A Characterization of the Generalized Lah Matrix

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Abstract

The purpose of this note is to give the power formula of the generalized Lah matrix and show $\mathcal{L}[x,y] = \mathcal{F}\mathcal{Q}[x,y]$, where $\mathcal{F}$ is the Fibonacci matrix and $\mathcal{Q}[x,y]$ is the lower triangular matrix. From it, several combinatorial identities involving the Fibonacci numbers are obtained.

Key Words: Generalized Lah matrix, Fibonacci number, Factorization of matrix

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Recently, in [4], two kinds of the Generalized Lah matrix were introduced and their algebraic properties were considered. Here we extend these two kinds of generalized Lah matrix as follows.

Let x and y be two nonzero real numbers. The generalized Lah numbers are defined by $L_{n,k}(x,y) = x^n y^k \binom{n-1}{k-1}\frac{n!}{k!}$. The generalized $n \times n$ Lah matrix $\mathcal{L}[x,y]$ is defined by

$$\mathcal{L}[x,y] = [L_{i,j}(x,y)]_{i,j=1,2,\ldots,n} = \begin{cases} x^i y^j \binom{i-1}{j-1}\frac{i!}{j!}, & if\ \ i \geq j, \\ 0, & otherwise. \end{cases} \quad (1)$$

Clearly, the Lah number $L_{n,k} = L_{n,k}(-1,1)$. The Lah matrix $\mathcal{L} = \mathcal{L}(-1,1)$. The unsigned Lah number $L^+_{n,k} = L_{n,k}(1,1)$. The unsigned Lah matrix $\mathcal{L}^+ = \mathcal{L}(1,1)$. The matrices $\mathcal{L}$ and $\mathcal{L}^+$ were introduced in [4].

The purpose of this note is to give the power formula of the generalized Lah matrix and show $\mathcal{L}[x,y] = \mathcal{F}\mathcal{Q}[x,y]$, where $\mathcal{F}$ is the Fibonacci matrix and $\mathcal{Q}[x,y]$ is the lower triangular matrix. From it, several combinatorial identities involving the Fibonacci numbers are obtained.

At first, it is easy to see that the following theorem holds by some simple computations.

$$\begin{aligned}
&\mathcal{L}[x,y]\mathcal{L}[\omega,\nu] \\
&= \begin{cases} \mathcal{L}\left[x(1+y\omega), \frac{y\omega\nu}{1+y\omega}\right], & if\ y\omega \neq -1 \\ diag\left\{-x\nu,\ x^2\nu^2,\ \ldots,\ (-1)^n x^n \nu^n\right\}, & if\ y\omega = -1, \end{cases}
\end{aligned} \tag{2}$$

and

$$\mathcal{L}[x,y]^{-1} = \mathcal{L}\left[-\frac{1}{y}, -\frac{1}{x}\right]. \tag{3}$$

By applying induction, we have

Theorem 1. Let k be a positive integer and x_i, y_i $(i = 1,\ 2,\ \ldots,\ k)$ be nonzero real numbers. Then

$$\prod_{i=1}^{k} \mathcal{L}\left[x_i, y_i\right] = \mathcal{L}\left[x_1\left(1+t_k\right), \frac{y_1\cdots y_k x_2 \cdots x_k}{1+t_k}\right], \tag{4}$$

where $t_j = \sum_{i=1}^{j-1} y_1 \cdots y_i x_2 \cdots x_{i+1} \neq -1$, for $j = 2,\ \ldots,\ k$.

Proof. We argue by induction on k. By (2), for $k = 1, 2$, the theorem is right obviously. Now carrying out the inductive step. We assume the theorem is true for k. Let $t_k = 1 + \sum_{i=1}^{k} y_1 \cdots y_i x_2 \cdots x_{i+1}$. Then

$$\begin{aligned}
\prod_{i=1}^{k+1} \mathcal{L}\left[x_i, y_i\right] &= \left(\prod_{i=1}^{k} \mathcal{L}\left[x_i, y_i\right]\right) \mathcal{L}\left[x_{k+1}, y_{k+1}\right] \\
&= \mathcal{L}\left[x_1 t_{k-1}, \frac{y_1 \cdots y_k x_2 \cdots x_k}{t_{k-1}}\right] \mathcal{L}\left[x_{k+1}, y_{k+1}\right] \\
&= \mathcal{L}\left[x_1 t_{k-1}\left(1 + \frac{y_1 \cdots y_k x_2 \cdots x_k}{t_{k-1}} x_{k+1}\right), \frac{\frac{y_1 \cdots y_k x_2 \cdots x_k}{t_{k-1}} x_{k+1} y_{k+1}}{1 + \frac{y_1 \cdots y_k x_2 \cdots x_k}{t_{k-1}} x_{k+1}}\right] \\
&= \mathcal{L}\left[x_1\left(t_{k-1} + y_1 \cdots y_k x_2 \cdots x_k x_{k+1}\right), \frac{y_1 \cdots y_k y_{k+1} x_2 \cdots x_k x_{k+1}}{t_{k-1} + y_1 \cdots y_k x_2 \cdots x_k x_{k+1}}\right]
\end{aligned}$$

$$= \mathcal{L}\left[x_1 t_k, \frac{y_1 \cdots y_{k+1} x_2 \cdots x_{k+1}}{t_k}\right]$$
$$= \mathcal{L}\left[x_1\left(1+\sum_{i=1}^{k} y_1 \cdots y_i x_2 \cdots x_{i+1}\right), \frac{y_1 \cdots y_{k+1} x_2 \cdots x_{k+1}}{1+\sum_{i=1}^{k} y_1 \cdots y_i x_2 \cdots x_{i+1}}\right]$$

by applying (2). Here the theorem is also true for $k+1$. By induction, we complete the proof of the theorem. □

In Theorem 1, taking $x_1 = x_2 = \ldots = x_k = x$, $y_1 = y_2 = \ldots = y_k = y$ and applying (2) then we can obtain the striking simplicity of the powers of the generalized Lah matrix $\mathcal{L}[x, y]$:

Corollary 2. Let k be a positive integer. Then

$$\mathcal{L}[x,y]^k = \begin{cases} \mathcal{L}\left[x\frac{1-x^k y^k}{1-xy},\ (1-xy)\frac{x^{k-1}y^k}{1-x^k y^k}\right], & \text{if } xy \neq -1, \\ I, & \text{if } xy = -1 \text{ and } k \text{ even}, \\ \mathcal{L}[x,y], & \text{if } xy = -1 \text{ and } k \text{ odd}. \end{cases}$$

where I is an identity matrix.

Corollary 3. Let the Lah matrix $\mathcal{L} = \mathcal{L}(-1, 1)$ and the unsigned Lah matrix $\mathcal{L}^+ = \mathcal{L}(1, 1)$. Then

$$\mathcal{L}^k = \begin{cases} \mathcal{L}, & \text{if } k \text{ odd}, \\ I, & \text{if } k \text{ even}, \end{cases}$$

and

$$(\mathcal{L}^+)^k = \mathcal{L}\left[k,\ \frac{1}{k}\right].$$

Next we establish the relation between the generalized Lah matrix $\mathcal{L}(x, y)$ and the Fibonacci matrix $\mathcal{F}$.

The Fibonacci numbers have been discussed in so many papers and books, see [1]. Let F_n be the n-th Fibonacci number. The Fibonacci $n \times n$ matrix $\mathcal{F}$ $(i, j = 1, 2, \ldots, n)$ is defined by

$$\mathcal{F} = [f_{i,j}] = \begin{cases} F_{i-j+1}, & \text{if } i-j+1 \geq 0, \\ 0, & \text{if } i-j+1 < 0, \end{cases} \tag{5}$$

which was studied in [2] and [3].

In [2], Lee, etc. gave the Cholesky factorization of the Fibonacci matrix and they also discussed the eigenvalues of the symmetric Fibonacci matrix $\mathcal{F}\mathcal{F}^T$. Also, gave the inverse of $\mathcal{F}$ as follows: if $\mathcal{F}^{-1} = [f'_{i,j}]$ $(i, j = 1, 2, \dots, n)$, then

$$f'_{i,j} = \begin{cases} 1, & if\ i = j, \\ -1, & if\ i-2 \leq j \leq i-1, \\ 0, & otherwise. \end{cases} \tag{6}$$

We define the $n \times n$ matrix $\mathcal{Q}[x, y] = [q_{i,j}(x, y)]$ $(i, j = 1, 2, \dots, n)$ as follows:

$$\begin{aligned} & q_{i,j}(x, y) \\ = & \left(x^i \binom{i-1}{j-1} i! - x^{i-1} \binom{i-2}{j-1} (i-1)! - x^{i-2} \binom{i-3}{j-1} (i-2)! \right) \frac{y^j}{j!} \end{aligned} \tag{7}$$

From the definition of $\mathcal{Q}[x, y]$, it is easy to see that $q_{1,1}(x, y) = xy$, $q_{1,j}(x, y) = 0$ for $j \geq 2$, $q_{2,1}(x, y) = xy(2x-1)$, $q_{2,2}(x, y) = x^2y^2$, $q_{2,j}(x, y) = 0$ for $j \geq 3$, $q_{i,1}(x, y) = (i-2)!x^{i-2}y\,(i(i-1)x^2 - (i-1)x - 1)$ and $q_{i,2}(x, y) = \frac{1}{2}x^{i-2}y^2(i-2)!\,[x^2 i(i-1)^2 - x(i-1)(i-2) - (i-3)]$ for $i \geq 3$.

Using the definition of the generalized Lah matrix $\mathcal{L}[x, y]$, $\mathcal{F}$ and $\mathcal{Q}[x, y]$, we can derive the following theorem.

Theorem 4.

$$\mathcal{L}[x, y] = \mathcal{F}\mathcal{Q}[x, y]. \tag{8}$$

Proof. To prove $\mathcal{L}[x, y] = \mathcal{F}\mathcal{Q}[x, y]$, it suffices to prove $\mathcal{F}^{-1}\mathcal{L}[x, y] = \mathcal{Q}[x, y]$ in view of the invertibility of the matrix $\mathcal{F}$. Let $\mathcal{F}^{-1} = [f'_{i,j}]$ be the inverse of $\mathcal{F}$. Since $f'_{1,j} = 0$ $(j \geq 2)$, then we have $f'_{1,1}L_{1,1}(x, y) = xy$ and $q_{1,1}(x) = xy = \sum_{k=0}^{n} f'_{1,k}L_{k,1}(x, y)$.

By reason of $L_{1,j}(x, y) = 0$ and $f'_{1,j} = 0$ $(j \geq 2)$, $\sum_{k=0}^{n} f'_{1,k}L_{k,j}(x) = f'_{1,1}L_{1,j}(x) = 0 = q_{1,j}(x, y)$ $(j \geq 2)$.

Since $f'_{2,j} = 0$ $(j \geq 3)$, $f'_{2,1} = -1$ and $f'_{2,2} = 1$, then we have $\sum_{k=0}^{n} f'_{2,k}L_{k,1}(x, y) = f'_{2,1}L_{1,1}(x, y) + f'_{2,2}L_{2,1}(x, y) = xy(2x-1) =$

$q_{2,1}(x,y)$. From (6), we have, for $i=3,4,\ldots,n$, $\sum_{k=0}^{n} f'_{i,k}L_{k,1}(x,y) = q_{i,1}(x,y)$.

Now, we consider $i \geq 2$ and $j \geq 1$. by (6) and the definition of $L_{i,j}$, we have $\sum_{k=0}^{n} f'_{i,k}L_{k,j}(x,y) = f'_{i,i}L_{i,j}(x,y) + f'_{i,i-1}L_{i-1,j}(x,y) + f'_{i,i-2}L_{i-2,j}(x,y) = L_{i,j}(x,y) - L_{i-1,j}(x,y) - L_{i-2,j}(x,y) = q_{i,j}(x,y)$.

Hence, we have $\mathcal{F}^{-1}\mathcal{L}[x,y] = \mathcal{Q}[x,y]$, the proof is completed. □

From the theorem, we have the following curious identities involving the Fibonacci numbers.

Corollary 5. We have

$$n!x^n\binom{n-1}{r-1} = r!x^rF_{n-r+1} + r!x^r(r(r+1)x-1)F_{n-r} +$$

$$+\sum_{k=r+2}^{n} F_{n-k+1}x^{k-2}k!\binom{k-1}{r-1}\left[x^2 - \frac{k-r}{k(k-1)}\left(x + \frac{k-r-1}{(k-1)(k-2)}\right)\right].$$

Specially,

$$n!\binom{n-1}{r-1} = r!F_{n-r+1} + r!(r(r+1)-1)F_{n-r} +$$

$$+\sum_{k=r+2}^{n} F_{n-k+1}k!\binom{k-1}{r-1}\left[1 - \frac{k-r}{k(k-1)}\left(1 + \frac{k-r-1}{(k-1)(k-2)}\right)\right].$$

and

$$n!(-1)^n\binom{n-1}{r-1} = r!(-1)^rF_{n-r+1} - r!(-1)^r(r(r+1)+1)F_{n-r} +$$

$$+\sum_{k=r+2}^{n} F_{n-k+1}(-1)^kk!\binom{k-1}{r-1}\left[1 - \frac{k-r}{k(k-1)}\left(\frac{k-r-1}{(k-1)(k-2)} - 1\right)\right].$$

If $r=1$, then we have

Corollary 6.

$$\begin{aligned} x^n n! &= xF_n + x(2x-1)F_{n-1} + \sum_{k=3}^{n} F_{n-k+1}x^{k-2}(k-2)! \times \\ &\quad \times \left(k(k-1)x^2 - (k-1)x - 1\right). \end{aligned}$$

In particular,

$$n! = F_{n+1} + \sum_{k=3}^{n} \frac{k-2}{k-1} k! F_{n-k+1},$$

and

$$F_n + (-1)^n n! = 3F_{n-1} + \sum_{k=3}^{n} (-1)^k (k^2-2)(k-2)! F_{n-k+1}.$$

If $r = 2$, then we have

Corollary 7.

$$\begin{aligned} x^n(n-1)n! &= 2F_{n-1}x^2 + \sum_{k=3}^{n} F_{n-k+1} x^{k-2} (k-2)! \times \\ &\times \left(k(k-1)^2 x^2 - (k-1)(k-2)x - (k-3) \right). \end{aligned}$$

In particular,

$$(n-1)n! = 2F_{n-1} + \sum_{k=3}^{n} (k-2)! \left(k^3 - 3k^2 + 3k + 1 \right) F_{n-k+1},$$

and

$$\begin{aligned} (n-1)n! &= 2F_{n-1} + \sum_{k=3}^{n} (-1)^{n-k} (k-2)! \times \\ &\times \left(k^3 - k^2 - 3k + 5 \right) F_{n-k+1}. \end{aligned}$$

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