All graphs are set reconstructible if all 2-connected graphs are set reconstructible*

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Abstract

A graph is called set reconstructible if it is determined uniquely (up to isomorphism) by the set of its vertex-deleted subgraphs. We prove that some classes of separable graphs with a unique endvertex are set reconstructible and show that all graphs are set reconstructible if all 2-connected graphs are set reconstructible.

1. Introduction

In this paper all graphs considered are simple. We use the terminology in Harary [4]. The degree of a vertex v of a graph G is denoted by **deg v** (or **$\deg_G$ v**). The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex v with deg v = m is referred to as an **m-vertex**. A 1-vertex is called an **endvertex** and the unique neighbour of a 1-vertex is called its **base**. The degree sequence of a graph G is denoted by **DS(G)**. **NDS(v)** denotes the sequence of degrees of the neighbours (neighbourhood degree sequence) of v in G. Maximal connected nonseparable subgraphs of G are called **blocks** of G. The complement $\overline{G}$ of a graph G is defined as the graph having the same vertex set as G and uw is an edge of $\overline{G}$ iff it is not an edge of G. For any graph G, we define the **pruned graph** of G denoted by **P(G)** as the maximal subgraph of G without endvertices.

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A vertex-deleted unlabeled subgraph G-v of a graph G is called a **card** of G. A graph is said to be **set reconstructible (set-rec)** if it is determined uniquely up to isomorphism from the set S of its (non isomorphic) cards. If a property (parameter) Q of a graph G is uniquely determined by the set of cards of G then Q is called a set-recognizable property (set-reconstructible parameter).

In this paper, we study the following strong form of **Ulam's Conjecture** for graphs.

Harary's Conjecture [3]. All graphs with at least four vertices are set reconstructible.

It is known [5,6,7] that many parameters and several classes of graphs like graphs with less than 12 vertices, disconnected graphs, trees and separable graphs without endvertices are set-rec. Arjomandi and Corneil [1] have proved that unicyclic graphs are set-rec. Outerplanar graphs have been set reconstructed by Giles [2].

In this paper we address the set reconstructibility of only connected graphs having at least 12 vertices. We prove that some classes of separable graphs with a unique endvertex are set-rec and show that all graphs are set-rec if all 2-connected graphs are set-rec.

2. P-graphs.

The following results are proved in Manvel [6].

Theorem 1. DS(G) of any graph G with $\delta(G) \leq 3$ is set-rec.

Theorem 2. The connectivity of G is set-rec.

Theorem 3. The number of cutvertices of G is set-rec.

Theorem 4. Separable graphs without endvertices are set-rec.

Theorem 5. G is set-rec iff $\overline{G}$ is set-rec.

Definition [8]. A graph G with p vertices is called a **P-graph** if
- (i) there exist only two blocks in G and one of them has just two vertices (denote the endvertex by x and its base by r) and
- (ii) there exists a vertex $u \neq r$ with deg $u = p-2$.

Throughout this paper, u, r and x are used in the sense of the above definition.

Remarks:

For P-graphs G, the following hold.

1. G-x is the only card without endvertices in S (the set of cards of G).
2. G-r is the only disconnected card in S.
3. G-u is the only (p-2)-vertex deleted connected card in S.
4. The degree sequence of a P-graph is set-rec by Theorem 1.
5. P-graphs are set-recognizable (Recognizability of (i) follows by Theorems 1, 2 and 3. Existence of u as in (ii) is guaranteed by the existence of a (p-2)-vertex deleted connected card in S (by Remark 3)).

We will not always spell set-recognizability out, but all cases (and subcases) based on properties of G treated below are set-recognizable.

3. Set reconstruction of P-graphs.

In this section we prove that P-graphs are set-rec if all 2-connected graphs are set-rec. This result will be useful while proving our main result.

Lemma 1. A P-graph G having no 2-vertices is set-rec.

Proof. The P-graphs under consideration are set-recognizable by Remarks 5 and 4. Moreover, when a card has an endvertex, it is x. Now G can be obtained uniquely from G-u (which is known by Remark 3) by adding a vertex and joining it to all the vertices of G-u other than x. □

Lemma 2. A P-graph G having a 2-vertex adjacent with r is set-rec.

Proof. For a P-graph G, the hypothesis occurs iff the unique disconnected card G-r of G has at least one endvertex.

In G, no 2-vertex has a 2-vertex neighbour as otherwise both these neighbouring 2-vertices are adjacent to u and u becomes a cutvertex of G leading to a contradiction. Hence in all 2-vertex deleted cards, x is identifiable as the only endvertex and r is identifiable as the base of x.

Now the card G-s, where s is a 2-vertex adjacent to r in G, is identifiable as a 2-vertex deleted card in which the identifiable vertex r has degree ($\deg_G$ r)-1. Now all graphs obtained by adjoining a new vertex to G-s and joining it to 'a (p-3)-vertex other than r' and to r are isomorphic and the graph thus obtained is G. □

Notation. Let G be a P-graph and T denote the set of neighbours of the 2-vertices of G other than u and r. Unless otherwise stated **we use the letters t and s to denote respectively a member of T and a 2-vertex neighbour of t.**

Note that the set of degrees of vertices in T can be derived from S.

Lemma 3. Let G be a P-graph. If G has a 2-vertex and a (p-2)-vertex other than u and r, then G is set-rec.

Proof. From DS(G) and deg r, we can set-recognize the hypothesis.

Now G can be obtained uniquely (up to isomorphism) by augmenting any G-s (by adjoining a vertex w to G-s and joining it to two (p-3)-vertices other than the base of the endvertex). □

Theorem 6. A P-graph having exactly one 2-vertex is set-rec.

Proof. By Lemmas 2 and 3, we can assume that $t \neq r$ and $3 \leq \deg t \leq p-3$. If $\deg r \neq \deg t$, then in G-u we can distinguish x from the other endvertex by their bases and hence G is set-rec. So, assume that deg r = deg t.

Clearly deg r ≥ 3. If deg r = 3 (and hence deg t=3) then in the unique 2-vertex deleted card G-s, u is identifiable as the only vertex of degree p-3 (≥ 9) and t as the only 2-vertex and hence G can be obtained uniquely by augmenting G-s.

So, consider the case deg r >3. In G-x, the vertices u and s (and hence t) are identifiable respectively as the only (p-2)-vertex and the only 2-vertex and hence (G-x)-u is known from G-x. Hence t is known in (G-x)-u as the base of the unique endvertex.

P((G-x)-u) = P(G-u) = G-{u, x, s} as deg r = deg t > 3 (Figure 1).

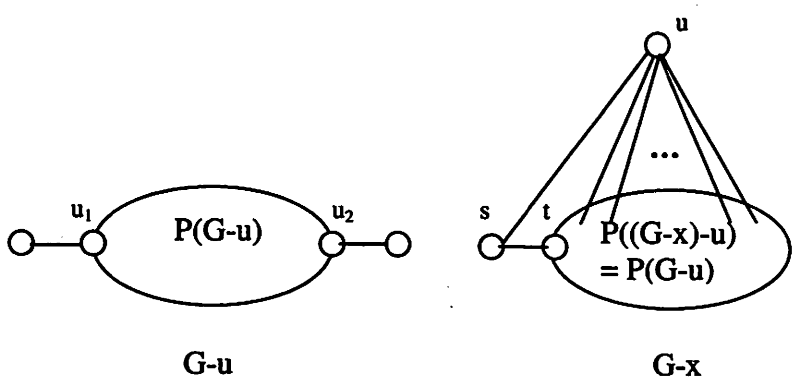


Figure 1.

Now, out of the bases u_1 and u_2 of the two endvertices in G-u, one must be the actual t of G and if it is known, x is known in G-u and hence G is the graph obtained from G-u by adjoining a new vertex and joining it to all vertices of G-u other than x. --- (1)

If there is an automorphism of G-u taking u_1 to u_2, then this automorphism takes u_2 to u_1 since u_1 and u_2 are the only vertices of G-u that occur as bases of endvertices (Figure 1). Hence the two choices for G constructed as in (1) by taking u_1 for t and u_2 for t respectively are isomorphic and so G is set-rec.

Now, let there be no automorphism of G-u taking u_1 to u_2. --- (2)

Obviously, there exists an isomorphism from G-x-u to an induced subgraph of G-u and this isomorphism should map t to u_1 or t to u_2. Without loss of generality, **let α be such an isomorphism with $\alpha(t) = u_1$**. If there exists another isomorphism β from G-x-u to an induced subgraph of G-u with $\beta(t) = u_2$, then $\beta\alpha^{-1}$ restricted to $G-u-u_1'-u_2'$ is an automorphism of $G-u-u_1'-u_2'$ taking u_1 to u_2, where u_i' is the endvertex of G-u adjacent to u_i, i=1,2. Hence $\beta\alpha^{-1}$ gives an automorphism of G-u taking u_1 to u_2, leading to a contradiction of (2). Hence all isomorphisms from G-x-u to an induced subgraph of G-u take t to u_1 so that **u_1 in G-u is the actual t of G**. Hence G is set-rec as in (1). □

Lemma 4. A P-graph G with at least two 2-vertices is set-rec if there is a t adjacent to all 2-vertices of G.

Proof. Now u and t are the only vertices adjacent to all 2-vertices in G. By Lemma 3, we can assume that deg t $\leq$ p-3 so that the hypothesis is set-recognizable, because in this case there are exactly two cards in S each having exactly k+1 endvertices, where k is the number of 2-vertices in G (k is known from DS(G)).

Since G has at least two 2-vertices, in every G-s, the vertices u and t are the only vertices adjacent to all 2-vertices and hence G can be obtained uniquely from a G-s by augmentation. □

Theorem 7. A P-graph G having at least two 2-vertices is set-rec if there is a t with deg t $\geq$ p-3.

Proof. By Lemma 2, we can assume that no 2-vertex is adjacent to r.

Throughout this proof, t stands for a member of T with deg t $\geq$ p-3.

Clearly deg $t \leq$ p-2. By Lemmas 3 and 4, we can assume that deg t = p-3 and t is nonadjacent to at least one 2-vertex of G. .... (1)

We have two subcases.

Case 1. t is adjacent to at least two 2-vertices of G.

Now G has at least three 2-vertices (by (1)).

Any vertex v other than u, t, r and x is not adjacent to x and not adjacent to the 2-vertices that are adjacent to t so that deg $v \leq$ p-4. Since r is adjacent to none of the 2-vertices (and there are at least three 2-vertices in G), deg $r \leq$ p-4. deg x = 1 < p-4. Hence

deg $w \leq$ p-4 for all $w \notin \{u,t\}$. ... (2)

A card obtained by deleting a 2-vertex adjacent to t can be located in S as a G-s such that 'the number of vertices in G-s adjacent to 2-vertices is equal to the number of vertices in G adjacent to 2-vertices' and NDS(s) = (p-2 , p-3).

In this G-s, u is identifiable as the only (p-3)-vertex (by (2)) and t is **identifiable up to 'similarity'** (as in (i) below) and hence G can be obtained uniquely (up to isomorphism) from G-s by augmentation.

(i) In this G-s, t is a (p-4)-vertex adjacent to a 2-vertex (by our hypothesis for Case 1), say s_1 and nonadjacent to a 2-vertex (by(2)), say s_2 and thus t is adjacent to all the vertices of G-s other than s_2 and x. If possible, let there be one more such t, say t' . Then t' is adjacent to all the vertices of G-s other than s_1 and x and hence the mapping (t t') (s_1 s_2) of G-s is an automorphism of G-s that fixes u.

Case 2. t is adjacent to exactly one 2-vertex of G.

By (1), G has exactly two 2-vertices and t is adjacent to exactly one of them.

A card obtained by deleting a 2-vertex adjacent to t can be located in S as a G-s such that it has exactly one 2-vertex and NDS(s) = (p-2 , p-3).

Let s_1 be the 2-vertex and t_1 be the neighbour of s_1 other than u in G-s.

In this G-s, the vertices u as well as t are identifiable up to 'similarity' using their degrees alone (as in (i) and (ii) below).

(i) In this G-s, u is a (p-3)-vertex adjacent to a 2-vertex and nonadjacent to the unique endvertex x. Hence u is adjacent to all the vertices of G-s other than x. The only other vertex which can have degree p-3 in G-s is t_1. If $\deg_{G-s} t_1 = p-3$, then t_1 is adjacent to all the vertices of G-s other than x. Hence t_1 and u have the same neighbours in G-s and hence ($u\ t_1$) is an automorphism of G-s.

(ii) In this G-s, s_1 is the only 2-vertex, x is the only endvertex and t is a (p-4)-vertex nonadjacent to s_1 and hence t is adjacent to all the vertices of G-s other than s_1 and x. Also any other (p-4)-vertex v which is adjacent neither to s_1 nor to x in G-s must be adjacent to all other vertices of G-s. Hence such a v and our t have the same neighbours in G-s and (t v) is an automorphism of G-s.

Two graphs G_1 and G_2 obtained from G-s by adjoining a vertex w and joining it with one among u and t_1 (if there is a tie) and with one among t and v (if there is a tie) are isomorphic under the mapping f given below.

	edges joined to get G_1	edges joined to get G_2	$f : V(G_1) \to V(G_2)$ (vertices moved by f)
a.	uw, wt	uw, wv	f(t)=v , f(v)=t
b.	uw, wt	t_1w, wt	f(u)= t_1 , f(t_1)=u
c.	uw, wt	t_1w, wv	f(u)= t_1 , f(t_1)=u f(t)=v , f(v)=t

Each map f is an isomorphism because the vertices u and t_1 and the vertices t and v respectively have the same neighbours in G-s. □

Theorem 8. A P-graph G with at least two 2-vertices and $\deg t \leq p-4$ for all $t \in T$ is set-rec if all 2-connected graphs are set-rec.

Proof. We will use induction on p, the number of vertices of G.

By Lemma 2, we can assume that no 2-vertex is adjacent to r.

Now $\deg w \leq p-4$ for all $w \notin \{u,r\}$ since a vertex $w \notin T \cup \{u, r, x\}$ is adjacent neither to x nor to the 2-vertices. Clearly the degree of r must be strictly less than p-2 as it is adjacent to none of the 2-vertices in G. Also if deg r = p-3 then G must have exactly two 2-vertices and hence G-u has exactly three endvertices with base of x (=r) different from the base of the other endvertices so that x can be distinguished from other endvertices in G-u by their bases as deg_{G-u} r = p-4 and deg_{G-u} t < p-4 for t ∈ T. Hence, in case deg r = p-3, G can be obtained uniquely by augmenting G-u.

So, we can assume that deg w ≤ p-4 ∀ w ≠ u.

Let G' = (G-x)-ur. Now we determine the set of cards of G' from S.

In each G-w ∈ S-{G-u, G-x, G-r}, the vertices u, x and r are identifiable respectively as the only (p-3)-vertex, the only vertex nonadjacent with u and the only neighbour of x. Hence for C ∈ S-{G-u, G-x, G-r}, C-x-ur is known. Also G'-u = (G-x-ur)-u = (G-x)-u where (G-x)-u is known (since u is the only (p-2)-vertex in G-x). Similarly, G'-r = (G-x-ur)-r = (G-r)-x where (G-r)-x is known (since x is the only isolated vertex in G-r). Thus the set

S' = {C-x-ur : C ∈ S-{G-u, G-x, G-r}} ∪ { G'-u, G'-r}

is the set of cards of G'.

Obviously, G' is connected. If G' is 2-connected, it is set-rec by hypothesis.

Now let G' be separable. r is not a cutvertex of G' as u is adjacent to all the vertices of G' other than r. Hence r and all its neighbours in G' are confined to a single block, say B. If u was a cutvertex of G' then all other blocks except possibly B contain u. Let B_1 ≠ B be an endblock of G'. Then B_1 is an endblock of G' +ur = G-x and hence u is a cutvertex of G-x leading to a contradiction (since G-x is a block). So u is not a cutvertex of G' and hence all the neighbours of u in G' are in the block of G' containing u. Now since $deg_{G'}$ u=p-3, G' has just two blocks one of which has just two vertices. That is, G' is a P-graph on p-1 vertices with r as the unique endvertex. If G' has a t with deg t ≥ (p-1)-3, then G' is set-rec by Theorem 7. Otherwise in G', deg t ≤ (p-1)-4 for every t ∈ ' T of G' ' and by induction hypothesis G' is set-rec.

Now r in G' is identifiable as the only vertex nonadjacent with the identifiable vertex u and hence G can be obtained uniquely from G' by augmentation. □

Theorem 9. P-graphs are set-rec if all 2-connected graphs are set-rec.

Proof. Follows by Lemma 1, Theorems 6, 7 and 8. □

4. Main result.

We now prove our main theorem.

Theorem 10. All connected graphs are set reconstructible iff all 2-connected graphs are set reconstructible.

Proof. The necessary part is obvious.
Sufficient part. Assume that all 2-connected graphs are set-rec. Let G be a separable graph on p (≥ 12) vertices. If G has no endvertex then G is set-rec (by Theorem 4) and hence we can assume that G has an endvertex and a (p-2)-vertex (because of Theorem 5, Theorem 4 and the hypothesis). ...(1)
So DS(G) is set-rec by Theorem 1.

We have two subcases.

Case 1. G has at least two endvertices.

Now $\overline{G}$ has at least two (p-2)-vertices. ... (2)

Let u_1 and u_2 be two (p-2)-vertices in $\overline{G}$. By (2), $\overline{G}$ has at most two endvertices and (1) now gives that $\overline{G}$ has either one or two endvertices.

Case 1.1. $\overline{G}$ has exactly one endvertex, say y.

Now $\overline{G}$ is a P-graph (as in (i) and (ii) below) and hence is set-rec by Theorem 9.

(i) If y is not adjacent to u_i, $i = 1,2$, then in $\overline{G}$-y, u_1 and u_2 are (p-2)-vertices and hence $\overline{G}$-y is a block as $\overline{G}$-y has only p-1 vertices. Hence $\overline{G}$ is a P-graph.

(ii) If y is adjacent to u_1 (say), then in $\overline{G}$-y, u_2 is adjacent to all the vertices and hence no vertex other than u_2 can be a cutvertex of $\overline{G}$-y. Also if u_2 was a cutvertex of $\overline{G}$-y, then u_1 and all its p-3 neighbours in $\overline{G}$-y are confined to a single block with p-2 vertices and the only other vertex of $\overline{G}$-y must be an endvertex adjacent u_2. Thus $\overline{G}$ has two endvertices, leading to a contradiction. Hence $\overline{G}$-y has no cutvertex and $\overline{G}$ is a P-graph.

Case 1.2. $\overline{G}$ has exactly two endvertices.

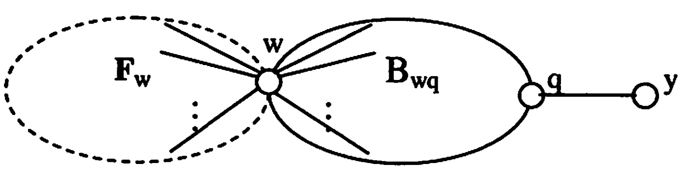
Now the bases of the two endvertices in $\overline{G}$ are different (otherwise (2) will be contradicted). Any vertex other than the bases of the endvertices can not have degree p-2. Hence $\overline{G}$ has at most two (p-2)-vertices and (2) now gives that $\overline{G}$ has exactly two (p-2)-vertices, which are the bases of the endvertices. In this case $\overline{G}$ is clearly set-recognizable from DS($\overline{G}$) and is set-rec by augmenting an endvertex-deleted card $\overline{G}$-y (by adding a vertex to $\overline{G}$-y and joining it to a (p-3)-vertex).

Case 2. G has exactly one endvertex, say y.

If G has more than one (p-2)-vertex, then G is a P-graph and hence is set-rec by Theorem 9. Hence let G have exactly one (p-2)-vertex, say w.

Case 2.1. w and y are nonadjacent in G.

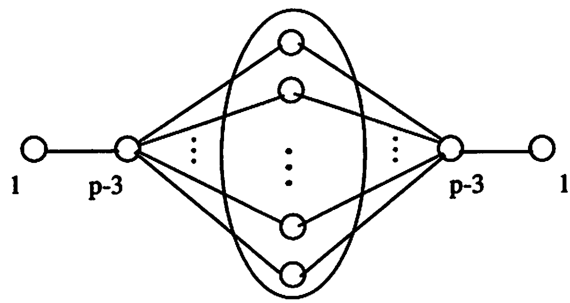
Now we can assume that w is a cutvertex of G as otherwise G is a P-graph and hence is set-rec. So w and q (the base of y) are the only cutvertices of G. Hence G is the union of three subgraphs B_{wq} (the nonendblock containing w and q), F_w (the union of endblocks containing w) and the endblock B_y ($\cong K_2$) containing y. (Figure 2).



G

Figure 2

If deg q = p-3 then $F_w \cong K_3$ (because G has only one endvertex). Consider a 2-vertex deleted card G-z with exactly two endvertices (the deleted 2-vertex cannot be from B_{wq} as every 2-vertex in B_{wq} is adjacent to w and q so that no additional endvertex is created). Such a G-z will be as in Figure 3 having an automorphism that interchanges the two endvertices, interchanges the two bases and fixes all other vertices. Hence all augmentations of G-z by introducing a 2-vertex so that the resulting graph has only one endvertex and only one endblock isomorphic to K_3 are isomorphic.



G-z

Figure 3

If deg q $\neq$ p-3 then deg q < p-3 (because $|F_w| \geq 3$). Now in the cards G-v that are connected and have at least one endvertex (cards for which deleted vertex is not one of w, y and q), the vertices w, y and q are identifiable as the only (p-3)-cutvertex, the only endvertex nonadjacent with w and the base of y respectively.

Among these cards G-v, if we choose one, say G_1 such that

(i) w and q are in the same block and

(ii) the block containing w and q has maximum number of vertices,

then the nonendblock of G_1 is B_{wq}.

Hence B_{wq} is known with w and q labeled. ... (3)

The only endvertex-deleted card in S is G-y and its only cutvertex is w. By (3), there is an isomorphism α from B_{wq} on to a block of G-y such that $\alpha(w)=w$. The graph G_α obtained from G-y by adding a vertex and joining it only with $\alpha(q)$ is a candidate for G. If β is another such isomorphism and G_β is the corresponding augmented graph, then $G_\alpha \cong G_\beta$ under the mapping ψ where

$$\psi = \beta\alpha^{-1} \text{ on vertices of } \alpha(B_{wq})$$
$$= \alpha\beta^{-1} \text{ on vertices of } \beta(B_{wq})$$
$$= \text{identity on all other vertices}$$

when $\alpha(B_{wq})$ and $\beta(B_{wq})$ are different blocks of G-y

and $\psi = \beta\alpha^{-1}$ on vertices of $\alpha(B_{wq})$
= identity on all other vertices
when $\alpha(B_{wq})$ and $\beta(B_{wq})$ are one and the same block of G-y.
Hence G is known up to isomorphism.

Case 2.2. w and y are adjacent in G.

Now in $\overline{G}$, w is the only endvertex and y is the only (p-2)-vertex and they are not adjacent. Hence $\overline{G}$ is set-rec as in Case 2.1.
This completes the proof. □

Conclusion. We observe that " reconstructibility " of P-graphs turns out to be of great use while shuttling between a graph and its complement in order to " reconstruct " it.

Harary's conjecture is stronger than Ulam's conjecture. So if at all Ulam's conjecture is false, then there exists a pair of non-isomorphic 2-connected graphs having the same set of cards.

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