in the even rows of L_3 are $3j'$+q′ where q'=2,3 and j'=$\lceil \frac{d+1}{2}\rceil,\cdots,d$. 3j+q $\neq 3j'+q'$ for any j, j'.

Further the values in odd rows of L_3 are $6d - 3j + p$ where $p = 4, 6$ and $j = \lfloor \frac{d+3}{2} \rfloor, \cdots, d$ and the values in the odd rows of L_4 are $6d - 3j'+4$ where $j' = d+1, d+2, \cdots, \lfloor \frac{3d+1}{2} \rfloor$.
Therefore $6d - 3j + p \neq$ 6d-3j'+3 for any j, j'.

Similarly the values in the even rows of L_3 are $3j + 3$ where $j = \lceil \frac{d+1}{2} \rceil, \cdots$,d-1 and values in L_4 are 3j'-1 where $j' = d+1, d+2, \cdots, \lceil \frac{3d-1}{2} \rceil$ respectively. Therefore $3j + 3 \neq$ 3j'-1 for any j, j'. Thus the values in L_1, L_2, L_3 and L_4 are distinct. Each value in the matrix L is associated with a vertex in tree $T_{4,d}$. We now show that the edge differences obtained from the given above labeling are distinct and therefore the labeling is graceful.

$| R - R_0 | = | 6d+4 \quad 6d+3 \quad 6d+2 \quad 6d+1 |$
$| R_1 - R_0 | = | 6d-2 \quad 6d-1 \quad 6d |$

¿From L_2 we calculate the absolute difference :
$| R_{2j} - R_{2j-1} | = | 6d-6j+1 \quad 6d-6j+2 \quad 3d-6j+3 |$
$| R_{2j+1} - R_{2j} | = | 6d-6j-2 \quad 6d-6j-1 \quad 6d-6j |$

¿From L_3 we calculate the absolute difference :

$| R_{2j} - R_{2j-1} | = | 6d-6j+2 \quad 6d-6j+3 |$
$| R_{2j+1} - R_{2j} | = | 6d-6j-1 \quad 6d-6j |$

¿From L_4 we calculate the absolute difference :

$| R_{2j} - R_{2j-1} | = | 6d-6j+5 \quad 6d-6j+3 |$
$| R_{2j+1} - R_{2j} | = | 6d-6j+2 \quad 6d-6j |$

The matrix $D = [D_r]$ can be described in the following manner. Each D_r where r=1,2, 3,4 has the first r-1 columns blank.

Specifically D is:

D_1	$R - R_0$ $R_1 - R_0$	$6d+4$	$6d+3$ $6d-2$	$6d+2$ $6d-1$	$6d+1$ $6d$	
D_2	$R_{2j} - R_{2j-1}$	$\diamond$	$6d-6j+1$	$6d-6j+2$	$6d-6j+3$	$j = 1, 2, \cdots, \lceil \frac{d-1}{2} \rceil$
	$R_{2j+1} - R_{2j}$	$\diamond$	$6d-6j-2$	$6d-6j-1$	$6d-6j$	$j = 1, 2, \cdots, \lceil \frac{d-2}{2} \rceil$
D_3	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$6d-6j+2$	$6d-6j+3$	$j = \lceil \frac{d+2}{2} \rceil, \cdots, d-1$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$6d-6j-1$	$3d-6j$	$j = \lceil \frac{d}{2} \rceil, \cdots, d-1$
D_4	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$6d-6j+5$	$j = d+1, d+2, \cdots, \lfloor \frac{3d}{2} \rfloor$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$\diamond$	$6d-6j+2$	$j = d, d+1, \cdots, \lfloor \frac{3d-1}{2} \rfloor$

$d \times 4$

Therefore for k=4 and any d $\in$ N the tree $T_{4,d}$ has a graceful labeling.

<u>The Labeling:</u>
We consider the case $k = 5$.
Tree $T_{k,d}$ is a quasistar with 5 paths of length 1, 1+d, 1+2d, 1+3d and 1+4d.
$N = 5 + 10d$. Table 3.1 gives a graceful labeling for $k = 5$.
R_{2j} and R_{2j-1} are the even rows and odd rows of the label matrix L respectively.

Table 3.1

R	$5+10d$	$5+10d$	$5+10d$	$5+10d$	$5+10d$	
R_0	$5j$	$5j+1$	$5j+2$	$5j+3$	$5j+4$	$j=0$
R_{2j-1}		$10d-5j+3$	$10d-5j+5$	$10d-5j+7$	$10d-5j+9$	$j=1,\cdots,\lceil\frac{d}{2}\rceil$
R_{2j}		$5j$	$5j+1$	$5j+2$	$5j+3$	$j=1,\cdots,\lfloor\frac{d}{2}\rfloor$
R_{2j-1}			$10d-5j+5$	$10d-5j+7$	$10d-5j+9$	$j=\lfloor\frac{d+3}{2}\rfloor,\cdots,d$
R_{2j}			$5j+1$	$5j+2$	$5j+3$	$j=\lfloor\frac{d+1}{2}\rfloor,\cdots,d-1$
R_{2d}			$10d+3$	$5d+1$	$5d+4$	
R_{2d+1}				$5d$	$5d+2$	
R_{2j-1}				$10d-5j+5$	$10d-5j+9$	$j=d+2,\cdots,\lceil\frac{3d}{2}\rceil$
R_{2j}				$5j-2$	$5j+1$	$j=d+1,\cdots,\lfloor\frac{3d}{2}\rfloor$
R_{2j-1}					$10d-5j+9$	$j=\lceil\frac{3d+2}{2}\rceil,\cdots,2d$
R_{2j}					$5j+1$	$j=\lfloor\frac{3d+2}{2}\rfloor,\cdots,2d$

$$L_1 = \begin{pmatrix} 5+10d & 5+10d & 5+10d & 5+10d & 5+10d \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}_{2\times 5}$$

$$L_2 = \left(\begin{array}{ccccc|l} \diamond & 10d-5j+3 & 10d-5j+5 & 10d-5j+7 & 10d-5j+9 & j=1,2,3,\cdots,\lceil\frac{d}{2}\rceil \\ \diamond & 5j & 5j+1 & 5j+2 & 5j+3 & j=1,2,3,\cdots,\lfloor\frac{d}{2}\rfloor \end{array}\right)_{d\times 5}$$

$$L_3 = \left(\begin{array}{ccccc|c} \diamond & \diamond & 10d-5j+5 & 10d-5j+7 & 10d-5j+9 & j=\lfloor\frac{d+3}{2}\rfloor,\cdots,d \\ \diamond & \diamond & 5j+1 & 5j+2 & 5j+3 & j=\lfloor\frac{d+1}{2}\rfloor,\cdots,d-1 \\ \diamond & \diamond & 10d+3 & 5d+1 & 5d+4 & \end{array}\right)_{d\times 5}$$

$$L_4 = \left(\begin{array}{ccccc|l} \diamond & \diamond & \diamond & 5d & 5d+2 & \\ \diamond & \diamond & \diamond & 10d-5j+5 & 10d-5j+9 & j=d+2,\cdots,\lceil\frac{3d}{2}\rceil \\ \diamond & \diamond & \diamond & 5j-2 & 5+-1 & j=d+1,\cdots,\lfloor\frac{3d}{2}\rfloor \end{array}\right)_{d\times 5}$$

$$L_5 = \left(\begin{array}{ccccc|c} \diamond & \diamond & \diamond & \diamond & 10d-5j+9 & j=\lceil \frac{3d+2}{2} \rceil, \cdots, 2d \\ & & & & 5+-1 & j=\lfloor \frac{3d}{2} \rfloor, \cdots, 2d \end{array} \right)_{d\times 5}$$

In the matrix L_1 the first row corresponds to the label given to the common vertex θ and the second row has distinct values. The values in odd rows of the matrices L_r where $r = 2,3,4,5$ are of the form $10d - 5j + p$ where $p = 3,5,7,9$ and $j = 1,2,3,\cdots, \frac{d}{2}$. The values are obviously distinct. In L_2 the entries in the even rows are of the form $3j + q$ where $q = 0,1,2,3$ and $j = 1,2,3,\cdots,\frac{d}{2}$. We now prove that the values in columns are distinct.
If $10d - 5j + p = 3j + q$ then the congruence $10d - 8j + p - q \equiv 0\ mod(10)$ is not true for any $d \in N$. Also values in L_1 and L_2 are distinct as $5 + 10d \neq 10d - 5j + p$ for any $p = 3,5,7,9$ and $j = 1,2,3,\cdots, \frac{d}{2}$. Further the values in odd rows of L_2 are of the form $10d - 5j + p$ where $p = 3,5,7,9$ and $j = 1,2,3,\cdots, \frac{d}{2}$ and the values in the odd rows of L_3 are $10d - 5j' + p'$ where $p' = 5,7,9$ and $j' = \frac{d}{2}+1, \frac{d}{2}+2,\cdots, d$. $10d - 5j + p \neq 10d - 5j' + p'$ for any j or j'. Similarly the values in the even rows of L_2 and L_3 are $5j + q$ where $q = 0,1,2,3$ and $j = 1,2,3,\cdots,\frac{d}{2}$ and $5j' + q'$ where $q' = 1,2,3$ and $j' = \frac{d}{2}+1, \frac{d}{2}+2,\cdots, d-1$ respectively $.5j + q \neq 5j' + q'$ for any j or j'. Further the values in odd rows of L_3 are of the form $10d-5j+p$ where $p = 5,7,9$ and $j = \frac{d}{2}+1, \frac{d}{2}+2,\cdots,d$ and the values in the odd rows of L_4 are $10d-5j'+p'$ where $p' = 5,9$ and $j' = d+2,\cdots,\frac{3d}{2}$. $10d - 5j + p \neq 10d - 5j' + p'$ for any j or j'. Similarly the values in the even rows of L_3 and L_4 are 5j+q where $q = 1,2,3$ and $j = \frac{d}{2}+1, \frac{d}{2}+2,\cdots,d-1$ and $5j' + q'$ where $q' = -2,1$ and $j' = d+1, d+2,\cdots,\frac{3d}{2}$ respectively. $5j + q \neq 5j' + q'$ for any j *or* j'. Also $10d + 3, 5d + 1, 5d + 4$ in L_3 are not equal to any value in L_4. The values in the odd rows of L_4 are $10d - 5j + p$ where $p = 5,9$ and $j = d+2,\cdots,\frac{3d}{2}$ and values of the odd rows in L_5 are $10d - 5j' + 9$ where $j' = \frac{3d}{2}+1, \frac{d}{2}+2,\cdots,2d$, therefore $10d - 5j + p \neq 10d - 5j + 9$ for any j *and* j'.
The values in the even rows of L_4 are $5j + q$ where $q = -2,1$ and $j = d+1, d+2,\cdots,\frac{3d}{2}$ and the values in the even rows of L_5 are $5j' + 1$ where $j' = \frac{3d}{2}+1, \frac{3d}{2}+2,\cdots,2d$. Therefore $5j + 1 \neq 5j' + 1$ for any j *and* j'. Thus the values in L_1, L_2, L_3, L_4 and L_5 are distinct.
Each value in the matrix L is associated with a vertex in tree $T_{k,d}$. We now show that the differences obtained from the labeling given above is graceful.
We construct matrix $D_{k-1)d\times k} = D$ which gives disitnct edge differences for the labeling described above.
The values in the matrix D will be the edge differences of the tree $T_{k,d}$. The common vertex θ is assigned the value N. The matrix D can thus be described in the following manner. Each D_r where $r = 1,2,3,4,5$ has $r-1$ entries blank, $D = [\ D_r\ ]$

$$D_1 = \begin{pmatrix} 5+10d & 4+10d & 3+10d & 2+10d & 1+10d \end{pmatrix}_{1\times 5}$$

Note that the differences in D_1 are between R and R_0.

$$| R_1 - R_0 | = | 10d-3 \quad 10d-2 \quad 10d-1 \quad 10d |$$

¿From L_2 we calculate the absolute difference :

$$| R_{2j} - R_{2j-1} | = | \; 10d-10j+3 \quad 10d-10j+4 \quad 10d-10j+5 \quad 10d-10j+6 \, |$$
$$| R_{2j+1} - R_{2j} | = | \; 10d-10j-2 \quad 10d-10j-1 \quad 10d-10j \quad 10d-10j+1 \, |$$

$$D_2 = \left(\begin{array}{c|ccccc|c} R_{2j}-R_{2j-1} & \diamond & 10d-10j+3 & 10d-10j+4 & 10d-10j+5 & 10d-10j+6 & j=1,\cdots,\lfloor \frac{d}{2} \rfloor \\ R_{2j+1}-R_{2j} & \diamond & 10d-10j-2 & 10d-10j-1 & 10d-10j & 10d-10j+1 & j=1,\cdots,\lfloor \frac{d-1}{2} \rfloor \end{array} \right)_{d-1\times 5}$$

¿From L_3 we calculate the absolute difference :

$$| R_{2j} - R_{2j-1} | = | 10d-10j+4 \;\; 10d-10j+5 \;\; 10d-10j+6 \, |$$
$$| R_{2j+1} - R_{2j} | = | 10d-10j-1 \;\; 10d-10j \;\; 10d-10j+1 \, |$$

$$D_3 = \left(\begin{array}{c|ccccc|c} R_{2j}-R_{2j-1} & \diamond & \diamond & 10d-10j+4 & 10d-10j+5 & 10d-10j+6 & j=\lceil \frac{d+1}{2} \rceil,\cdots,d-1 \\ R_{2j+1}-R_{2j} & \diamond & \diamond & 10d-10j-1 & 10d-10j & 10d-10j+1 & j=\lfloor \frac{d+1}{2} \rfloor,\cdots,d-1 \\ R_{2d}-R_{2d-1} & \diamond & \diamond & 5d-2 & 6 & 5 & \end{array} \right)_{d\times 5}$$

¿From L_4 we calculate the absolute difference :

$$| R_{2d+1} - R_{2d} | = | \quad 1 \quad 2 \, |$$

$$| R_{2j} - R_{2j-1} | = | 10d-10j+710d-10j+8 \, |$$
$$| R_{2j+1} - R_{2j} | = | 10d-10j+2 \quad 10d-10j+3 \, |$$

$$D_4 = \left(\begin{array}{c|ccccc|c} R_{2d+1}-R_{2d} & \diamond & \diamond & \diamond & 1 & 2 & \\ R_{2j}-R_{2j-1} & \diamond & \diamond & \diamond & 10d-10j+7 & 10d-10j+8 & j=d+1,\cdots,\lfloor \frac{3d}{2} \rfloor \\ R_{2j+1}-R_{2j-1} & \diamond & \diamond & \diamond & 10d-10j+2 & 10d-10j+3 & j=d,\cdots,\lfloor \frac{3d-1}{2} \rfloor \end{array} \right)_{d\times 5}$$

¿From L_5 we calculate the absolute difference :

$$| R_{2j} - R_{2j-1} | = | 10d-10j+8 \, |$$
$$| R_{2j+1} - R_{2j} | = | 10d-10j+3 \, |$$

$$D_5 = \left(\begin{array}{c|ccccc|c} R_{2j}-R_{2j-1} & \diamond & \diamond & \diamond & \diamond & 10d-10j+8 & j=\lceil \frac{3d+1}{2} \rceil,\cdots,2d \\ R_{2j+1}-R_{2j-1} & \diamond & \diamond & \diamond & \diamond & 10d-10j+3 & j=\lceil \frac{3d}{2} \rceil,\cdots,2d-1 \end{array} \right)_{d\times 5}$$

The matrix $D = [D_r]$ can be described in the following manner. Each D_r where $r = 1,2,3,4,5$ has the first r-1 columns blank.

Specifically D is:

D_1	$R-R_0$	$10d+5$	$10d+4$	$10d+3$	$10d+2$	$10d+1$	
	R_1-R_0		$10d-3$	$10d-2$	$10d-1$	$10d$	
D_2	$R_{2j}-R_{2j-1}$	$\diamond$	$10d-10j+3$	$10d-10j+4$	$10d-10j+5$	$10d-10j+6$	$j=1,\cdots,\lfloor\frac{d}{2}\rfloor$
	$R_{2j+1}-R_{2j}$	$\diamond$	$10d-10j-2$	$10d-10j-1$	$10d-10j$	$10d-10j+1$	$j=1,\cdots,\lfloor\frac{d-1}{2}\rfloor$
D_3	$R_{2j}-R_{2j-1}$	$\diamond$	$\diamond$	$10d-10j+4$	$10d-10j+5$	$10d-10j+6$	$j=\lceil\frac{d+1}{2}\rceil,\cdots,d-1$
	$R_{2j+1}-R_{2j}$	$\diamond$	$\diamond$	$10d-10j-1$	$10d-10j$	$10d-10j+1$	$j=\lfloor\frac{d+1}{2}\rfloor,\cdots,d-1$
	$R_{2d}-R_{2d-1}$	$\diamond$	$\diamond$	$5d-2$	6	5	
D_4	$R_{2d+1}-R_{2d}$	$\diamond$	$\diamond$	$\diamond$	1	2	
	$R_{2j}-R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$10d-10j+7$	$10d-10j+8$	$j=d+1,\cdots,\lfloor\frac{3d}{2}\rfloor$
	$R_{2j+1}-R_{2j}$	$\diamond$	$\diamond$	$\diamond$	$10d-10j+2$	$10d-10j+3$	$j=d,\cdots,\lfloor\frac{3d-1}{2}\rfloor$
D_5	$R_{2j}-R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$10d-10j+8$	$j=\lceil\frac{3d+1}{2}\rceil,\cdots,2d$
	$R_{2j+1}-R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$10d-10j+3$	$j=\lceil\frac{3d}{2}\rceil,\cdots,2d-1$

$d-1\times 5$

We observe that the differences in D_1 are distinct as they are 10d+p where $p=-3,-2,\ldots,5$ and the values in the remaining D_r where $r=2,3,\ldots,5$ are $10d-10j+q$ where $q=-2,-3,\ldots,8$. In each D_r j takes different values so the values $10d-10j+q$ are distinct. Therefore for any $d\in$ N the tree $T_{5,d}$ has a graceful labeling.

Table 4.1 gives a labeling (which will be shown to be graceful) of T_6,d (for any $d\in$ N).

R_{2j} and R_{2j-1} are the even rows and odd rows of the label matrix L .

R_0	0	1	2	3	4	5	
R_{2j-1}		$15d-5j+2$	$15d-5j+4$	$15d-5j+6$	$15d-5j+8$	$15d-5j+10$	$j=1,\cdots,\lceil\frac{d}{2}\rceil$
R_{2j}		$5j+1$	$5j+2$	$5j+3$	$5j+4$	$5j+5$	$j=1,\cdots,\lfloor\frac{d}{2}\rfloor$
R_{2j-1}			$15d-5j+4$	$15d-5j+6$	$15d-5j+8$	$15d-5j+10$	$j=\lfloor\frac{d+3}{2}\rfloor,\cdots,d$
R_{2j}			$5j+2$	$5j+3$	$5j+4$	$5j+5$	$j=\lceil\frac{d+1}{2}\rceil,\cdots,d-1$
R_{2d}				$15d+4$	$5d+3$	$5d+4$	$5d+5$

	Table	4.1	contd				
R_{2j-1}				$15d-5j+10$	$15d-5j+8$	$15d-5j+6$	$j=d+1,\cdots,\lfloor\frac{3d-1}{2}\rfloor$
R_{2j-1}				$15d-5j+10$	$15d-5j+8$	$15d-5j+7$	$j=\lceil\frac{3d}{2}\rceil$ (d odd)
R_{2j-1}				$15d-5j-3$	$15d-5j+6$	$15d-5j+8$	$j=\lceil\frac{3d}{2}\rceil$ (d even)
R_{2j}				$5j+5$	$5j+4$	$5j+3$	$j=d+1,\cdots,\lfloor\frac{3d-1}{2}\rfloor$
R_{2j}				$5j+5$	$5j+10$	$5j+3$	$j=\lceil\frac{3d-1}{2}\rceil$($d$ even)
R_{2j}				$5j+6$	$5j+3$	$5j+4$	$j=\lceil\frac{3d-1}{2}\rceil$($d$ odd)
R_{2j-1}					$15d-5j+6$	$15d-5j+7$	$j=\lceil\frac{3d+2}{2}\rceil$($d$ even)
R_{2j-1}					$15d-5j+6$	$15d-5j+2$	$j=\lceil\frac{3d+2}{2}\rceil$($d$ odd)
R_{2j-1}					$15d-5j+6$	$15d-5j+2$	$j=\lceil\frac{3d+2}{2}\rceil,\cdots,2d$
R_{2j}					$5j+2$	$5j-1$	$j=\lceil\frac{3d+1}{2}\rceil$($d$ even)
R_{2j}					$5j-3$	$5j$	$j=\lceil\frac{3d}{2}\rceil$($d$ odd)
R_{2j}					$5j+2$	$5j-1$	$j=\lceil\frac{3d+2}{2}\rceil,\cdots,2d-1$
R_{2d}					$15d+2$	$10d-1$	
R_{2j-1}						$15d-5j+6$	$j=2d+1,\cdots,\lceil\frac{5d}{2}\rceil$
R_{2j}						$5j-3$	$j=2d+1,\cdots,\lfloor\frac{5d}{2}\rfloor$

The matrix $D=[D_r]$ can be described in the following manner. Each D_r where $r=1,2,3,4$ has the first r-1 columns blank. Specifically D is:

D_1	$R-R_0$ R_1-R_0	$15d+6$	$15d+5$ $15d-4$	$15d+4$ $15d-3$	$15d+3$ $15d-2$	$15d+2$ $15d-1$	$15d+1$ $15d$
D_2	$R_{2j}-R_{2j-1}$	$\diamond$	$15d-10j+1$	$15d-10j+2$	$15d-10j+3$	$15d-10j+4$	$15d-10j+5$ $j=1,\cdots,\lfloor\frac{d}{2}\rfloor$
	$R_{2j+1}-R_{2j}$	$\diamond$	$15d-10j-4$	$15d-10j-3$	$15d-10j-2$	$15d-10j-1$	$10d-10j$ $j=1,\cdots,\lfloor\frac{d-1}{2}\rfloor$
D_3	$R_{2j}-R_{2j-1}$	$\diamond$	$\diamond$	$15d-10j+2$	$15d-10j+3$	$15d-10j+4$	$15d-10j+5$ $j=\lceil\frac{d+1}{2}\rceil,\cdots,d-1$
	$R_{2j+1}-R_{2j}$	$\diamond$	$\diamond$	$15d-10j-3$	$15d-10j-2$	$15d-10j-1$	$15d-10j$ $j=\lfloor\frac{d+1}{2}\rfloor,\cdots,d-1$
	$R_{2d}-R_{2d-1}$	$\diamond$	$\diamond$	$5d$	$5d+3$	$5d+4$	$5d+5$

D_4	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$15d-10j+5$	$15d-10j+4$	$15d-10j+3$ $j=d+1,\cdots,\frac{3d-2}{2}$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$\diamond$	$15d-10j$	$15d-10j-1$	$15d-10j-2$ $j=d+1,\cdots,\frac{3d-4}{2}$
	$R_{2(\frac{3d-2}{2})+1} - R_{2(\frac{3d-2}{2})}$	$\diamond$	$\diamond$	$\diamond$	3	7	10
	$R_{2(\frac{3d}{2})} - R_{2(\frac{3d}{2})-1}$	$\diamond$	$\diamond$	$\diamond$	8	4	5
D_5	$R_{2(\frac{3d}{2})+1} - R_{2(\frac{3d}{2})}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	9	1
	$R_{2(\frac{3d+2}{2})} - R_{2(\frac{3d+2}{2})-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	6	2
	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$15d-10j+4$	$15d-10j+3$ $j=\lceil\frac{3d+4}{2}\rceil,\cdots,2d-1$
	$R_{2j+1} - R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$15d-10j+4$	$15d-10j+3$ $j=\lceil\frac{3d+2}{2}\rceil,\cdots,2d-1$
	$R_{2(2d)} - R_{2(2d)-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$10d-4$	$5d-3$
D_6	$R_{2(2d)+1} - R_{2(2d)}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$5d-2$
	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$15d-10j+9$ $j=2d+1,\cdots,\lfloor\frac{5d}{2}\rfloor$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$15d-10j+4$ $j=2d+1,\cdots,\lfloor\frac{5d}{2}\rfloor$

$d-1\times 6$

We observe that the values in D_1 are distinct. The values in D_2 are $15d-10j+p$ where $p=-4,-3,\ldots,5$ which are clearly distinct. Comparing the values of D_2 with D_r where $r=3,4,5,6$ we observe that even though the values are $15d-10j+p$ for $p=-3,-4,\ldots,5$ and 9 the values in D_r, where $r=2,3,4,5,6$ are distinct as j takes different values in each D_r. Note that the remaining values in the D_r are distinct as they have no j component. Thus all the values in D are distinct and correspond to the edge differences of tree T_6, d. Therefore for $k=6$ and any even $d \in N$ the tree $T_{6,d}$ has a graceful labeling.

D_1	$R-R_0$ R_1-R_0	$15d+6$	$15d+5$ $15d-4$	$15d+4$ $15d-3$	$15d+3$ $15d-2$	$15d+2$ $15d-1$	$15d+1$ $15d$
D_2	$R_{2j} - R_{2j-1}$	$\diamond$	$15d-10j+1$	$15d-10j+2$	$15d-10j+3$	$15d-10j+4$	$15d-10j+5$ $j=1,\cdots,\lfloor\frac{d}{2}\rfloor$
	$R_{2j+1} - R_{2j}$	$\diamond$	$15d-10j-4$	$15d-10j-3$	$15d-10j-2$	$15d-10j-1$	$10d-10j$ $j=1,\cdots,\lfloor\frac{d-1}{2}\rfloor$
D_3	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$15d-10j+2$	$15d-10j+3$	$15d-10j+4$	$15d-10j+5$ $j=\lceil\frac{d+1}{2}\rceil,\cdots,d-1$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$15d-10j-3$	$15d-10j-2$	$15d-10j-1$	$15d-10j$ $j=\lfloor\frac{d+1}{2}\rfloor,\cdots,d-1$
	$R_{2d} - R_{2d-1}$	$\diamond$	$\diamond$	$5d$	$5d+3$	$5d+4$	$5d+5$
D_4	$R_{2j} - R_{2j-1}$	$\diamond$	$\diamond$	$\diamond$	$15d-10j+5$	$15d-10j+4$	$15d-10j+3$ $j=d+1,\cdots,\lfloor\frac{3d-3}{2}\rfloor$
	$R_{2j+1} - R_{2j}$	$\diamond$	$\diamond$	$\diamond$	$15d-10j$	$15d-10j-1$	$15d-10j-2$ $j=d,\cdots,\lfloor\frac{3d-3}{2}\rfloor$
	$R_{2(\frac{3d-1}{2})} - R_{2(\frac{3d-1}{2})-1}$	$\diamond$	$\diamond$	$\diamond$	9	10	7
	$R_{2(\frac{3d-1}{2})+1} - R_{2(\frac{3d-1}{2})}$	$\diamond$	$\diamond$	$\diamond$	4	5	3

			Table		contd		
D_5	$R_{2(\frac{3d+1}{2})} - R_{2(\frac{3d+1}{2})-1}$	◇	◇	◇	◇	6	2
	$R_{2(\frac{3d+1}{2})+1} - R_{2(\frac{3d+1}{2})}$	◇	◇			1	8
	$R_{2j} - R_{2j-1}$	◇	◇	◇	◇	$15d - 10j + 4$	$15d - 10j + 3$ $j = \lceil\frac{3d+3}{2}\rceil, \cdots, 2d-1$
	$R_{2j+1} - R_{2j-1}$	◇	◇	◇	◇	$15d - 10j + 4$	$15d - 10j + 3$ $j = \lceil\frac{3d+3}{2}\rceil, \cdots, 2d-1$
	$R_{2(2d)} - R_{2(2d)-1}$	◇	◇	◇	◇	$10d - 4$	$5d - 3$
D_6	$R_{2(2d)+1} - R_{2(2d)}$	◇	◇	◇	◇	◇	$5d - 2$
	$R_{2j} - R_{2j-1}$	◇	◇	◇	◇	◇	$15d - 10j + 9$ $j = 2d+1, \cdots, \lfloor\frac{5d-1}{2}\rfloor$
	$R_{2j+1} - R_{2j}$	◇	◇	◇	◇	◇	$15d - 10j + 4$ $j = d, \cdots, \lfloor\frac{3d-3}{2}\rfloor$

We observe that the values in D_1 are distinct. The values in D_2 are $15d - 10j + p$ where $p = -4, -3, \ldots, 5$ which are clearly distinct. Comparing the values of D_2 with D_r where $r = 3, 4, 5, 6$ we observe that even though the values are $15d - 10j + p$ for $p = -3, -4, \ldots, 5$ and 9 the values in D_r, where $r = 2, 3, 4, 5, 6$ are distinct as j takes different values in each D_r. Note that the remaining values in the D_r are distinct as they have no j component. Thus all the values in D are distinct and correspond to the edge differences of tree T_6, d. Therefore for $k = 6$ and any odd $d \in$ N the tree $T_{6,d}$ has a graceful labeling.

References:

1. J.A.Gallian *A dynamic survey of graph labeling*, Electronic Journal of Combinatorics, 5(2002), ds6.
2. R. Badrinath and P.Panigrahi *Graceful labeling of Balanced star of Paths.* To appear in Proceedings of R.C.Bose centenary conference, Indian Statistical Institute Kolkatta(2002).
3. A. Rosa *On certain valuations of the vertices of a graph, in Theórie des Graphes,* Dunod, Paris(1968) 349-355,MR 36-6319.