The K-behaviour of p-trees

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Abstract

Let $G = (V, E)$ be a graph with n vertices. The *clique graph* of G is the intersection graph $K(G)$ of the set of all (maximal) cliques of G and K is called the *clique operator*. The *iterated clique graphs* $K^i(G)$ are recursively defined by $K^0(G) = G$ and $K^i(G) = K(K^{i-1}(G))$, $i > 0$. A graph is *K-divergent* if the sequence $|V(K^i(G))|$ of all vertex numbers of its iterated clique graphs is unbounded, otherwise it is *K-convergent*. The long-run behaviour of G, when we repeatedly apply the clique operator, is called the *K-behaviour* of G.

In this paper we characterize the K-behaviour of the class of graphs called p-trees, that has been extensively studied by Babel. Among many other properties, a p-tree contains exactly $n-3$ induced P_4s. In this way we extend some previous result about the K-behaviour of cographs, i.e. graphs with no induced P_4s. This characterization leads to a polynomial time algorithm for deciding the K-convergence or K-divergence of any graph in the class.

1 Introduction

Given a graph $G = (V, E)$ a subgraph H of G is a *complete* if every pair of distinct vertices of H are adjacent. A *clique* is a maximal complete subgraph of G. The *clique graph* $K(G)$ of a graph G is the intersection graph of the cliques of G. It is obtained by representing each clique of G by a vertex of $K(G)$ and connecting two vertices by an edge if and only if their corresponding cliques intersect. The *iterated clique graphs* $K^i(G)$ are defined by $K^0(G) = G$ and $K^i(G) = K(K^{i-1}(G))$, $i > 0$. We refer to [14] and [16] for the literature on iterated clique graphs. Graphs behave in a

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variety of ways when we repeatedly apply the clique operator K, the main distinction being between K-convergence and K-divergence. A graph G is said to be *K-divergent* if the sequence $|V(K^i(G))|$ of all vertex numbers of its iterated clique graphs is unbounded, otherwise G is *K-convergent*. In particular if $\lim_{i\to\infty} |V(K^i(G))| = 1$ we say that G is *K-null*.

The first examples of K-divergent graphs were found in the class of the complete multipartite graphs, denoted by $K_{p_1,p_2,\ldots,p_q}$, whose vertex-sets can be partitioned into q disjoint stable sets S_i of cardinality p_i and for every $u \in S_i$ and $v \in S_j, i \neq j$, the edge uv belongs to G. In [6, 11] it was proved that $K_{2,2,\ldots,2}$ is K-divergent. Moreover, Neumann-Lara in [11] showed that all complete multipartite graphs $K_{p_1,\ldots,p_q}$, with $q \geq 3$ and $p_i \geq 2$, $1 \leq i \leq q$, are K-divergent with superexponential growth. The remaining ones are the cliques K_n on n vertices, the bipartite graphs K_{p_1,p_2} and the multipartite graphs with a universal vertex $K_{1,p_2,\ldots,p_q}$ that were previously known to be K-convergent.

The question whether the K-convergence of a graph is algorithmically decidable is an open problem. For restricted families of graphs containing both K-convergent and K-divergent graphs, their K-behaviour has been characterized for complements of cycles [11], clockwork graphs [8], regular Whitney triangulations of closed surfaces [9] and cographs [7]. However, in all these cases the K-behaviour can be decided in polynomial time.

In our paper we extend the results presented in [7] about the K-behaviour of *cographs*, i.e. graphs without any induced chordless path on four vertices, termed P_4. We consider the class of p-trees, introduced by Babel in [1], as the class of graphs where each induced subgraph contains a vertex that belongs to at most one P_4. Many characterization of p-trees by forbidden configurations, in terms of the number of P_4s and by the uniqueness of the p-chains connecting any two vertices of a p-tree are given in [1, 3]. In particular it is proved that a p-tree contains exactly $n - 3$ P_4s. To decide the K-behaviour of p-trees we use a more recent characterization of p-trees given by Babel in [2] based on special properties of the unique modular decomposition tree associated to each graph of the class.

Using the modular decomposition tecnique, the K-behaviour of cographs, has been completely characterized in [7]. In this paper, the same tecnique is used to decide the K-behaviour of p-trees.

The modular decomposition tree of any graph can be computed in linear time [10] and therefore it is the natural framework for finding polynomial time algorithms of many problems.

In section 2 we give some definitions and recall some general results given in [4], [11] and [7] that allow to derive the K-behaviour of the whole graph from the K-behaviour of some suitable subgraph of itself. In section 3 we characterize the K-behaviour of p-trees. This characterization leads to a linear time algorithm for deciding the K-behaviour of any graph in the class.

2 Preliminaries and definitions

All graphs in this paper are finite and simple. Let $G = (V, E)$ be a graph with vertex-set $V(G)$ and edge-set $E(G)$. We denote by n the cardinality of V. If v is a vertex of G then $N(v)$ is the set of vertices which are adjacent to v and $N[v]$ is the set $N(v) \cup \{v\}$. For any u and v in V, we say that u is *dominated* by v or v is *dominating* u in G, if $N[u] \subseteq N[v]$. If v is a vertex dominating every other vertex of G, then we say that v is a ***universal*** vertex. A *stable set* is a set of pairwise nonadjacent vertices.

The *complement graph* of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. Given a subset U of V, let $G[U]$ stand for the subgraph of G induced by U. Let X be a subset of V and x any vertex of X. The *quotient graph* G/X is defined as $V(G/X) = (V(G) - X) \cup \{x\}$ and $E(G/X) = E(G[V(G) - X]) \cup \{xv \mid uv \in E(G),\ u \in X,\ v \in V(G) - X\}$.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic*, denoted $G \cong G'$, if there is a bijection $f : V \to V'$ satisfying, for all $u, v \in V$, $uv \in E$ if and only if $f(u)f(v) \in E'$.

Let G and G' be two vertex disjoint graphs. We can define the *parallel composition* of G and G' as the graph $G \cup G'$ so that $V(G \cup G') = V(G) \cup V(G')$ and $E(G \cup G') = E(G) \cup E(G')$. The *serial composition* of G and G' is the graph $G + G'$ defined by $V(G + G') = V(G) \cup V(G')$ and $E(G + G') = E(G) \cup E(G') \cup \{vv' \mid v \in V(G),\ v' \in V(G')\}$.

Let P_4 denote the chordless path with vertices u, v, w, x and edges uv, vw, wx. The vertices v and w are called *midpoints* whereas the vertices u and x are called *endpoints*. Following the terminology of Jamison and Olariu [5], a graph G is *p-connected* (or, more extensively, *P_4-connected*) if, for each partition V_1, V_2 of V into two sets, there exists an induced P_4 which contains vertices from V_1 and V_2. Such P_4 is a *crossing* between V_1 and V_2. The *p-connected components* of a graph G are the maximal induced p-connected subgraphs.

A *module* of a graph G is a subset M of vertices of $V(G)$ such that each vertex in $V(G) \setminus M$ either is adjacent to all vertices of M, or is adjacent to no vertex in M. The empty set, the subsets formed by single vertices of G and the set $V(G)$ are *trivial* modules. A graph is *prime* if it only contains trivial modules. Say that M is a *strong* module if, for any other module A, the intersection of M and A is empty or contains one of the modules. The unique partition of the vertex set of a graph G into maximal strong modules is used recursively to define its unique *modular decomposition tree* $T(G)$. The module M is *parallel* (P) if $G[M]$ is disconnected; M is *serial* (S) if $\overline{G}[M]$ is disconnected; M is *neighbourhood* (N) if both $G[M]$ and $\overline{G}[M]$

are connected. Similarly, say that G[M] is parallel, serial or neighbourhood when M is respectively so. The leaves of $T(G)$ are the vertices of G and the internal vertices of $T(G)$ are modules labeled with P, S or N (for parallel, serial, or neighbourhood module, respectively).

We will often identify the modules M_i with the induced subgraphs $G_i = G[M_i]$.

For disconnected G, the maximal strong modules are the connected components. In this case $G = G_1 \cup G_2 \cup \ldots \cup G_p$.

If $\overline{G}$ is disconnected, the maximal strong modules of G are the connected components of $\overline{G}$. In this case $G = G_1 + G_2 + \ldots + G_p$.

If G is a serial graph and each G_i has a modular decomposition of the form

$$G_i = \cup_{j=1}^{p_i} G_{ij}, \ p_i \geq 2,$$

we say that G is a *parallel-decomposable serial graph.*

Note that any connected cograph without universal vertices is a parallel-decomposable serial graph, since cographs have no neighbourhood modules.

To study the K-behaviour of a graph G, we shall also use some powerful results that allow to predict the K-behaviour of G from the K-behaviour of some suitable subgraph of itself.

If H is a subgraph of G and $H = G - \{v_1, v_2, \ldots, v_k\}$, where v_i is a dominated vertex of $G - \{v_1, v_2, \ldots, v_{i-1}\}$, $i = 1, \ldots, k$, we say that H is a *strong retract* of G.

The following result is given in [4].

Theorem 1 *Let G be a graph. If H is a strong retract of G, G and H have the same K-behaviour.*

It is easy to see that if v is a dominated vertex of a module M of G, then v is a dominated vertex of G.

By using a weaker concept than dominance a sufficient condition for K-divergency is given in [11].

Let G, H be graphs. A mapping f from $V(G)$ to $V(H)$ is a *morphism* if $f(u)$ and $f(v)$ either coincide or are adjacent in H whenever u and v are adjacent in G. A *retraction* is a morphism f from a graph G to a subgraph H of itself such that the restriction $f|_H$ of f to $V(H)$ is the identity map. In this case we say that H is a *retract* of G.

It's useful to notice that, if $v \in V(G)$, there is always a *total retraction* from G to v.

The following theorem given in [11], describes the relationship between retracts and K-divergence.

Theorem 2 *If G has a K-divergent retract H, then G is K-divergent.*

The following lemmas, proved in [7], are useful to find a retraction of G, once its modular decomposition is known.

Lemma 1 *Let G be a graph and M a module of G. If R is a retract of $G[M]$, then $G[(V(G) - M) \cup V(R)]$ is also a retract of G.*

Lemma 2 *Let G be a graph and M a module of G. Then the quotient graph G/M is a retract of G.*

Lemma 3 *Let G be a graph. If $P = S_1 \cup S_2 \cup \ldots \cup S_q$ is a parallel module of G and some S_i is a single vertex v, then $G - \{v\}$ is a retract of G.*

Most of the results on convergence of iterated clique graphs are on the domain of clique-Helly graphs. A graph is *clique-Helly* if its cliques satisfy the Helly property: each family of mutually intersecting cliques has non-trivial intersection. Clique-Helly graphs are always K-convergent [6] and can be recognized in polynomial time [15]. It will be useful to proof one of our results to use the following theorem given in [15].

Let T be a triangle of a graph G. The *extended triangle* of G, relative to T, is the subgraph $\widehat{T}$ of G induced by the vertices which form a triangle with at least one edge of T.

Theorem 3 *A graph G is clique-Helly if and only if every extended triangle has a universal vertex.*

3 The K-behaviour of p-trees

The purpose of this section is to characterize the K-behaviour of a subclass of the P_4-connected graphs, called p-trees, since they are provided with structural properties that can be expressed in a quite analogous way to the characterization of ordinary trees.

To investigate the K-behaviour of p-trees we shall use the characterization given in [2], based on the structure of the p-chains in a p-tree, that we recall for reader's convenience.

A *p-chain* is a sequence of vertices such that every four consecutive ones induce a P_4. A p-chain $X = \{v_1, v_2, \ldots, v_k\}$ is *simple* if the only P_4s contained in $G[X]$ are induced by the set of vertices $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for $i = 1, 2, \ldots, k-3$. In other words a p-chain is simple if and only if the vertices of the p-chain induce precisely $k - 3$ P_4s.

The graphs P_k ($k \geq 4$), the graphs R_5, R_6 and R_7 (see Figure 1), the split graphs Q_k ($k \geq 5$) (see Figure 2) and their complements are simple p-chains. Remind that a graph is called *split graph* if its vertex set can be partitioned in a complete and a stable set. A split graph Q_k, has vertex set $V = \{v_1, v_2, \ldots, v_k\}$, where $A = \{v_{2i-1}\}$ is the stable set, $B = \{v_{2i}\}$ is

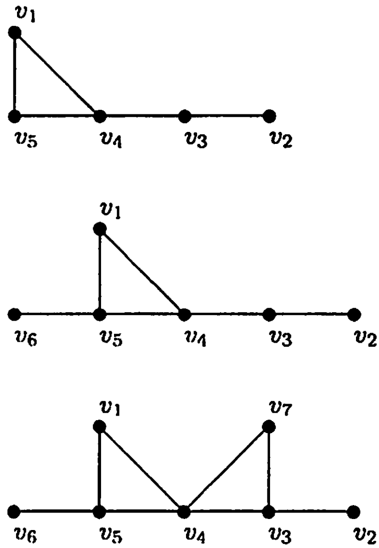

Figure 1: The graphs R_5, R_6, R_7

the complete and the edges connecting each vertex of B to the vertices of A are $\{v_{2i}v_{2i-1}$ and $v_{2i}v_{2j+1}, j > i\}$.

It has been proved in [2] that every simple p-chain is isomorphic to one of the above graphs.

A vertex is a *p-end-vertex* if it belongs to exactly one P_4.

Obviously simple p-chains are p-trees and it turns out that every p-tree can be generated starting from a simple p-chain extended by a number of p-end-vertices which can eventually be replaced by cographs.

Note that Q_5 and R_5 are isomorphic to a P_4 with one endpoint replaced by the cographs $\overline{K}_2$ and K_2 respectively.

A ***spiked p-chain*** P_k is a $P_k = (v_1, v_2, \dots, v_k)$, $k \geq 6$, extended introducing two additional vertices x and y such that x is adjacent to v_2 and v_3 and y is adjacent to v_{k-1} and v_{k-2}; moreover we request that x and y do not belong to a common P_4. One or both of the vertices x and y may be missing. In the following we shall refer to P_5, R_6 and R_7 as spiked p-chains P_5 and their vertices, from now on, will be named by $v_1, \dots, v_5, x, y$.

A ***spiked p-chain*** Q_k is a $Q_k = (v_1, v_2, \dots, v_k)$, $k \geq 6$, with additional vertices $z_2, z_3, \dots, z_{k-5}$ such that

$$\begin{aligned} N(z_i) &= \{v_2, v_4, \dots, v_{i-1}, v_{i+1}\} \cup \{z_2, z_4, \dots, z_{i-1}\} \text{ for } i \text{ odd;} \\ \overline{N}(z_i) &= \{v_1, v_3, \dots, v_{i-1}, v_{i+1}\} \cup \{z_3, z_5, \dots, z_{i-1}\} \text{ for } i \text{ even.} \end{aligned}$$

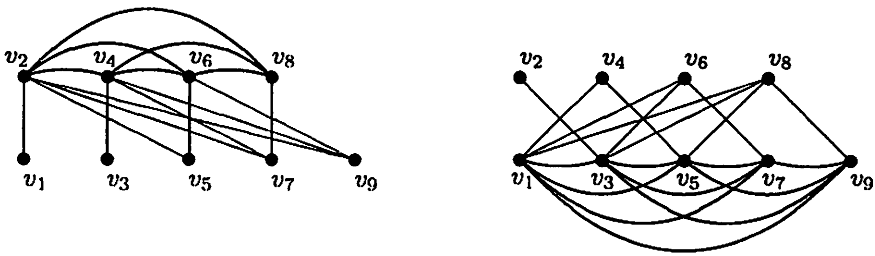

Figure 2: The graphs Q_9 and $\overline{Q}_9$

Figure 3: The spiked p-chain P_8

Any of the vertices $z_2, z_3, \ldots, z_{k-5}$ may be missing (see Figure 4).

A spiked p-chain $\overline{P}_k$ (or $\overline{Q}_k$) is the complement of a spiked p-chain P_k (or Q_k).

Finally we have the following characterization of p-trees.

Theorem 4 (Babel [2]) *A graph is a p-tree if and only if it is either a P_4 with one vertex replaced by a cograph or a spiked p-chain with the p-end-vertices replaced by cographs.*

It is easy to verify that v_1, x, y, v_k and $v_1, z_2, z_3, \ldots, z_{k-5}, v_k$ are the only p-end-vertices of spiked P_k ($\overline{P}_k$) and Q_k ($\overline{Q}_k$) respectively. In fact P_k, Q_k and their complements are simple p-chains and the unique P_4 containing z_i, x or y is induced by $\{z_i, v_{i+1}, v_{i+2}, v_{i+3}\}$, $\{x, v_3, v_4, v_5\}$ or $\{y, v_{n-2}, v_{n-3}, v_{n-4}\}$ respectively.

Following the "taxonomy" of the class of p-trees proposed in Theorem 4 we shall characterize their K-behaviour.

Theorem 5 *Let G be either a spiked p-chain P_k ($k \geq 5$) or Q_k ($k \geq 6$) with the p-end-vertices replaced by a cograph or a P_4 with an endpoint replaced by a cograph. Then G is K-null.*

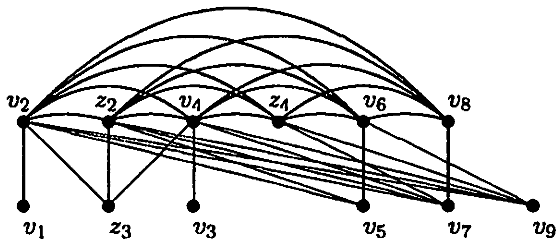

Figure 4: The spiked p-chain Q_9

Proof The vertices of any cograph replacing a p-end-vertex of the spiked p-chain or an endpoint of P_4 are dominated either from v_2 or v_{k-1}. After their elimination we obtain either a shorter path or a split graph. By repeated elimination of dominated vertices we obtain a strong retract of G isomorphic to K_1 and then by Theorem 1, G is K-null. ■

Notice that Theorem 5 holds more in general whenever we replace cographs by any graph.

From this point onwards, let us denote by H_v the cograph eventually replacing a p-end-vertex v.

Theorem 6 *Let G be a spiked p-chain $\overline{Q}_k$, $k \geq 6$, with the p-end-vertices replaced by cographs. Then G is K-null.*

Proof Vertex v_3 dominates any other vertex of G except the vertices of H_{v_1}, v_4 and v_5. After their elimination, we obtain a strong retract of G with a universal vertex v_5. Then, by Theorem 1, G is K-null. ■

The following lemma will be very useful in the next pages; even if it is a particular case of Theorem 11 in [7], we will give a direct proof.

Lemma 4 *Let G be a P_3 with the midpoint replaced by a cograph H. Then G is K-convergent if and only if each connected component of H contains a universal vertex. Furthermore G is K-null if H is connected.*

Proof Let $P_3 = (v_1, v_2, v_3)$. Without loss of generality, we can assume that we replace the vertex v_2 with the cograph $H = H_1 \cup H_2 \cup \cdots \cup H_q$, $q \geq 1$.

If each H_i, $i = 1, \ldots, q$, contains a universal vertex u_i then u_i is a strong retract of H_i and $G[v_1, v_2, u_1, u_2, \ldots, u_q] \cong K_{2,q}$ is a K-convergent strong retract of G. Hence G is also K-convergent by Theorem 1. Furthermore G is K-null if $q = 1$.

Otherwise at least one H_i is a parallel decomposable serial graph. Without loss of generality let us assume that $i = 1$. Then by Lemma 2 we can retract each H_j, $j \geq 2$, to a single vertex u_j, and by Lemma 3 we can retract H to H_1. Let $H_1 = M_1 + M_2 + \cdots + M_r$, $r \geq 2$, and $M_i = \cup_{j=1}^{p_i} M_{i,j}$, $p_i \geq 2$, be the modular decomposition of H_1. From Lemma 2 we can retract each $M_{i,j}$ to a single vertex and therefore $K_{p_1,p_2,\ldots,p_r}$, $r \geq 2$ is a retract of H. Since v_1 and v_3 are not connected by any edge and both are connected to every vertex of H, then $K_{p_1,p_2,\ldots,p_r,2}$, $r \geq 2$, is a K-divergent retract of G. Hence G is also K-divergent by Theorem 2. ■

Lemma 5 *Let G be isomorphic to a $\overline{P}_k$ $(k \geq 5)$. Then G is K-convergent if and only if $k = 5, 6$ or $k = 3h + 1$, with $h \geq 2$. Furthermore in the last case G is K-null.*

Proof Vertex v_1 dominates vertex v_3. After deleting v_3, by iterating the process, we can successively delete from $\overline{P}_k$ the vertices v_{3i}, $i = 2, \ldots, h$, with $h = \lfloor \frac{k}{3} \rfloor$, since each one of them becomes dominated by the vertex v_{3i-2}. Then, if $k = 3h$, $h \geq 2$, or $k = 3h+2$, $h \geq 1$, the strong retract of $\overline{P}_k$ is $H = K_{\underbrace{2,2,\ldots,2}_{\lceil \frac{k}{3} \rceil}}$, that is K-convergent if and only if $k = 5, 6$. Otherwise

$k = 3h + 1$, $h \geq 2$, and the strong retract of $\overline{P}_k$ is $H = K_{\underbrace{2,2,\ldots,2}_{\lfloor \frac{k}{3} \rfloor}} + K_1$

that is K-null. By Theorem 1, G and H have the same K-behaviour. ■

Let us denote by P_3^* a graph isomorphic to a P_3 with the midpoint replaced by a cograph H.

Theorem 7 *Let G be a P_4 with a midpoint replaced by a cograph H. Then G is K-convergent if and only if each connected component of H contains a universal vertex. Furthermore G is K-null if H is connected.*

Proof Without loss of generality, we may assume replacing the vertex v_2 with a cograph H. The vertex v_3 dominates vertex v_4. Then P_3^* is a strong retract of G. Then G has the same K-behaviour of P_3^* by Theorem 1. Hence the proof follows from Lemma 4. ■

Theorem 8 *Let G be a $\overline{P}_5$ with the p-end-vertices replaced by cographs. Then G is K-convergent if and only if each connected component of the cographs H_{v_1} and H_{v_5} has a universal vertex.*

Proof Let G be isomorphic to $\overline{P}_5$ with the p-end-vertices v_1 and v_5 replaced by the cographs H_{v_1} and H_{v_5}. Let $H_{v_1} = H_{v_1}^1 \cup H_{v_1}^2 \cup \cdots \cup H_{v_1}^{q_1}$ and

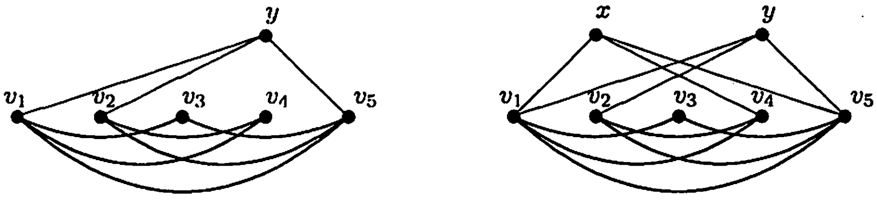

Figure 5: The graphs $\overline{R}_6$ and $\overline{R}_7$

$H_{v_5} = H^1_{v_5} \cup H^2_{v_5} \cup \cdots \cup H^{q_5}_{v_5}$. Let us assume that each connected component of H_{v_1} and H_{v_5} contains a universal vertex. Let u^i_1, $1 \le i \le q_1$, and u^j_5, $1 \le j \le q_5$, be the universal vertices of the connected components of H_{v_1} and H_{v_5} respectively. Then $G[v_2, v_3, v_4, u^1_1, \ldots, u^{q_1}_1, u^1_5, u^2_5, \ldots, u^{q_5}_5]$ is a strong retract of G. The only triangles of $G[v_2, v_3, v_4, u^1_1, \ldots, u^{q_1}_1, u^1_5, u^2_5, \ldots, u^{q_5}_5]$ are of the form v_3, u^i_1, u^j_5, $1 \le i \le q_1$, $1 \le j \le q_5$. For each of the above triangles, the extended triangle is $G[v_3, u^1_1, \ldots, u^{q_1}_1, u^1_5, u^2_5, \ldots, u^{q_5}_5]$ that contains the universal vertex v_3. Then, by Theorem 3, $G[v_2, v_3, v_4, u^1_1, \ldots, u^{q_1}_1, u^1_5, u^2_5, \ldots, u^{q_5}_5]$ is clique-Helly and therefore K-convergent. Then G is also K-convergent by Theorem 1.

Otherwise at least one connected component of either H_{v_1} or H_{v_5}, say of H_{v_5}, does not contain a universal vertex. By Lemma 2 we can retract each $H^i_{v_1}$, $1 \le i \le q_1$, to a single vertex v_i, and by Lemma 3 we can retract H_{v_1} to v_1. Then v_1 dominates v_3 and $(\{v_1\} \cup \{v_2\}) + (\{v_4\} \cup H_{v_5})$ is a retract of G isomorphic to a P^*_3 with the midpoint replaced by the cograph $H = \{v_4\} \cup H_{v_5}$.

By Lemma 4, P^*_3 is K-divergent. Hence, by Theorem 2, G is also K-divergent. ■

In the following theorems we will assume, for the cases of $\overline{P}_6$ and $\overline{R}_6$, that the additional vertex present is the vertex y.

Theorem 9 *Let G be a $\overline{R}_6$ with the p-end-vertices replaced by cographs. Then G is K-convergent if and only if H_k or H_y has a universal vertex.*

Proof Let G be isomorphic to $\overline{R}_6$ (see Figure 5) with the p-end-vertices v_5 and y replaced by the cographs H_{v_5} and H_y. The vertex v_1 dominates the vertex v_3. Then the graph $\overline{K}_2 + ((H_{v_5} + H_y) \cup \{v_4\})$ is a strong retract of G isomorphic to a P^*_3 with the midpoint replaced by the cograph $H = (H_{v_5} + H_y) \cup \{v_4\}$. By Lemma 4, P^*_3 is K-convergent if and only if each connected component of H contains a universal vertex, that happens if and only if either H_{v_5} or H_y contains a universal vertex. By Theorem 1, G behaves like P^*_3. ■

Theorem 10 *Let G be a $\overline{R_7}$ with the p-end-vertices replaced by cographs. Then G is K-convergent.*

Proof The p-end-vertices are x and y (see Figure 5). The vertex v_1 dominates every vertex of H_x and the vertex v_5 dominates every vertex of H_y. Then by Theorem 1 G behaves like a $\overline{P_5}$ and it is therefore K-convergent by Lemma 5. ■

Theorem 11 *Let G be a spiked p-chain $\overline{P_6}$ with the p-end-vertices replaced by cographs. Then G is K-convergent if and only if H_{v_6} has a universal vertex.*

Proof The p-end-vertices are v_1, v_6 and y. If H_{v_6} contains a universal vertex u_6, then it dominates v_4 and every vertex of H_{v_6}, H_y and after their deletion v_3 dominates every vertex of H_{v_1}. Hence $K_{2,2}$ is a K-convergent strong retract of G and by Theorem 1, G is also K-convergent. Otherwise we can retract H_{v_1} to a single vertex v_1, then v_1 dominates v_3 and $\overline{K_2} + ((H_{v_6} + (H_y \cup \{v_4\})) \cup \{v_5\})$ is a retract of G isomorphic to a P_3^* with the midpoint replaced by the cograph $H = (H_{v_6} + (H_y \cup \{v_4\})) \cup \{v_5\}$. Since H_{v_6} does not contain a universal vertex H has a connected component that is a parallel decomposable serial graph. By Lemma 4, P_3^* is K-divergent. Hence, by Theorem 2, G is also K-divergent. ■

Theorem 12 *Let G be a spiked p-chain $\overline{P_k}$, $k \geq 7$, with the p-end-vertices replaced by cographs. Then G is K-convergent if and only if $k = 3h+1$, with $h \geq 2$ and the cographs H_1 and H_k have a universal vertex. Furthermore if G is K-convergent then it is K-null.*

Proof Let G be a spiked p-chain $\overline{P_k}$, $k \geq 7$ (see Figure 3).

If $k = 3h + 1$, with $h \geq 2$, we must consider two cases. Both the cographs H_{v_1} and H_{v_k} have a universal vertex, say u_1 and u_k respectively, then the vertex u_1 dominates every vertex of H_{v_1} and H_x and the vertex u_k dominates every vertex of H_{v_k} and H_y. Hence $\overline{P_k}$ is a K-null strong retract of G by Lemma 5 and, by Theorem 1, G is also K-null.

At least one of the cographs H_{v_1} or H_{v_k}, say H_{v_k}, does not contain a universal vertex. By Lemma 2, we can retract H_{v_1} to a single vertex v_1 and delete every vertex of H_x since they are dominated vertices. After, we can delete one after the other the vertices v_{3i}, $i = 1, 2, \ldots, h$, with $h = \lfloor \frac{k}{3} \rfloor$ and every vertex of H_y since each one of them becomes a dominated vertex. Then $K_{\underbrace{2,2,\ldots,2}_{h}} + H_{v_k}$ is a retract of G. Since H_{v_k} does not contain a universal vertex it is either a disconnected graph or a parallel decomposable serial graph. If H_{v_k} is a disconnected graph, by Lemma 2, we can retract each component to a single vertex and, by Lemma 3, we obtain a $\overline{K}_2$ as a retract of H_{v_k}. If H_{v_k} is a parallel decomposable serial graph, in the same way we can retract each parallel module to a $\overline{K}_2$ obtaining a K-divergent retract $K_{\underbrace{2,2,\ldots,2}_{h'}}$ with $h' \geq 3$. Hence, by Theorem 2, G is K-divergent.

If $k = 3h$, with $h \geq 3$ or $k = 3h + 2$, with $h \geq 2$, by Lemma 2 and Lemma 3, we can retract H_{v_1} and H_{v_k} to a single vertex v_1 and v_k respectively. Then the vertex v_1 dominates every vertex of H_x and the vertex v_k dominates every vertex of H_y. Hence $\overline{P}_k$ is a K-divergent retract of G and by Theorem 2, G is also K-divergent. ■

Finally, we note that the characterization given in the above theorems leads to a linear time algorithm for deciding the K-behaviour of any p-tree. In fact, it is known that a p-tree can be recognized in linear time [2] and the conditions of the above theorems can also be tested in linear time.

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