# THE LAST DESCENT IN SAMPLES OF GEOMETRIC RANDOM VARIABLES AND PERMUTATIONS

ARNOLD KNOPFMACHER† AND HELMUT PRODINGER‡

ABSTRACT. For words of length $n$, generated by independent geometric random variables, we study the average initial and end heights of the last descent in the word. In addition we compute the average initial and end height of the last descent in a random permutation of $n$ letters.

## 1. INTRODUCTION

Let $X$ denote a geometrically distributed random variable, i. e., $\mathbb{P}\{X = k\} = pq^{k-1}$ for $k \in \mathbb{N}$ and $q = 1 - p$. The combinatorics of $n$ geometrically distributed independent random variables $X_1, \dots, X_n$ has attracted recent interest, especially because of applications in computer science. Two such areas in which they arise, are skip lists [2, 15, 18, 8] and probabilistic counting [3, 6, 7, 9].

One of the first combinatorial questions investigated for words $a_1 \dots a_n$, with the letters $a_i$ independently generated according to the geometric distribution, was the number of left-to-right maxima in [16]. In [10] the authors began a study of descents in samples of geometrically distributed independent random variables. A descent corresponds to a pair of geometic random variables $a_i, a_{i+1}$, with $a_i > a_{i+1}$ for strict descents and $a_i \geq a_{i+1}$ in the case of weak descents. We call $a_i$ the initial height and $a_{i+1}$ the end height of the descent. The size of a descent is defined to be the initial height minus the end height.

Recently the authors [11] studied the height of the *first* descent in a string of $n$ geometrically distributed independent random letters. We continue the study of descent heights in this article, by considering the initial and

---

*Date*: October 25, 2004.

1991 *Mathematics Subject Classification.* Primary: 05A15; Secondary: 60C05.

*Key words and phrases.* geometric random variables, generating function, permutations, descents.

†This material is based upon work supported by the National Research Foundation under grant number 2053740.

‡This material is based upon work supported by the National Research Foundation under grant number 2053748.

end heights and hence the size of the *last* descent in such a string. For example in $w = 2251144311$ the initial and end heights of the last descent are respectively 3 and 1 (strict case) and 1 and 1 (weak case). The size of the last descent in our example is 2 in the strict case and 0 in the weak case. In this paper strict descents are considered in Section 2 and weak descents in Section 3. In each case the limit $q \to 1$ is studied in Section 4.

By comparing the results of Section 4 to the corresponding results for the first descent in [11], we obtain an interesting conclusion relating first descents and last descents in geometric samples:

**Theorem 1.** *The ratio of the expected size of the first strict (weak) descent to the expected size of the last strict (weak) descent tends to* $\sum_{m\geq 2} \frac{H_m}{m!} = 1.165382215\ldots$ *as* $q \to 1$.

Many permutation statistics can be deduced from the corresponding geometric random variable statistic by letting $q \to 1$. However, this does not apply in the case of descent heights, so we consider separately the initial and end heights of the last descent for a random permutation of $n$ letters in Section 5. In this case a simple bijection shows that the expected size of the first descent equals the expected size of the last descent.

## 2. Strict descents

**2.1. The end height of the last strict descent.** For the end height of the last strict descent in a sample of geometrically distributed random variables we use the following decomposition for the set of all words:

$$\{\textit{all words}\} = \bigcup_{h\geq 1} \Big( \{\textit{any word}\}\{\textit{letter} > h\}\cdot \cdot \{\textit{weakly increasing word starting with } h\}\Big) \cup \{\textit{weakly increasing word}\}.$$

We now consider a probability generating function $F(z,u)$, where $z$ labels the number of random variables (=letters), and $u$ marks the end height of the last strict descent.

This leads to

$$\begin{aligned} F(z,u) &= \frac{1}{1-z} \sum_{h\geq 1} \Big( \sum_{i=h+1}^{\infty} pq^{i-1}z \Big) pq^{h-1}u^h z \prod_{j=h}^{\infty} \frac{1}{1-pq^{j-1}z} \\ &\qquad + \prod_{j\geq 1} \frac{1}{1-pq^{j-1}z} \qquad (1) \\ &= \frac{pz^2}{1-z} \sum_{h\geq 1} q^{2h-1}u^h \prod_{j=h}^{\infty} \frac{1}{1-pq^{j-1}z} + \prod_{j\geq 1} \frac{1}{1-pq^{j-1}z}. \end{aligned}$$

We check that $F(z,1)$ produces all words:

$$F(z,1) = \frac{pz^2}{1-z}\sum_{h\geq 1} q^{2h-1} \prod_{j=h-1}^{\infty} \frac{1}{1-pq^j z} + \prod_{j\geq 0} \frac{1}{1-pq^j z} = \frac{1}{1-z},$$

using Theorem 2 below.

**Theorem 2.**

$$pz^2 \sum_{h\geq 1} q^{2h-1} \prod_{j=h-1}^{\infty} \frac{1}{1-pq^j z} = 1-(1-z)\prod_{j\geq 0}\frac{1}{1-pq^j z}. \tag{2}$$

*Proof.* It is beneficial to use the notation $(x)_n := (1-x)(1-xq)\dots(1-xq^{n-1})$. We will make use of Heine's formula

$$\sum_{m\geq 0} \frac{(a)_m (b)_m t^m}{(q)_m (c)_m} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{m\geq 0} \frac{(c/b)_m (t)_m}{(q)_m (at)_m} b^m, \tag{3}$$

see e. g. [1].

We wish to evaluate

$$A := \frac{1}{(pz)_\infty} \sum_{h\geq 0} pz^2 q^{2h+1} (pz)_h = \frac{pqz^2}{(pz)_\infty} B,$$

with

$$B := \sum_{h\geq 0} q^{2h} (pz)_h.$$

Apply Heine's formula with $t = q^2$, $a = q$, $b = pz$, $c = 0$:

$$\begin{aligned}
B &= \frac{(pz)_\infty (q^3)_\infty}{(q^2)_\infty} \sum_{m\geq 0} \frac{(q^2)_m}{(q)_m (q^3)_m} (pz)^m \\
&= (pz)_\infty \sum_{m\geq 0} \frac{1-q^{m+1}}{(q)_{m+2}} (pz)^m \\
&= \frac{(pz)_\infty}{p^2 z^2} \sum_{m\geq 2} \frac{1}{(q)_m} (pz)^m - \frac{q(pz)_\infty}{p^2 q^2 z^2} \sum_{m\geq 2} \frac{1}{(q)_m} (-pqz)^m \\
&= \frac{(pz)_\infty}{p^2 z^2} \left[ \frac{1}{(pz)_\infty} - 1 + z \right] - \frac{(pz)_\infty}{p^2 q z^2} \left[ \frac{1}{(-pqz)_\infty} - 1 + qz \right] \\
&= \frac{(pz)_\infty}{pqz^2} - \frac{1}{pqz^2} - \frac{1}{pqz}.
\end{aligned}$$

And finally

$$A = \frac{pqz^2}{(pz)_\infty} B = 1 - \frac{1-z}{(pz)_\infty}.$$

□

For the mean end height of the last descent we must compute $\frac{\partial}{\partial u}F(z,1)$. Now

$$\frac{\partial}{\partial u}F(z,1)=\frac{pz^2}{1-z}\sum_{h\geq 1}hq^{2h-1}\prod_{j=h}^{\infty}\frac{1}{1-pq^{j-1}z}. \tag{4}$$

Since the dominant pole is at $z=1$ we have by means of singularity analysis [4], [5],

$$\begin{aligned}[z^n]\frac{\partial}{\partial u}F(z,1) &\sim \sum_{h\geq 1}hpq^{2h-1}\prod_{j=h-1}^{\infty}\frac{1}{1-pq^j}\\ &=\frac{1}{(p)_\infty}\sum_{h\geq 0}(h+1)pq^{2h+1}(p)_h.\end{aligned}$$

Therefore we have shown

**Theorem 3.** *The expected end height of the last strict descent is asymptotically as $n\to\infty$,*

$$\frac{1}{(p)_\infty}\sum_{h\geq 0}(h+1)pq^{2h+1}(p)_h. \tag{5}$$

**2.2. The initial height and size of the last strict descent.** We now consider a probability generating function $F(z,u)$, where $z$ labels the number of random variables, and $u$ marks the initial height of the last strict descent.

Let us use the following decomposition again

$$\begin{aligned}\{\textit{all words}\} = \bigcup_{h\geq 1}\Big(&\{\textit{any word}\}\{\textit{letter} > h\}\cdot\\ &\cdot\{\textit{weakly increasing word starting with } h\}\Big)\\ &\cup\{\textit{weakly increasing word}\}.\end{aligned}$$

This leads to

$$\begin{aligned}F(z,u)&=\frac{1}{1-z}\sum_{h\geq 1}\Big(\sum_{i=h+1}^{\infty}pq^{i-1}u^iz\Big)pq^{h-1}z\prod_{j=h}^{\infty}\frac{1}{1-pq^{j-1}z}\\ &\qquad+\prod_{j\geq 1}\frac{1}{1-pq^{j-1}z}\\ &=\frac{p^2z^2}{(1-z)(1-qu)}\sum_{h\geq 1}q^{2h-1}u^{h+1}\prod_{j=h}^{\infty}\frac{1}{1-pq^{j-1}z}+\prod_{j\geq 1}\frac{1}{1-pq^{j-1}z}.\end{aligned} \tag{6}$$

For the mean initial height of the last descent we must compute $\frac{\partial}{\partial u}F(z,1)$. Now

$$\frac{\partial}{\partial u}F(z,1) = \frac{z^2}{1-z}\sum_{h\geq 1} q^{2h}\prod_{j=h}^{\infty}\frac{1}{1-pq^{j-1}z} + \frac{pz^2}{1-z}\sum_{h\geq 1}(h+1)q^{2h-1}\prod_{j=h}^{\infty}\frac{1}{1-pq^{j-1}z}. \tag{7}$$

Since the dominant pole is at $z=1$ we have

$$\begin{aligned}[z^n]\frac{\partial}{\partial u}F(z,1) &\sim \sum_{h\geq 1} q^{2h}\prod_{j=h-1}^{\infty}\frac{1}{1-pq^j} + \sum_{h\geq 1}(h+1)pq^{2h-1}\prod_{j=h-1}^{\infty}\frac{1}{1-pq^j}\\ &= \frac{q}{p} + \frac{1}{(p)_\infty}\sum_{h\geq 1}(h+1)pq^{2h-1}(p)_{h-1}\end{aligned}$$

by using (2). Therefore we have shown

**Theorem 4.** *The expected initial height of the last strict descent is asymptotically as $n\to\infty$,*

$$\frac{q}{p} + \frac{1}{(p)_\infty}\sum_{h\geq 0}(h+2)pq^{2h+1}(p)_h. \tag{8}$$

The mean size of the last descent is obtained by subtracting the mean end height from the mean initial height of the last descent. From Theorems 3 and 4 and identity (2) we obtain

**Theorem 5.** *As $n\to\infty$, the expected size of the last strict descent is asymptotically $1/p$.*

Figure 1 illustrates the dependence on $q$ of the initial height, end height and size of the last strict descent.

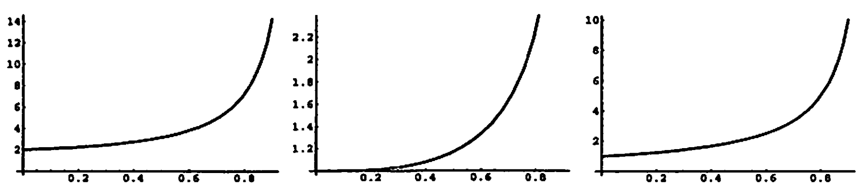


FIGURE 1. *Mean initial height, end height and size of the last strict descent as a function of $q$.*

## 3. Weak descents

The computational steps in the case of weak descents are very much parallel to the corresponding steps for strict descents. Hence we include only the main steps in the derivations below.

**3.1. The end height of the last weak descent.** We now consider a probability generating function $F(z,u)$, where $z$ labels the number of random variables, and $u$ marks the end height of the last weak descent.

We use the following decomposition

$$\{\textit{all words}\} = \bigcup_{h\geq 1}\Big(\{\textit{any word}\}\{\textit{letter} \geq h\}\cdot \\ \cdot\{\textit{strictly increasing word starting with } h\}\Big) \\ \cup\{\textit{strictly increasing word}\}.$$

This leads to

$$F(z,u) = \frac{pz^2}{1-z}\sum_{h\geq 0} q^{2h}u^{h+1}\prod_{j=h+1}^{\infty}(1+pq^jz) + \prod_{j\geq 0}(1+pq^jz). \tag{9}$$

Once again $F(z,1) = \dfrac{1}{1-z}$, in view of

**Theorem 6.**

$$pz^2\sum_{h\geq 0} q^{2h}\prod_{j=h+1}^{\infty}(1+pq^jz) = 1-(1-z)\prod_{j\geq 0}(1+pq^jz). \tag{10}$$

*Proof.* We make use of the following identity proved in [11],

$$\sum_{h\geq 1} pz^2q^{2h}\prod_{j=1}^{h}\frac{1}{1-pq^jz} = \prod_{j\geq 1}\frac{1}{1-pq^jz} - 1 - qz.$$

Therefore

$$\sum_{h\geq 1} pz^2q^{2h}\prod_{j=0}^{h}\frac{1}{1-pq^jz} = \prod_{j\geq 0}\frac{1}{1-pq^jz} - \frac{1+qz}{1-pz}.$$

Replacing $z$ by $-z$ and multiplying through by $(-pz)_\infty$,

$$\sum_{h\geq 1} pz^2q^{2h}\prod_{j=h+1}^{\infty}(1+pq^jz) = 1 - \frac{1-qz}{1+pz}(-pz)_\infty.$$

Hence

$$\sum_{h\geq 0} pz^2q^{2h} \prod_{j=h+1}^{\infty}(1+pq^jz) = 1 - \frac{1-qz}{1+pz}(-pz)_\infty + \frac{pz^2}{1+pz}(-pz)_\infty$$
$$= 1-(1-z)(-pz)_\infty.$$

□

Now

$$\frac{\partial}{\partial u}F(z,1) = \frac{pz^2}{1-z}\sum_{h\geq 0}(h+1)q^{2h}\prod_{j=h+1}^{\infty}(1+pq^jz). \tag{11}$$

Since the dominant pole is at $z=1$ we have

$$[z^n]\frac{\partial}{\partial u}F(z,1) \sim p\sum_{h\geq 0}(h+1)q^{2h}\prod_{j=h+1}^{\infty}(1+pq^j).$$

Consequently we have

**Theorem 7.** *The expected end height of the last weak descent is asymptotically as $n\to\infty$,*

$$(-pq)_\infty \sum_{h\geq 0}(h+1)\frac{pq^{2h}}{(-pq)_h}. \tag{12}$$

**3.2. The initial height and size of the last weak descent.** We now consider a probability generating function $F(z,u)$, where $z$ labels the number of random variables, and $u$ marks the initial height of the last weak descent.

We use again the decomposition

$$\{\textit{all words}\} = \bigcup_{h\geq 1}\Big(\{\textit{any word}\}\{\textit{letter} \geq h\}\cdot$$
$$\cdot\{\textit{strictly increasing word starting with } h\}\Big)$$
$$\cup\{\textit{strictly increasing word}\}.$$

This leads to

$$F(z,u) = \frac{p^2z^2}{1-z}\sum_{h\geq 0}q^{2h}\frac{u^{h+1}}{1-qu}\prod_{j=h+1}^{\infty}(1+pq^jz) + \prod_{j\geq 0}(1+pq^jz). \tag{13}$$

Now

$$\begin{aligned}\frac{\partial}{\partial u}F(z,1) &= \frac{z^2}{1-z}\sum_{h\geq 0}q^{2h+1}\prod_{j=h+1}^{\infty}(1+pq^jz)\\ &+\frac{pz^2}{1-z}\sum_{h\geq 0}(h+1)\,q^{2h}\prod_{j=h+1}^{\infty}(1+pq^jz).\end{aligned} \tag{14}$$

From the dominant pole at $z = 1$ and by using (10) we find that

$$[z^n]\frac{\partial}{\partial u}F(z,1) \sim \frac{q}{p} + (-pq)_\infty \sum_{h\geq 0}(h+1)\,\frac{pq^{2h}}{(-pq)_h}.$$

Hence

**Theorem 8.** *The expected initial height of the last weak descent is asymptotically as $n \to \infty$,*

$$\frac{q}{p} + (-pq)_\infty \sum_{h\geq 0}(h+1)\,\frac{pq^{2h}}{(-pq)_h}. \tag{15}$$

From Theorems 7 and 8 we deduce

**Theorem 9.** *As $n \to \infty$, the expected size of the last weak descent is asymptotically $\frac{q}{p}$.*

Figure 2 illustrates the dependence on $q$ of the initial height, end height and size of the last weak descent.

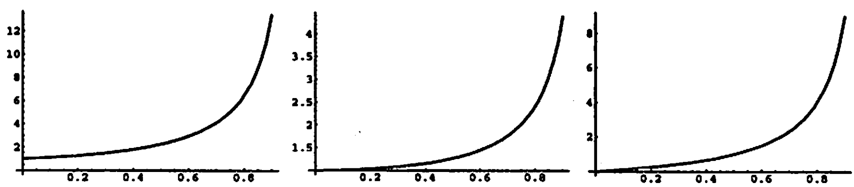


FIGURE 2. *Mean initial height, end height and size of the last weak descent as a function of $q$.*

## 4. THE BEHAVIOUR FOR $q \to 1$

The figures indicate that the quantities in Theorems 3, 4 and 5, as well as in Theorems 7, 8 and 9 tend to infinity as $q \to 1$. We will quantify this observation in this section.

We start our study with

$$R_1 := \sum_{h\geq 0}(h+1)pq^{2h+1}(p)_h = \frac{d}{dz}pqz\sum_{h\geq 0}(zq^2)^h(p)_h\Big|_{z=1}.$$

We use Heine with $t = zq^2$, $a = p$, $b = q$, $c = 0$:

$$R_2 := \sum_{h\geq 0}(zq^2)^h(p)_h = \frac{(q)_\infty(pqz^2)_\infty}{(zq^2)_\infty}\sum_{m\geq 0}\frac{(zq^2)_m}{(q)_m(pqz^2)_m}q^m.$$

We use Heine again; $t = q$, $a = zq^2$, $b = 0$, $c = pqz^2$. For this, $(c/b)_m b^m$ has to be interpreted as $(b-c)(b-cq)\dots(b-cq^{m-1})$, after which $b$ can be replaced by zero;

$$
\begin{aligned}
R_2 &= \frac{(q)_\infty (pqz^2)_\infty}{(zq^2)_\infty} \frac{(zq^3)_\infty}{(pqz^2)_\infty (q)_\infty} \sum_{m\ge 0} \frac{(-1)^m q^{\binom{m}{2}} (pq^2 z)^m (q)_m}{(q)_m (zq^3)_m} \\
&= \frac{1}{1-zq^2} \sum_{m\ge 0} \frac{(-1)^m q^{\binom{m}{2}} (pq^2 z)^m}{(zq^3)_m} \\
&= \frac{1}{pqz} \sum_{m\ge 1} \frac{(-1)^{m-1} q^{\binom{m}{2}} (pqz)^m}{(zq^2)_m}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
R_1 &= \frac{d}{dz} \sum_{m\ge 1} \frac{(-1)^{m-1} q^{\binom{m}{2}} (pqz)^m}{(zq^2)_m} \Bigg|_{z=1} \\
&= \sum_{m\ge 1} \frac{(-1)^{m-1} q^{\binom{m}{2}} (pq)^m}{(q^2)_m} \left[ m + \sum_{i=2}^{m+1} \frac{q^i}{1-q^i} \right].
\end{aligned}
$$

Thus

$$
\lim_{q\to 1} R_1 \cdot (1-q) = \sum_{m\ge 1} \frac{(-1)^{m+1}(H_{m+1}-1)}{(m+1)!} = \sum_{m\ge 1} \frac{(-1)^m H_m}{m!} - \frac{1}{e} + 1.
$$

Since $1/(p)_\infty \to e$ as $q \to 1$, we infer that the quantity in Theorem 3 for the end height behaves for $q \to 1$ as

$$
\frac{e \sum_{m\ge 1} \frac{(-1)^m H_m}{m!} + e - 1}{1-q} = \frac{0.400379677\dots}{1-q}
$$

and the quantity in Theorem 4 for the initial height behaves for $q \to 1$ as

$$
\frac{e \sum_{m\ge 1} \frac{(-1)^m H_m}{m!} + e}{1-q} = \frac{1.400379677\dots}{1-q}.
$$

Next we consider

$$
\begin{aligned}
R_3 &:= \sum_{h\ge 0} (h+1) pq^{2h} \frac{1}{(-pq)_h} = \frac{p(1+p)}{q^2} \sum_{h\ge 1} hq^{2h} \frac{1}{(-p)_h} \\
&= \frac{p(1+p)}{q^2} \frac{d}{dz} \sum_{h\ge 0} (zq^2)^h \frac{1}{(-p)_h} \Bigg|_{z=1}.
\end{aligned}
$$

We apply Heine, with $t = zq^2$, $a = 0$, $b = q$, and $c = -p$:

$$
R_4 := \sum_{h\ge 0} (zq^2)^h \frac{1}{(-p)_h} = \frac{(q)_\infty}{(-p)_\infty (zq^2)_\infty} \sum_{m\ge 0} \frac{(\frac{-p}{q})_m (zq^2)_m}{(q)_m} q^m.
$$

A further application of Heine, with $t = q$, $a = zq^2$, $b = -p/q$, and $c = 0$ turns this into

$$R_4 = \frac{(q)_\infty}{(-p)_\infty (zq^2)_\infty} \frac{(-\frac{p}{q})_\infty (zq^3)_\infty}{(q)_\infty} \sum_{m\geq 0} \frac{(q)_m}{(q)_m (zq^3)_m} \left(\frac{-p}{q}\right)^m$$
$$= \frac{1+p/q}{1-zq^2} \sum_{m\geq 0} \frac{(-p)^m}{q^m (zq^3)_m}$$
$$= \frac{p-q}{p} \sum_{m\geq 1} \frac{(-1)^m p^m}{q^m (zq^2)_m}.$$

Thus

$$R_3 = \frac{(1+p)(p-q)}{q^2} \frac{d}{dz} \sum_{m\geq 1} \frac{(-1)^m p^m}{q^m (zq^2)_m} \bigg|_{z=1}$$
$$= \frac{(1+p)(p-q)}{q^2} \sum_{m\geq 1} \frac{(-1)^m p^m}{q^m (q^2)_m} \sum_{i=2}^{m+1} \frac{q^i}{1-q^i}.$$

At this stage, we can perform the limit for $q \to 1$ and find

$$\lim_{q\to 1} R_3 \cdot (1-q) = \sum_{m\geq 1} \frac{(-1)^{m+1}}{(m+1)!} \sum_{i=2}^{m+1} \frac{1}{i} = \sum_{m\geq 2} \frac{(-1)^m (H_m - 1)}{m!}$$
$$= \sum_{m\geq 1} \frac{(-1)^m H_m}{m!} - \frac{1}{e} + 1.$$

Since $(-pq)_\infty \to e$ as $q \to 1$, we infer that the quantity in Theorem 7 for the weak end height behaves for $q \to 1$ as

$$\frac{e \sum_{m\geq 1} \frac{(-1)^m H_m}{m!} + e - 1}{1-q} = \frac{0.400379677\ldots}{1-q}$$

and the quantity in Theorem 8 for the weak initial height behaves for $q \to 1$ as

$$\frac{e \sum_{m\geq 1} \frac{(-1)^m H_m}{m!} + e}{1-q} = \frac{1.400379677\ldots}{1-q}.$$

Finally, in [11] it is shown that the size of the first strict (weak) descent is asymptotic to $\frac{L}{1-q}$ as $q \to 1$, where $L = \sum_{m\geq 2} \frac{H_m}{m!} = 1.165382215\ldots$. By comparing this with Theorems 5 and 9 we deduce Theorem 1.

## 5. The height of the last descent in permutations

**5.1. End height of last descent in permutations.** Various permutation statistics, such as the number of descents [10] can be deduced from the corresponding geometric random variable statistic by letting $q \to 1$. However, it is not possible to deduce the end height of the last descent in a permutation of $n$ letters from the corresponding geometric random variable statistic. We therefore consider this question separately.

Suppose that this end descent height is $m$. If this occurs in position $n-j$ then we have $n-m$ choices for the element that precedes $m$ in the permutation and $\binom{n-m-1}{j}$ choices for the elements that form the strictly increasing sequence of $j$ values that follow $m$. There are $(n-j-2)!$ arrangements of the remaining elements to complete the permutation. Hence the total number of permutations whose last descent has end height $m$ is

$$\sum_{j=0}^{n-m-1}(n-m)\binom{n-m-1}{j}(n-j-2)!\,. \tag{16}$$

As all permutations of $n$ letters other than $12\ldots n$ have a descent, if we sum (16) over $m$ we obtain the identity

$$\sum_{m=1}^{n-1}(n-m)\sum_{j=0}^{n-m-1}\binom{n-m-1}{j}(n-j-2)! = n!-1. \tag{17}$$

A direct proof is as follows

$$\begin{aligned} A &:= \sum_{m=1}^{n-1}(n-m)\sum_{j=0}^{n-m-1}\binom{n-m-1}{j}(n-j-2)! \\ &= \sum_{j=0}^{n-2}\frac{(n-j-2)!}{j!}\sum_{m=1}^{n-j-1}\frac{(n-m)!}{(n-m-j-1)!} \\ &= \sum_{j=0}^{n-2}(n-j-2)!(j+1)\sum_{m=1}^{n-j-1}\binom{n-m}{j+1} \\ &= \sum_{j=0}^{n-2}(n-j-2)!(j+1)\binom{n}{j+2} \\ &= n!\sum_{j=0}^{n-2}\left[\frac{1}{(j+1)!}-\frac{1}{(j+2)!}\right] = n!-1. \end{aligned}$$

The average end height is then given by the expression

$$\frac{g(n)}{n!} := \frac{1}{n!}\sum_{m=1}^{n-1}m(n-m)\sum_{j=0}^{n-m-1}\binom{n-m-1}{j}(n-j-2)!\,. \tag{18}$$

**Theorem 10.** *We have*

$$\frac{g(n)}{n!} = (n+1)\left[3-\sum_{j=0}^{n}\frac{1}{j!}-\frac{2}{(n+1)!}\right] \sim (3-e)(n+1), \tag{19}$$

*as* $n \to \infty$, *where* $3 - e = 0.2817181715\ldots$ .

*Proof.* We have

$$\begin{aligned}
B &:= \frac{1}{n!}\sum_{m=1}^{n-1} m(n-m) \sum_{j=0}^{n-m-1} \binom{n-m-1}{j}(n-j-2)! \\
&= \frac{1}{n!}\sum_{j=0}^{n-2} \frac{(n-j-2)!}{j!} \sum_{m=1}^{n-j-1} m \frac{(n-m)!}{(n-m-j-1)!} \\
&= \frac{1}{n!}\sum_{j=0}^{n-2}(n-j-2)!(j+1) \sum_{m=1}^{n-j-1} m\binom{n-m}{j+1} \\
&= (n+1)\frac{1}{n!}\sum_{j=0}^{n-2}(n-j-2)!(j+1) \sum_{m=1}^{n-j-1} \binom{n-m}{j+1} \\
&\quad - \frac{1}{n!}\sum_{j=0}^{n-2}(n-j-2)!(j+1) \sum_{m=1}^{n-j-1} (n-m+1)\binom{n-m}{j+1} \\
&= (n+1)\Big(1-\frac{1}{n!}\Big) - \frac{1}{n!}\sum_{j=0}^{n-2}(n-j-2)!(j+1)(j+2) \sum_{m=1}^{n-j-1} \binom{n-m+1}{j+2} \\
&= (n+1)\Big(1-\frac{1}{n!}\Big) - \frac{1}{n!}\sum_{j=0}^{n-2}(n-j-2)!(j+1)(j+2)\binom{n+1}{j+3} \\
&= (n+1)\Big(1-\frac{1}{n!}\Big) - (n+1)\sum_{j=0}^{n-2}\frac{(j+1)(j+2)}{(j+3)!} \\
&= (n+1)\Big(1-\frac{1}{n!}\Big) - (n+1)\sum_{j=0}^{n-2}\Big[\frac{1}{(j+1)!} - \frac{2}{(j+2)!} + \frac{2}{(j+3)!}\Big] \\
&= (n+1)\Big[3 - \sum_{j=0}^{n}\frac{1}{j!} - \frac{2}{(n+1)!}\Big] \sim (n+1)(3-e).
\end{aligned}$$

□

From [11] we see that the ratio of the expected end heights, for the first descent and last descent tends to $\frac{e-2}{2(3-e)} = 1.274823389\ldots$ as $n \to \infty$.

The $g(n)$ sequence

0, 1, 6, 33, 202, 1419, 11358, 102229, 1022298, 11245287, 134943454, 1754264913, ...

is not in Sloane [19].

**5.2. Initial height of last descent in permutations.** It is also not possible to deduce the initial height of the last descent in a permutation of $n$-letters from the corresponding geometic random variable statistic, as $q \to 1$.

Suppose that the end descent height is $m$ and the initial descent height is $k > m$. If $m$ occurs in position $n-j$ then we have $k-1$ choices for $m$ and $\binom{n-m-1}{j}$ choices for the elements that form the strictly increasing sequence of $j$ values that follow $m$. There are $(n-j-2)!$ arrangements of the remaining elements to complete the permutation. Hence the total number of permutations whose last descent has initial height $k$ is

$$\sum_{m=1}^{k-1} \sum_{j=0}^{n-m-1} \binom{n-m-1}{j} (n-j-2)! \,. \tag{20}$$

As all permutations of $n$ letters other than $12 \ldots n$ have a descent, if we sum (20) over $k$ we obtain the identity

$$\sum_{k=2}^{n} \sum_{m=1}^{k-1} \sum_{j=0}^{n-m-1} \binom{n-m-1}{j} (n-j-2)! = n! - 1. \tag{21}$$

A direct proof is

$$\begin{aligned} C &:= \sum_{k=2}^{n} \sum_{m=1}^{k-1} \sum_{j=0}^{n-m-1} \binom{n-m-1}{j} (n-j-2)! \\ &= \sum_{m=1}^{n-1} \sum_{k=m+1}^{n} \sum_{j=0}^{n-m-1} \binom{n-m-1}{j} (n-j-2)! \\ &= \sum_{m=1}^{n-1} (n-m) \sum_{j=0}^{n-m-1} \binom{n-m-1}{j} (n-j-2)! = A. \end{aligned}$$

The average initial height is then given by the expression

$$\frac{h(n)}{n!} := \frac{1}{n!} \sum_{k=2}^{n} k \sum_{m=1}^{k-1} \sum_{j=0}^{n-m-1} \binom{n-m-1}{j} (n-j-2)! \,. \tag{22}$$

**Theorem 11.** *We have*

$$\frac{h(n)}{n!} = \frac{n+1}{2} \left[ 4 - \sum_{j=0}^{n} \frac{1}{j!} - \frac{1}{n!} - \frac{2}{(n+1)!} \right] \sim (n+1)\left(2 - \frac{e}{2}\right), \tag{23}$$

*as* $n \to \infty$, *where* $2 - e/2 = 0.6408590857\ldots$ .

*Proof.* We have

$$
\begin{aligned}
D &:= \frac{1}{n!}\sum_{k=2}^{n} k \sum_{m=1}^{k-1} \sum_{j=0}^{n-m-1} \binom{n-m-1}{j}(n-j-2)! \\
&= \frac{1}{n!}\sum_{m=1}^{n-1} \sum_{k=m+1}^{n} k \sum_{j=0}^{n-m-1} \binom{n-m-1}{j}(n-j-2)! \\
&= \frac{1}{2n!}\sum_{m=1}^{n-1} m(n-m) \sum_{j=0}^{n-m-1} \binom{n-m-1}{j}(n-j-2)! \\
&\quad + \frac{n+1}{2n!}\sum_{m=1}^{n-1} (n-m) \sum_{j=0}^{n-m-1} \binom{n-m-1}{j}(n-j-2)! \\
&= \frac{B}{2} + \frac{1}{2n!}(n!-1) \\
&= \frac{n+1}{2}\left[4 - \sum_{j=0}^{n}\frac{1}{j!} - \frac{1}{n!} - \frac{2}{(n+1)!}\right] \sim (n+1)\frac{4-e}{2}.
\end{aligned}
$$

□

From [11] we see that the ratio of the expected initial heights, for the first descent and last descent tends to $\frac{e-2}{2-e/2} = 1.120810868\ldots$ as $n \to \infty$.

The $h(n)$ sequence for $n \geq 1$ is

$0, 2, 13, 74, 458, 3226, 25835, 232550, 2325544, 25581038, 306972521, 3990642850, \ldots$

is also not in Sloane [19].

We deduce from Theorems 10 and 11 that the mean size of the last descent is asymptotic to $(e/2-1)(n+1)$ as $n \to \infty$.

In [11] it was shown that the mean size of the first descent is also asymptotic to $(e/2-1)(n+1)$ as $n \to \infty$. Indeed we can give a simple combinatorial proof that the expected sizes of the first and last descents over all permutations of $n$ are the same. Given a permutation $\pi(1), \pi(2), \ldots, \pi(n)$ the mapping that sends $\pi(i)$ to $n+1-\pi(n+1-i)$ for $1 \leq i \leq n$, is easily seen to be a size preserving bijection from the last descent to the first descent.

## References


[1] G. E. Andrews, *The theory of partitions*, Encyclopaedia of mathematics and its applications, 2, Addison-Wesley, 1976.

[2] L. Devroye. A limit theory for random skip lists. *Advances in Applied Probability*, 2:597–609, 1992.

[3] P. Flajolet and G. N. Martin. Probabilistic counting algorithms for data base applications. *Journal of Computer and System Sciences*, 31:182–209, 1985.

[4] P. Flajolet and A M. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3:216-240, 1990.
[5] P.Flajolet and R. Sedgewick. *Analytic Combinatorics*. Book in preparation.
[6] P. Kirschenhofer and H. Prodinger. On the analysis of probabilistic counting. In E. Hlawka and R. F. Tichy, editors, *Number–theoretic Analysis*, volume 1452 of *Lecture Notes in Mathematics*, pages 117–120, 1990.
[7] P. Kirschenhofer and H. Prodinger. A result in order statistics related to probabilistic counting. *Computing*, 51:15–27, 1993.
[8] P. Kirschenhofer and H. Prodinger. The path length of random skip lists. *Acta Informatica*, 31:775–792, 1994.
[9] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Analysis of a splitting process arising in probabilistic counting and other related algorithms. *Random Structures and Algorithms*, 9:379–401, 1996.
[10] A. Knopfmacher and H. Prodinger. Descents in samples of geometric random variables. Mathematics and Computer Science: Algorithms, Trees, Combinatorics and Probabilities, Drmota, Flajolet, Gardy and Gittenberger (Eds.), 339–350, 2004.
[11] A. Knopfmacher and H. Prodinger. The first descent in samples of geometric random variables and permutations. Submitted, 2004.
[12] D. E. Knuth. *The art of computer programming*, volume 3: Sorting and Searching. Addison-Wesley, 1973. Second edition, 1998.
[13] D. E. Knuth. *The art of computer programming*, volume 4: Chapter 7. Preliminary version available at *http://www-cs-faculty.stanford.edu/ knuth/news.html*.
[14] G. Louchard and H. Prodinger. Ascending runs of sequences of geometrically distributed random variables: a probabilistic analysis. *Theoretical Computer Science*, 304:59–86, 2003.
[15] T. Papadakis, I. Munro, and P. Poblete. Average search and update costs in skip lists. *BIT*, 32:316–332, 1992.
[16] H. Prodinger. Combinatorics of geometrically distributed random variables: Left-to-right maxima. *Discrete Mathematics*, 153:253–270, 1996.
[17] H. Prodinger. Combinatorics of geometrically distributed random variables: Length of ascending runs. Proceedings of LATIN'2000, *Lecture Notes in Comp. Sci.* 1776, Gonnet, Panario and Viola (Eds.), 473–482, 2000.
[18] W. Pugh. Skip lists: a probabilistic alternative to balanced trees. *Communications of the ACM*, 33:668–676, 1990.
[19] N.J.A. Sloane and S. Plouffe, *The encyclopaedia of integer sequences*, Academic Press, 1995. Online edition available at *http://www.research.att.com/~njas/sequences/*.

Arnold Knopfmacher, The John Knopfmacher Centre for Applicable Analysis and Number Theory, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa
*E-mail address*: arnoldkn@cam.wits.ac.za

Helmut Prodinger, The John Knopfmacher Centre for Applicable Analysis and Number Theory, Department of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa
*E-mail address*: helmut@maths.wits.ac.za