The k-exponents of Primitive, Nearly Reducible Matrices *

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Abstract

Let $D = (V, E)$ be a primitive, minimally strong digraph. In 1982, J. A. Ross studied the exponent of D and obtained that $\exp(D) \leq n + s(n-3)$, where s is the length of a shortest circuit in D([7]). In this paper, the k-exponent of D is studied. Our principle result is that

$$\exp_D(k) \leq \begin{cases} k+1+s(n-3), & \text{if } 1 \leq k \leq s, \\ k+s(n-3), & \text{if } s+1 \leq k \leq n, \end{cases}$$

with equality if and only if D isomorphic to the digraph $D_{s,n}$ with vertex set $V(D_{s,n}) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(D_{s,n}) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_s, v_1), (v_n, v_2)\}$. If $(s, n-1) \neq 1$, then

$$\exp_D(k) < \begin{cases} k+1+s(n-3), & \text{if } 1 \leq k \leq s, \\ k+s(n-3), & \text{if } s+1 \leq k \leq n, \end{cases}$$

and if $(s, n-1) = 1$, then $D_{s,n}$ is a primitive, minimally strong digraph on n vertices with the k-exponent

$$\exp_D(k) = \begin{cases} k+1+s(n-3), & \text{if } 1 \leq k \leq s, \\ k+s(n-3), & \text{if } s+1 \leq k \leq n. \end{cases}$$

Moreover, We provide a new proof of Theorem 1 in [6] and Theorem 2 in [14] by applying this result.

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1 Introduction

Exponents of primitive, nearly reducible matrices have been studied in [3, 5, 7, 9, 11, 13]. In 1990, from the background of a memoryless communication system, R. A. Brualdi and B. L. Liu ([2]) generalized the concept of exponent for a primitive matrices(or a primitive digraph) and introduced the concept of k-exponent. In this paper, we study the k-exponents of primitive, nearly reducible matrices. The definitions of irreducible matrices, nearly reducible matrices, primitive matrices and their exponents as well as the definitions of strong digraphs, minimally strong digraphs, primitive digraphs and their exponents can be found in [3] and [7]. For any $n \times n$ nonnegative matrix A, we define its associated digraph $D(A) = (V, E)$ to be the digraph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and arc set $E = \{(v_i, v_j) : a_{ij} > 0\}$. Clearly $D(A)$ depends only on the zero-nonzero pattern of A. It is well known that with this correspondence of matrices and digraphs, we have the following results: A irreducible $\Longleftrightarrow$ $D(A)$ strong, A nearly reducible $\Longleftrightarrow$ $D(A)$ minimally strong, A primitive $\Longleftrightarrow$ $D(A)$ primitive and in this case A and $D(A)$ have the same k-exponent $\exp_A(k) = \exp_{D(A)}(k)$ (which we shall define later). So we can use the graph theoretical language to formulate and prove our results.

Let $D = (V, E)$ denote a digraph with n vertices. A walk π of length p from u to v in D is a sequence of vertices $u, u_1, \dots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \dots, (u_{p-1}, v)$, where the vertices and arcs need not to be distinct, and denoted by $\pi = (u, u_1, \dots, u_{p-1}, v)$. The initial vertex of π is u, the terminal vertex is v, and $u_1, \dots, u_{p-1}$ are the internal vertices of π. If $u = v$, then π is a circuit (or a closed walk). A circuit whose length is 1 is called a loop. A circuit π is elementary if the vertices are distinct except for $u = v$. A path is a walk with distinct vertices.

We recall that a digraph D is primitive if there exists an integer $k > 0$ such that for all ordered pairs of vertices $v_i, v_j \in V$, there exists a walk of length k from v_i to v_j in D, and the least such k is called the exponent of D, denoted by $\exp(D)$. We know that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its elementary circuits is 1.

Let $D = (V, E)$ be a primitive digraph with n vertices $v_1, v_2, \dots, v_n$. For any v_i, $v_j \in V$, let $\exp_D(v_i, v_j) :=$ the smallest integer p such that there is a walk of length t from v_i to v_j for each integer $t \geq p$. Clearly

$\exp(D) = \max\{\exp_D(u_i, v_j) : u_i, v_j \in V\}$. Let the exponent of vertex v_i be defined by $\exp_D(v_i) := \max\{\exp_D(v_i, v_j) : v_j \in V\}(i = 1, 2, \ldots, n)$. Then $\exp_D(v_i)$ is the smallest integer p such that there is a walk of length p from v_i to each vertex of D. It follows also that $\exp(D) = \max\{\exp_D(u_i) : u_i \in V\}$. We choose to order the vertices of D in such a way that $\exp_D(v_{i_1}) \leq \exp_D(v_{i_2}) \leq \cdots \leq \exp_D(v_{i_n})$, and we call the number $\exp_D(v_{i_k})$ the k-point exponent of D (the k-exponent for short), which is denoted by $\exp_D(k)$. Clearly $\exp(D) = \exp_D(n)$.

Let $D = (V, E)$ be a primitive, minimally strong digraph with n vertices ($PMSD_n$). We define $\exp^m(n) = \max\{\exp_D(n) : D \in PMSD_n\}$ and $\exp^m(n,k) = \max\{\exp_D(k) : D \in PMSD_n\}(k = 1, 2, \ldots, n)$. Clearly $\exp^m(n, n) = \exp^m(n)$. In 1982, J. A. Ross [7, Theorem 4.9] obtained that $\exp_D(n) \leq n + s(n-3)$, where s is the length of a shortest circuit in D. In 1999, B. L. Liu obtained

Theorem A *([6], Theorem 1)*

$$\exp^m(n,k) = \begin{cases} n^2 - 5n + 7 + k, & \text{if} \quad 1 \leq k \leq n-2, \\ n^2 - 4n + 5, & \text{if} \quad k = n-1, \\ n^2 - 4n + 6, & \text{if} \quad k = n. \end{cases}$$

In 2002, B. Zhou characterized primitive, minimally strong digraphs on n vertices whose k-exponent ($1 \leq k \leq n$) achieve the maximum value $\exp^m(n,k)$, and obtained

Theorem B *([14], Theorem 2) Let D be primitive, minimally strong digraph on n vertices, and let $n > 4$ and $1 \leq k \leq n$. Then $\exp_D(k) = \exp^m(n,k)$ if and only if D is isomorphic to the digraph $D_{n-2,n} = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1)\} \cup \{(v_{n-2}, v_1), (v_n, v_2)\}$.*

In this paper, using the same constructions employed in [7] by J. A. Ross, we prove

Theorem 1.1 *Let D be a primitive, minimally strong digraph on n vertices, and let $s = s(D)$ be the length of a shortest circuit of D. Then*

$$\exp_D(k) \leq \begin{cases} k + 1 + s(n-3), & \text{if} \quad 1 \leq k \leq s, \\ k + s(n-3), & \text{if} \quad s+1 \leq k \leq n, \end{cases}$$

with equality if and only if D isomorphic to the digraph $D_{s,n} = (V, E)$, where vertex set $V(D_{s,n}) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(D_{s,n}) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_s, v_1), (v_n, v_2)\}$. If $(s, n-1) \neq 1$, then

$$\exp_D(k) < \begin{cases} k + 1 + s(n-3), & \text{if} \quad 1 \leq k \leq s, \\ k + s(n-3), & \text{if} \quad s+1 \leq k \leq n. \end{cases}$$

And if $(s,n-1)=1$, then $D_{s,n}$ is a primitive, minimally strong digraph on n vertices with k-exponent

$$\exp_{D_{s,n}}(k)=\begin{cases} k+1+s(n-3), & \text{if } 1\le k\le s,\\ k+s(n-3), & \text{if } s+1\le k\le n.\end{cases}$$

Moreover, we give a new proof of Theorems A and B by using Theorem 1.1.

2 Preliminaries

Let $D=(V,E)$ be a primitive digraph for which $r_1,r_2,\ldots,r_\lambda$ are the distinct lengths of the elementary circuits. Write $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$. For $x,y\in V(D)$, the relative distance $d_{L(D)}(x,y)$ from x to y is defined to be the length of the shortest walk from x to y that meets at least one circuit of each length r_i for $i=1,2,\ldots,\lambda$.

Let $a_1,a_2,\ldots,a_k$ be distinct positive integers with $gcd(a_1,a_2,\ldots,a_k)=1$. The Frobenius number $\phi(a_1,a_2,\ldots,a_k)$ is defined to be the least integer m such that every integer with $t\ge m$ can be expressed as $t=z_1a_1+z_2a_2+\cdots+z_ka_k$, where $z_1,z_2,\ldots,z_k$ are nonnegative integers. We know that $\phi(a_1,a_2,\ldots,a_k)$ is finite if $gcd(a_1,a_2,\ldots,a_k)=1$, and $\phi(a_1,a_2)=(a_1-1)(a_2-1)$.

Lemma 2.1 *([10], Theorem 2.2) Let D be a digraph on n vertices, and let $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$, then*

$$\exp_D(x,y)\le d_{L(D)}(x,y)+\phi_{L(D)},$$

where $\phi_{L(D)}=\phi(r_1,r_2,\ldots,r_\lambda)$. Furthermore, we have

$$\exp_D(x)\le \max\{d_{L(D)}(x,y):y\in V\}+\phi_{L(D)},$$

$$\exp_D(n)\le \max\{d_{L(D)}(x,y):x,y\in V\}+\phi_{L(D)}.$$

An ordered pair of vertices x,y in a digraph D is said to have the unique walk property if every walk from x to y of length at least $d_{L(D)}(x,y)$ consists of some walk π of length $d_{L(D)}(x,y)$ from x to y augmented by a number of elementary circuits each of which has a vertex in common with π (note that the word "unique" in this definition refers to the length of the walk π rather than to the walk π itself).

Lemma 2.2 *Let D be a primitive digraph, $L(D)=\{r_1,r_2,\ldots,r_\lambda\}$. If the ordered pair of vertices x,y has the unique walk property, then*

$$\exp_D(x,y)=d_{L(D)}(x,y)+\phi_{L(D)}$$

Proof. If $\exp_D(x,y) < d_{L(D)}(x,y) + \phi_{L(D)}$, then there exists a walk from x to y of length $\omega = d_{L(D)}(x,y) + \phi_{L(D)} - 1$ by the definition of $\exp_D(x,y)$. By the definition of unique walk property, there exist nonnegative integer $z_1, z_2, \ldots, z_\lambda$ such that $\phi_{L(D)} - 1 = z_1r_1 + z_2r_2 + \cdots + z_\lambda r_\lambda$. This contradict that $\phi_{L(D)}$ is Frobenius number of $r_1, r_2, \ldots, r_\lambda$. Hence $\exp_D(x,y) \geq d_{L(D)}(x,y) + \phi_{L(D)}$. By Lemma 2.1, $\exp_D(x,y) = d_{L(D)}(x,y) + \phi_{L(D)}$. □

Lemma 2.3 *([7], Lemma 2.1) Let $D = (V,E)$ be a minimally strong digraph and let $X \subseteq V$. If the digraph D_X induced on X is strong, then D_X is minimally strong.*

In a strong digraph D=(V,E), each vertex x has indegree $\delta_D^-(x)$ and outdegree $\delta_D^+(x)$ at least one. A vertex x is called an antinode if $\delta_D^-(x) = \delta_D^+(x) = 1$; otherwise , x is called a node. We assure that D is minimally strong and π =(x_0, x_1, ..., x_k) is a path with $k \geq 2$ whose initial and terminal vertices are nodes and whose internal vertices are antinodes. Letting Y={x_1, ..., x_{k-1}}, we say π is a branch provided D_{V-Y} is strong. By Lemma 2.3 we know that D_{V-Y} is minimally strong, and we write $D_{V-Y} = D \sim \pi$. We say $D \sim \pi$ is formed from D by removing the branch π, or D is formed from $D \sim \pi$ by adding the branch π. Every minimally strong digraph which is not an elementary circuit contains a branch.

Lemma 2.4 *([7], Corollary 2.3) Let D be a minimally strong digraph which is not an elementary circuit, and D contains an elementary circuit of length s. Then there exists a branch π of D such that $D \sim \pi$ contains an elementary circuit of length s.*

Let $D = (V,E)$ be a digraph and let $x,y \in V$. We say x and y are connected in D if there exists a sequence of vertices $x_0, x_1, \ldots, x_k$ with $k \geq 0$ such that $x = x_0, y = x_k$, and either$(x_i, x_{i+1}) \in E$ or $(x_{i+1}, x_i) \in E$ for $i = 0, 1, \ldots, k-1$. This is clearly an equivalence relation, and the equivalence classes are the connected components of D. We denote by D^k the digraph with vertex set V and arc set $E^k = \{(x,y)$: there exists in D a walk of length k from x to $y\}$.

Let D denote a strong digraph of order n, and let $h = h(D)$ be the greatest common divisor of the lengths of the (elementary) circuits of D. We call the integer h the index of cyclicity.

Lemma 2.5 *([7], Lemma 4.4) Let D be a strong digraph with index of cyclicity h. Then for each integer $j \geq 1$, the digraph D^{hj} has h connected components. Moreover, each connected component is strongly connected and primitive. In particular, for every integer $j \geq 1$, D^j is strongly connected and primitive whenever D is.*

Lemma 2.6 *([7], Lemma 4.5) Let D be a strong digraph with index of cyclicity h. If (x_0, x_1, ..., x_h) is any walk in D, then for each integer $j \geq 1$,* *the vertices x_0, x_1, ..., x_{h-1} are all in different connected components of D^{hj}, and x_0, x_h are in the same component.*

Let D be a minimally strong digraph. If a branch is removed from D, the resulting minimally strong digraph either is an elementary circuit or contains a branch. Thus one may continue to remove branches from the resulting digraphs until an elementary circuit is obtained. The number of branches which must be removed to obtain an elementary circuit is an invariant of the digraph D ([1, 8]), and is denoted by $\mu(D)$.

Now let D be a minimally strong digraph which is not an elementary circuit. Suppose there exists a sequence D_0, D_1, ..., $D_\mu = D$ of minimally strong digraphs which satisfy

(2.1) D_0 is an elementary circuit,

(2.2) D_i is formed from $D_{i-1} (i = 1, 2, \ldots, \mu)$ by adding the branch $\pi_i=(x_0^i, x_1^i, \ldots, x_{k_i}^i)$ such that for $i = 2, 3, \ldots, \mu$, we have $x_0^i = x_{r_i}^{i-1}$ and $x_{k_i}^i = x_{s_i}^{i-1}$ where $1 \le s_i \le r_i \le k_{i-1} - 1$,

we say the digraph D is special. Note that for $i = 1, 2, \ldots, \mu - 1$, the path π_i is a branch of D_i but not a branch of D. Also, we notice that $\mu = \mu(D)$.

Let D be a minimally strong digraph which is not an elementary circuit. Then by [7, Lemma 2.5], the number of branches of D is greater than 2, and the number of branches of D equals 2 if and only if D is special.

Let $D = (V, E)$ be an arbitrary digraph, and let $Y \subseteq V$ with $Y \neq \Phi$. Let $y \notin V$ and form the digraph $D*Y=((V-Y) \cup \{y\}, E_Y)$, where $(u, v) \in E_Y$ if and only if one of the following holds

(2.3) $u, v \in V - Y$, and $(u, v) \in E$.

(2.4) $u \in V - Y$, $v = y$, and there exists $\omega \in Y$ such that $(u, \omega) \in E$.

(2.5) $u = y$, $v \in V - Y$, and there exists $\omega \in Y$ such that $(\omega, v) \in E$

We say $D * Y$ is the contraction of Y (in D).

Lemma 2.7 *([7], Lemma 4.3) Let D be a primitive, minimally strong digraph. Suppose that for any branch α of D, either $s(D \sim \alpha) > s(D)$ or every circuit in $D \sim \alpha$ has length divisible by $s(D)$. Then D is special, and its two branches π and ρ satisfy*

(2.6) $s(D \sim \rho) > s(D)$

(2.7) Every circuit in $D \sim \pi$ has length divisible by $s(D)$.

Moreover, D contains a unique circuit of length $s(D)$.

Remark The proof of following Lemma 2.8 is almost the same as in [7, Lemma 4.7]. However, the conclusion here is different from [7, Lemma 4.7]. So we include the proof here.

Lemma 2.8 *Let D be a primitive special digraph with $\mu(D) \ge 2$, and let $s = s(D)$. Suppose the branches ρ and π of D satisfy (2.6) and (2.7). Then*

$(D \sim \pi)^s$ has s connected components $A_1, A_2, \ldots, A_s$ which are strongly connected. Let

$$D^* = (\cdots((D^s * A_1) * A_2) \cdots) * A_s.$$

If $s \geq 3$ and D^* is not an elementary circuit, then $\exp_D(1) \leq s(n-3)$.

Proof. By [7, Lemma 4.7], it suffices to prove the last assertion. By the proof of [7, Lemma 4.7], we have that D^s and D^* are strongly connected, $|A_i| \geq 2$ $(i = 1, \ldots, s)$ and A_i $(i = 1, \ldots, s)$ contains a loop vertex in $(D \sim \pi)^s$. Let $\mathcal{Y}_i$ be the contraction of A_i in D^* for each $i = 1, \ldots, s$. By the proof of [7, Lemma 4.7] again, there exists a vertex x of D^* which must be an internal vertex of π such that $\delta^+_{D^*}(x) = 2$, and the terminal vertices of these arcs are an internal vertex z of π and $\mathcal{Y}_r$ for some $r = 1, \ldots, s$. Choose $\mathcal{Y}_p$ and $\mathcal{Y}_q$ such that $d_{D^*}(\mathcal{Y}_p, x) = \min\{d_{D^*}(\mathcal{Y}_i, x) : i = 1, \ldots, s\}$ and $d_{D^*}(z, \mathcal{Y}_q) = \min\{d_{D^*}(z, \mathcal{Y}_i) : i = 1, \ldots, s\}$. By the proof of [7, Lemma 4.7] again, we have that p, q, r are distinct. Then in D^* there exists a path from $\mathcal{Y}_p$ to any vertex which avoids either $\mathcal{Y}_r$ or $\mathcal{Y}_q$. Let $a_p \in A_p$ be a loop vertex in $(D \sim \pi)^s$ (notice that a_p is also a loop vertex in D^s). It follows that in D^s there exists a path from a_p to any vertex which avoids either A_r or A_q. Since $|A_r| \geq 2$ and $|A_q| \geq 2$, there exists a walk of length $n-3$ from a_p to each vertex in D^s. Also, there exists a walk of length $s(n-3)$ from a_p to each vertex in D. Hence $\exp_D(a_p) \leq s(n-3)$. Thus $\exp_D(1) \leq \exp_D(a_p) \leq s(n-3)$. □

Lemma 2.9 *([2], Lemma 3.3)Let D be a primitive digraph with n vertices. Then*

$$\exp_D(k) \leq \exp_D(k-1) + 1 \quad (2 \leq k \leq n),$$

$$\exp_D(k) \leq \exp_D(1) + (k-1) \quad (1 \leq k \leq n).$$

Lemma 2.10 *([6], Lemma 8)*

$$\exp_{D_{n-2,n}}(k) = \begin{cases} n^2 - 5n + 7 + k, & \text{if} \quad 1 \leq k \leq n-2, \\ n^2 - 4n + 5, & \text{if} \quad k = n-1, \\ n^2 - 4n + 6 & \text{if} \quad k = n, \end{cases}$$

where $D_{n-2,n} = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_{n-2}, v_1), (v_n, v_2)\}$.

3 Proofs of The Theorems

First we prove Theorem 1.1.
Proof. The proof is by induction on $\mu = \mu(D)$.

If $\mu = 1$, then D is isomorphic to the digraph $\hat{D}_{s,n}$ with the vertex set $V(\hat{D}_{s,n}) = \{v_1, v_2, \ldots, v_n\}$ and the arc set $E(\hat{D}_{s,n}) = \{(v_i, v_{i+1}) : 1 \le i \le n-1\} \cup \{(v_s, v_1), (v_n, v_{p+1})\}$. Clearly $L(\hat{D}_{s,n}) = \{s, n-p\}$. By the condition that D is a primitive minimally strong digraph and s is the length of the shortest circuit of D, we have that $n-p > s \ge 2$, $p \ge 1$ and $(s, n-p) = 1$. Observe that the ordered pair of vertices v_{s+1}, v_n in $\hat{D}_{s,n}$ have unique walk property. By Lemma 2.2, we have

$$\begin{aligned}
&\exp_{\hat{D}_{s,n}}(v_{s+1}) = \max\{\exp_{\hat{D}_{s,n}}(v_{s+1}, v) \colon v \in V\}\\
&= \exp_{\hat{D}_{s,n}}(v_{s+1}, v_n) = d_{L(\hat{D}_{s,n})}(v_{s+1}, v_n) + \phi_{L(\hat{D}_{s,n})}\\
&= n - s - 1 + n - p + (s-1)(n-p-1) = n + s(n-p-2).
\end{aligned}$$

Clearly

$$\begin{aligned}
&\exp_{\hat{D}_{s,n}}(v_s) < \exp_{\hat{D}_{s,n}}(v_{s-1}) < \cdots < \exp_{\hat{D}_{s,n}}(v_{p+1})\\
&< \exp_{\hat{D}_{s,n}}(v_p) = \exp_{\hat{D}_{s,n}}(v_n) < \exp_{\hat{D}_{s,n}}(v_{p-1})\\
&= \exp_{\hat{D}_{s,n}}(v_{n-1}) < \cdots < \exp_{\hat{D}_{s,n}}(v_1) = \exp_{\hat{D}_{s,n}}(v_{n-p+1})\\
&< \exp_{\hat{D}_{s,n}}(v_{n-p}) < \exp_{\hat{D}_{s,n}}(v_{n-p-1}) < \cdots < \exp_{\hat{D}_{s,n}}(v_{s+1})
\end{aligned}$$

Hence

$$\exp_D(k) = \exp_{\hat{D}_{s,n}}(k)$$

$$= \begin{cases} \exp_{\hat{D}_{s,n}}(v_{s-k+1}), & \text{if } 1 \le k \le s-p,\\ \exp_{\hat{D}_{s,n}}(v_{p-[\frac{l-1}{2}]}), & \text{if } k = s-p+l,\\ & \text{where } l \in \{1,2,\ldots,2p\}, [a] \text{ denotes}\\ & \text{the largest integer not exceeding } a.\\ \exp_{\hat{D}_{s,n}}(v_{n-k+s+1}), & \text{if } s+p+1 \le k \le n \end{cases}$$

$$= \begin{cases} k+p+s(n-p-2), & \text{if } 1 \le k \le s-p,\\ k+p-l+[\frac{l+1}{2}]+s(n-p-2), & \text{if } k = s-p+l,\\ k+s(n-p-2), & \text{if } s+p+1 \le k \le n \end{cases}$$

$$= \begin{cases} k+1+s(n-3)-(s-1)(p-1), & \text{if } 1 \le k \le s-p,\\ k+1+s(n-3)-(s-1)(p-1)+[\frac{1-l}{2}], & \text{if } k = s-p+l,\\ & \text{where } l \in \{1,2,\ldots,p\},\\ k+s(n-3)-(s-1)(p-1)+1+[\frac{1-l}{2}], & \text{if } k = s-p+l,\\ & \text{where } l \in \{p+1,\ldots,2p\},\\ k+s(n-3)-s(p-1) & \text{if } s+p+1 \le k \le n \end{cases}$$

$$\le \begin{cases} k+1+s(n-3), & \text{if } 1 \le k \le s,\\ k+s(n-3), & \text{if } s+1 \le k \le n, \end{cases}$$

with equility if and only if $p = 1$, namely $D \cong D_{s,n}$. This proves the theorem when $\mu = 1$.

Now suppose $\mu(D) \ge 2$, we must show that

$$\exp_D(k) \le \begin{cases} k+s(n-3), & \text{if } 1 \le k \le s,\\ k-1+s(n-3), & \text{if } s+1 \le k \le n. \end{cases}$$

First we assure that D contains a branch π such that $s(D \sim \pi) = s(D)$ and $D \sim \pi$ is primitive. Let l be the number of internal vertices of π, then we have $l \geq 1$ since D is a minimally strong digraph. By the inductive hypothesis, we have

$$\exp_{D\sim\pi}(k) \leq \begin{cases} k+1+s(n-l-3), & \text{if } \ 1 \leq k \leq s, \\ k+s(n-l-3), & \text{if } \ s+1 \leq k \leq n-l. \end{cases}$$

Let v_{i_k} $(k = 1, 2, \ldots, n-l)$ be a vertex in $D \sim \pi$ such that $\exp_{D\sim\pi}(k) = \exp_{D\sim\pi}(v_{i_k})$, then

$$\exp_D(v_{i_k}) \leq \exp_{D\sim\pi}(v_{i_k}) + l.$$

It follows that

$$\exp_D(k) \leq \max\{\exp_D(v_{i_t}) : 1 \leq t \leq k\}$$

$$\leq \max\{\exp_{D\sim\pi}(v_{i_t}) : 1 \leq t \leq k\} + l = \exp_{D\sim\pi}(k) + l$$

$$\leq \begin{cases} k+1+s(n-l-3)+l, & \text{if } \ 1 \leq k \leq s, \\ k+s(n-l-3)+l, & \text{if } \ s+1 \leq k \leq n-l \end{cases}$$

$$\leq \begin{cases} k+1+s(n-3)-l(s-1), & \text{if } \ 1 \leq k \leq s, \\ k+s(n-3)-l(s-1), & \text{if } \ s+1 \leq k \leq n-l \end{cases}$$

$$\leq \begin{cases} k+s(n-3), & \text{if } \ 1 \leq k \leq s, \\ k-1+s(n-3), & \text{if } \ s+1 \leq k \leq n-l. \end{cases}$$

When $n-l+1 \leq k \leq n$, let v_j be a vertex in D such that $\exp_D(k) = \exp_D(v_j)$. If $v_j \in V(D \sim \pi)$, and let k_1 be an integer such that $\exp_{D\sim\pi}(k_1) = \exp_{D\sim\pi}(v_j)$, then we have that $k_1 \leq n-l < n-l+1 \leq k$, and so

$$\exp_D(k) = \exp_D(v_j) \leq \exp_{D\sim\pi}(v_j) + l$$

$$\leq \begin{cases} k_1+1+s(n-l-3)+l, & \text{if } \ 1 \leq k_1 \leq s, \\ k_1+s(n-l-3)+l, & \text{if } \ s+1 \leq k_1 \leq n-l \end{cases}$$

$$= \begin{cases} k_1+1+s(n-3)-l(s-1), & \text{if } \ 1 \leq k_1 \leq s, \\ k_1+s(n-3)-l(s-1), & \text{if } \ s+1 \leq k_1 \leq n-l \end{cases}$$

$$\leq \begin{cases} k_1+s(n-3), & \text{if } \ 1 \leq k_1 \leq s, \\ k_1-1+s(n-3), & \text{if } \ s+1 \leq k_1 \leq n-l \end{cases}$$

$$\leq k-1+s(n-3).$$

Let $\pi = (v_0, v_1, v_2, \ldots, v_{l+1})$. Then $\exp_D(v_1) > \exp_D(v_2) > \ldots > \exp_D(v_l)$. For any nonnegative integer z, there exists a walk of length $\exp_{D\sim\pi}(v_{l+1})+z$ from vertex v_{l+1} to each vertex in $D \sim \pi$, hence there exists a walk of length $\exp_{D\sim\pi}(v_{l+1}) + l + z$ from vertex v_1 to each vertex in $D \sim \pi$. And moreover, there exists a walk of length $\exp_{D\sim\pi}(v_{l+1}) + l + t + z$ from vertex v_1 to vertex $v_t (1 \leq t \leq l)$. Hence

$$\exp_D(v_1) \le \exp_{D\sim\pi}(v_{l+1}) + 2l \le \exp_{D\sim\pi}(n-l) + 2l$$
$$\le n - l + s(n-l-3) + 2l = n - l + s(n-3) - l(s-2)$$
$$\le n - l + s(n-3) \le k - 1 + s(n-3).$$

Hence

$$\exp_D(v_l) < \exp_D(v_{l-1}) < \cdots < \exp_D(v_2) < \exp_D(v_1) \le k - 1 + s(n-3).$$

Thus, If v_j is one of the internal vertices of π and $\exp_D(k) = \exp_D(v_j)$, then we have

$$\exp_D(k) = \exp_D(v_j) \le k - 1 + s(n-3).$$

Therefore

$$\exp_D(k) \le k - 1 + s(n-3) \quad (n - l + 1 \le k \le n).$$

Hence we may assume that the removal of any branch α from D yields a digraph $D \sim \alpha$ with either $s(D \sim \alpha) > s(D)$ or $D \sim \alpha$ not primitive. By Lemma 2.4 there exists a branch π with $s(D \sim \pi) = s(D)$, so we have $h(D \sim \pi) = h > 1$.

First we assume that $h \ne s$, then $h < s$ and $h|s$, and so $s/h \ge 2$. By Lemma 2.5, $(D \sim \pi)^s$ has h connected components, each strongly connected and primitive. By Lemma 2.6 , $(D \sim \pi)^s$ has at least $s/h \ge 2$ loop vertices in each component.

Suppose that there exists some component A_i of $(D \sim \pi)^s$ containing three loop vertices. Let $x \in A_i$ be a loop vertex of D^s and let $y \in A_i$ be an exit vertex of D^s such that $d_{D^s}(x, y)$ is minimum. By the proof of [7, Theorem 4.9], there exists in D^s a walk of length $n-3$ from x to any vertex $v \in V$. Then there exists in D a walk of length $s(n-3)$ from x to any vertex $v \in V$. Hence $\exp_D(x) \le s(n-3)$. Therefore $\exp_D(1) \le \exp_D(x) \le s(n-3)$. By Lemma 2.9,

$$\exp_D(k) \le k - 1 + s(n-3) \le \begin{cases} k + s(n-3), & \text{if } 1 \le k \le s, \\ k - 1 + s(n-3), & \text{if } s+1 \le k \le n. \end{cases}$$

Suppose now that each component of $(D \sim \pi)^s$ contains exactly two loop vertices, then $D \sim \pi$ contains exactly one circuit of length s. Let $\sigma = (x_0, x_1, \ldots, x_{s-1}, x_0)$ be the circuit of $D \sim \pi$ of length s. Then there exists a vertex of σ, say x_{s-1}, such that $\delta^+_{D\sim\pi}(x_{s-1}) \ge 2$ Since $\mu(D) \ge 2$. By the proof of [7, Theorem 4.9], in D^s there exists a walk from x_0 to any vertex of length $n-3$. Then in D there exists a walk from x_0 to any vertex of length $s(n-3)$. Therefore $\exp_D(1) \leq \exp_D(x_0) \leq s(n-3)$. By Lemma 2.9,

$$\exp_D(k) \leq k-1+s(n-3) \leq \begin{cases} k+s(n-3), & \text{if } 1 \leq k \leq s, \\ k-1+s(n-3), & \text{if } s+1 \leq k \leq n. \end{cases}$$

Now assume that $h = s$, in this case the removal of any branch α from D yields a digraph $D \sim \alpha$ with either $s(D \sim \alpha) > s(D)$ or $h(D \sim \alpha) = s(D)$. By Lemma 2.7, D is special and its branches ρ and π satisfy (2.6) and (2.7), moreover, D contains a unique (elementary) circuit of length $s = s(D)$. By Lemma 2.6, each component of $(D \sim \pi)^s$ exactly contains one loop vertex. For the remainder of this proof we shall consider D to be such a digraph.

We first suppose $s \geq 3$, and let $\pi = (x_0, x_1, \ldots, x_k)$ and $\tau = (y_0, y_1, \ldots, y_l)$, where τ is the branch of $D \sim \pi$ which is not ρ, by (2.2) we have $x_0 = y_j$ and $x_k = y_i$ where $1 \leq i \leq j \leq l-1$. If $j - i \leq s-2$, then the digraph D^* is not an elementary circuit by the proof of [7, Theorem 4.9]. By Lemmas 2.8 and 2.9,

$$\begin{aligned} \exp_D(k) &\leq k-1+\exp_D(1) \leq k-1+s(n-3) \\ &\leq \begin{cases} k+s(n-3), & \text{if } 1 \leq k \leq s, \\ k-1+s(n-3), & \text{if } s+1 \leq k \leq n. \end{cases} \end{aligned}$$

If $j - i \geq s-1$, then by the proof of [7, Theorem 4.9], in D^s there exists an elementary circuit ξ which vertex set is $\{y_i, \ldots, y_j\} \cup \{x_1, x_2, \ldots, x_{k-1}\}$, and so for some $p \in \{1, \ldots, k-1\}$ and $q \in \{i, \ldots, j\}$, there exists an arc (x_p, y_q) of ξ. Let A_t be the connected component of $(D \sim \pi)^s$ which contains y_q, and let $a \in A_t$ be a loop vertex. If some arc with initial vertex x_p has terminal vertex in $V - A_t$, then the digraph D^* is not an elementary circuit. By Lemmas 2.8 and 2.9,

$$\begin{aligned} \exp_D(k) &\leq k-1+\exp_D(1) \leq k-1+s(n-3) \\ &\leq \begin{cases} k+s(n-3), & \text{if } 1 \leq k \leq s, \\ k-1+s(n-3), & \text{if } s+1 \leq k \leq n. \end{cases} \end{aligned}$$

If all arcs of D^s with initial vertex x_p have terminal vertex in A_t. Then by the proof of [7, Theorem 4.9], we have that $\max\{d_{D^s}(a,x) : x \in V\} \leq n-3$. Since $a \in A_t$ is a loop vertex, in D^s there exists a walk of length $n-3$ from a to any vertex $x \in V$. Furthermore in D there exists a walk of length $s(n-3)$ from a to any vertex $x \in V$. Therefore, $\exp_D(a) \leq s(n-3)$. By Lemma 2.9

$$\begin{aligned} \exp_D(k) &\leq k-1+\exp_D(1) \leq k-1+\exp_D(a) \leq k-1+s(n-3) \\ &\leq \begin{cases} k+s(n-3), & \text{if } 1 \leq k \leq s, \\ k-1+s(n-3), & \text{if } s+1 \leq k \leq n. \end{cases} \end{aligned}$$

The proof of the theorem when $s \geq 3$ is now complete.

Suppose now $s = 2$. Then, in the sense of isomorphism, the special digraph $D = (V, E)$ with $V = \bigcup_{i=1}^{\mu} \{x_1^i, x_2^i, \ldots, x_{s_{i+1}}^i, \ldots, x_{r_{i+1}}^i, \ldots, x_{k_i-1}^i\}$ $\cup\{a, b\}$ and $E = \bigcup_{i=1}^{\mu} \bigcup_{j=0}^{k_i-1} \{(x_j^i, x_{j+1}^i)\} \cup\{(a, b), (b, a)\}$, where $x_j^i (1 \leq i \leq \mu$, $1 \leq j \leq k_\mu - 1)$, a, b are distinct vertices, $|V| = n$; and $x_0^1 = x_{k_1}^1 = a$, $x_0^i = x_{r_i}^{i-1}$, $x_{k_i}^i = x_{s_i}^{i-1}$, $1 \leq s_i \leq r_i \leq k_{i-1} - 1$ for $i = 2, \ldots, \mu$. Let $\rho = (a, b, a)$ and $\pi_i = (x_0^i, x_1^i, \ldots, x_{k_i}^i)(i = 1, 2, \ldots, \mu)$. Then ρ and $\pi = \pi_\mu$ are two branches of D, and satisfy the following (3.1) and (3.2):

(3.1) $s(D \sim \rho) > 2$

(3.2) Length of any circuit in $D \sim \pi$ is an even integer, and ρ is a unique circuit of length 2 in D.

We want to show that

$$\exp_D(k) \leq \begin{cases} 2n - 5, & \text{if } \ k = 1, \\ 2n - 4, & \text{if } \ k = 2, \\ k - 1 + 2(n - 3), & \text{if } \ 3 \leq k \leq n. \end{cases}$$

For each $i = 1, \ldots, \mu$, we use p_i to denote the length of the elementary circuit in D containing π_i. By (3.2) and D primitive, we have that $2|p_i(i = 1, 2, \ldots, \mu - 1)$ and p_μ is an odd integer, and so $\phi_{L(D)} = \phi(2, p_\mu) = p_\mu - 1$. By Lemma 2.1, for any vertices x,$y \in V(D)$, we have

$$\exp_D(x, y) \leq d_{L(D)}(x, y) + \phi_{L(D)} = d_{L(D)}(x, y) + p_\mu - 1.$$

Let t_i and l_i be the number of internal vertices in paths $(x_0^i, x_1^i, \ldots, x_{s_{i+1}}^i)$ and $(x_{r_{i+1}}^i, \ldots, x_{k_i}^i)$ $(i = 1, 2, \ldots, \mu - 1)$ respectively. Let ω_i be the number of vertices in path $(x_{s_{i+1}}^i, \ldots, x_{r_{i+1}}^i)$ $(i = 1, 2, \ldots, \mu - 1)$, and let ω be the number of internal vertices in $\pi = \pi_\mu$. Put $X_\mu = \{x_1^\mu, x_2^\mu, \ldots, x_{k_\mu-1}^\mu\}$. It is easy to see that $\sum_{i=1}^{\mu-1} (t_i + l_i + \omega_i) + \omega + 2 = n$,and $\omega \geq 1$ since D is a minimally strong digraph.

We first estimate the exponent $\exp_D(a)$ of the vertex a. By Lemma 2.1,

$$\exp_D(a) \leq \max\{d_{L(D)}(a, v) : v \in V\} + p_\mu - 1$$

Clearly,

$$\max\{d_{L(D)}(a, v) : v \in X_\mu\} = d_{L(D)}(a, x_{k_\mu-1}^\mu),$$

and

$$\max\{d_{L(D)}(a, v) : v \in V - X_\mu\} = \begin{cases} d_{L(D)}(a, x_{s_2-1}^1), & \text{if } \ t_1 \geq 1, \\ d_{L(D)}(a, b), & \text{if } \ t_1 = 0. \end{cases}$$

Since

$$d_{L(D)}(a, x^{\mu}_{k_\mu - 1}) \le \sum_{i=1}^{\mu-1}(t_i + l_i + \omega_i) + \omega$$

$$= (n - \omega - 2) + \omega = n - 2,$$

$$d_{L(D)}(a, x^1_{s_2-1}) = \sum_{i=1}^{\mu-1}(t_i + l_i + \omega_i) + \sum_{i=1}^{\mu-2}\omega_i + 1 + t_1$$

$$= (n - \omega - 2) + \sum_{i=1}^{\mu-2}\omega_i + t_1 + 1,$$

$$d_{L(D)}(a, b) = \sum_{i=1}^{\mu-1}(t_i + l_i + \omega_i) + \sum_{i=1}^{\mu-2}\omega_i + 2$$

$$= (n - \omega - 2) + \sum_{i=1}^{\mu-2}\omega_i + 2,$$

$p_\mu = \omega_{\mu-1} + \omega$ and $\omega \ge 1$, then

$$\begin{aligned} d_{L(D)}(a, x^{\mu}_{k_\mu-1}) + p_\mu - 1 &= n - 2 + \omega_{\mu-1} + \omega - 1 \\ &= (n-3) + (\omega_{\mu-1} + \omega) \\ &= (n-3) + (n-2) - (\sum_{i=1}^{\mu-1} t_i + \sum_{i=1}^{\mu-1} l_i + \sum_{i=1}^{\mu-2}\omega_i) \\ &\le 2n - 5, \end{aligned}$$

$$\begin{aligned} d_{L(D)}(a, x^1_{s_2-1}) + p_\mu - 1 &= (n-2) + \sum_{i=1}^{\mu-1}\omega_i + t_1 \\ &\le (n-2) + (n - 2 - \omega) \\ &\le (n-2) + (n-3) = 2n - 5. \end{aligned}$$

$$\begin{aligned} d_{L(D)}(a, b) + p_\mu - 1 &= (n-2) + \sum_{i=1}^{\mu-1}\omega_i + 1 \\ &= (n-1) + \sum_{i=1}^{\mu-1}\omega_i \\ &= (n-1) + (n - 2 - \omega) - \sum_{i=1}^{\mu-1}(t_i + l_i). \end{aligned}$$

If $\sum_{i=1}^{\mu-1}(t_i + l_i) \ge 1$, then

$$(n - 2 - \omega) - \sum_{i=1}^{\mu-1}(t_i + l_i) \le n - 4.$$

If $\sum_{i=1}^{\mu-1}(t_i + l_i) = 0$, then $\omega_i (i = 1, \ldots, \mu - 1)$ is odd since p_i $(i = 1, \ldots, \mu - 1)$ are even, and ω is even since p_μ is odd. It follows that

$$(n - 2 - \omega) - \sum_{i=1}^{\mu-1}(t_i + l_i) \leq n - 4.$$

Therefore

$$d_{L(D)}(a, b) + p_\mu - 1 \leq 2n - 5$$

Consequently

$$\exp_D(a) \leq \max\{d_{L(D)}(a, v) : v \in V\} + p_\mu - 1 \leq 2n - 5.$$

Now we estimate the exponent $\exp_D(b)$ of the vertex b. Clearly, For any vertex $v \in V$, $d_{L(D)}(b, v) = d_{L(D)}(a, v) + 1$. Hence

$$\exp_D(b) = \exp_D(a) + 1 \leq 2n - 4.$$

Finally, we estimate the exponent $\exp_D(x_{r_2}^1)$ of the vertex $x_{r_2}^1$.
Since

$$\max\{d_{L(D)}(x_{r_2}^1, v) : v \in V - X_\mu \text{ and } v \neq b\}$$

$$\leq \sum_{i=1}^{\mu-1}(t_i + l_i + \omega_i) + \sum_{i=1}^{\mu-2}\omega_i + \max\{l_1, \ldots, l_{\mu-1}, t_1, \ldots, t_{\mu-1}\} + 1,$$

then

$$\max\{d_{L(D)}(x_{r_2}^1, v) : v \in V - X_\mu \text{ and } v \neq b\} + p_\mu - 1$$

$$\leq \sum_{i=1}^{\mu-1}(t_i + l_i + \omega_i) + \sum_{i=1}^{\mu-1}\omega_i + \omega + \max\{l_1, \ldots, l_{\mu-1}, t_1, \ldots, t_{\mu-1}\}$$

$$\leq (n - \omega - 2) + (n - 2) = 2n - 4 - \omega \leq 2n - 5.$$

Since

$$d_{L(D)}(x_{r_2}^1, b) = \sum_{i=1}^{\mu-1} l_i + \sum_{i=2}^{\mu-1} t_i + \sum_{i=1}^{\mu-1} \omega_i + \sum_{i=2}^{\mu-2} \omega_i + 2$$

$$= (n - \omega - 2) + \sum_{i=2}^{\mu-2} \omega_i + 2 - t_1$$

and

$$\omega \geq 1, \ \sum_{i=1}^{\mu-1} l_i + \sum_{i=1}^{\mu-1} t_i + \omega_1 \geq 1,$$

then

$$d_{L(D)}(x_{r_2}^1, b) + p_\mu - 1$$

$= (n-\omega-2) + \sum_{i=2}^{\mu-2} \omega_i + 2 - t_1 + \omega_{\mu-1} + \omega - 1$
$= (n-1) + \sum_{i=2}^{\mu-1} \omega_i - t_1$
$= (n-1) + (n-\omega-2) - (\sum_{i=1}^{\mu-1} l_i + \sum_{i=1}^{\mu-1} t_i + \omega_1) - t_1$
$\leq 2n-5.$

Notice that
$\max\{d_{L(D)}(x^1_{r_2}, v) : v \in X_\mu\} = d_{L(D)}(x^1_{r_2}, x^\mu_{k_\mu - 1})$
$\leq \sum_{i=1}^{\mu-1} (t_i + l_i + \omega_i) + \omega + 1$
$= (n-\omega-2) + \omega + 1 = n-1,$

hence
$\max\{d_{L(D)}(x^1_{r_2}, v) : v \in X_\mu\} + p_\mu - 1$
$\leq n - 1 + \omega_{\mu-1} + \omega - 1$
$\leq (n-1) + (n-2) - 1 = 2n-4.$

Therefore,
$\exp_D(x^1_{r_2}) \leq 2n-4.$

It follows that
$\exp_D(1) \leq \exp_D(a) \leq 2n-5,$
$\exp_D(2) \leq \max\{\exp_D(a), \exp_D(b)\} \leq 2n-4,$
$\exp_D(3) \leq \max\{\exp_D(a), \exp_D(b), \exp_D(x^1_{r_2})\} \leq 2n-4.$

By Lemma 2.9,
$\exp_D(k) \leq 1 + \exp_D(k-1) \leq \cdots \leq k - 3 + \exp_D(3)$
$\leq k - 3 + 2n - 4 = k - 1 + 2(n-3)$, for $3 \leq k \leq n$.

Thus we complete the proof of Theorem 1.1. □

As an application of Theorem 1.1, now we give a new proof of Theorems A and B.

Proof. Let $D \in PMSD_n$, then $s(D) \leq n-2$. By Theorem 1.1,

$$\exp_D(n) \leq n + s(n-3) \leq n + (n-2)(n-3) = n^2 - 4n + 6$$

with equality if and only if D is isomorphic to $D_{n-2,n}$, and

$$\exp_D(n-1) \leq (n-1) + s(n-3) \leq (n-1) + (n-2)(n-3) = n^2 - 4n + 5$$

with equality if and only if D is isomorphic to $D_{n-2,n}$.

It follows that Theorems A and B hold when $k = n-1$ and $k = n$.

Suppose that $1 \leq k \leq n-2$. If $s = n-2$, then D is isomorphic to $D_{n-2,n}$. By Lemma 2.10,
$\exp_D(k) = \exp_{D_{n-2,n}}(k) = n^2 - 5n + 7 + k.$

If $s \leq n-3$, then by Theorem 1.1,

$$\exp_D(k) \leq \begin{cases} k + 1 + s(n-3), & \text{if} \quad 1 \leq k \leq s, \\ k + s(n-3), & \text{if} \quad s+1 \leq k \leq n-2 \end{cases}$$

$$\leq \begin{cases} k+1+(n-3)(n-3), & \text{if } 1 \leq k \leq s, \\ k+(n-3)(n-3), & \text{if } s+1 \leq k \leq n-2 \end{cases}$$

$$\leq \begin{cases} s+1+(n-3)(n-3), & \text{if } 1 \leq k \leq s, \\ n-2+(n-3)(n-3), & \text{if } s+1 \leq k \leq n-2 \end{cases}$$

$$\leq \begin{cases} n-3+1+(n-3)(n-3), & \text{if } 1 \leq k \leq s, \\ n^2-5n+7, & \text{if } s+1 \leq k \leq n-2 \end{cases}$$

$$= n^2-5n+7 < n^2-5n+7+k.$$

Therefore Theorems A and B hold when $1 \leq k \leq n-2$.

The proof of Theorems A and B is complete. □

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