On imp-sets and kernels by monochromatic paths of the duplication

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Abstract

In [4] H.Galana-Sanchez introduced the concept of kernels by monochromatic paths which generalize the concept of kernels. In [6] they proved the necessary and sufficient conditions for the existence of kernels by monochromatic paths of the duplication of a subset of vertices of a digraph, where a digraph is without monochromatic directed circuits. In this paper we study independent by monochromatic paths sets and kernels by monochromatic paths of the duplication. We generalize result from [6] for an arbitrary edge coloured digraph.

Keywords: kernel, kernel by monochromatic paths, duplication
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1 Introduction

For concepts not defined here, see [3]. Let D be a finite, directed graph (for short: a digraph) where $V(D)$ is the set of vertices and $A(D)$ is the set of arcs of D. By a path from a vertex x_1 to a vertex x_n, $n \geq 2$, we mean a sequence of vertices $x_1, ..., x_n$ and arcs $(x_i, x_{i+1}) \in A(D)$, for $i = 1, ..., n-1$ and for simplicity we denote it by $x_1...x_n$. A circuit is a path with $x_1 = x_n$. A digraph D is said to be an edge m-coloured digraph if its arcs are coloured with m colours. A path is called monochromatic if all of its arcs are coloured alike. By $C_D(x)$ we mean a family of all monochromatic circuits in D including the vertex x.
A set $J \subset V(D)$ is said to be a kernel by monochromatic paths of the m-coloured digraph D if it satisfies the following properties:
1. J is independent by monochromatic paths i.e. for any two different vertices $x, y \in J$ there is no monochromatic path between them and
2. J is dominating by monochromatic paths i.e. for each $x \in V(D) \setminus J$ there exists monochromatic path from x to y, for some $y \in J$.
A subset containing only one vertex and the empty set are meant as independent by monochromatic paths. The set $V(D)$ is dominating by monochromatic paths of a digraph D. For convenience throughout this paper we will write an imp-set of D instead of independent by monochromatic

paths set of D and dmp-set of D instead of dominating by monochromatic paths set of D.
The concept of kernels by monochromatic paths generalize kernels in classical sense. The existence of kernels in edge-coloured digraphs have been investigated by several authors, see by example [4 - 8] and [11], [12].
Let D be an edge coloured digraph and X be a proper subset of $V(D)$. Let H be a digraph isomorphic to a subdigraph of D induced by X. A vertex from $V(H)$ corresponds to $x \in X$ we will denote by x^c. The duplication of X over D is the edge coloured digraph D^X and defined as follows: $V(D^X) = V(D) \cup V(H)$ and $A(D^X) = A(D) \cup A(H) \cup A_1 \cup A_2$ where $A_1 = \{(x^c, y) - \textit{coloured } \psi; x^c \in V(H), y \in V(D) \text{ and } (x, y) \in A(D) - \textit{coloured } \psi\}$ and $A_2 = \{(y, x^c) - \textit{coloured } \psi; y \in V(D), x^c \in V(H) \text{ and } (y, x) \in A(D) - \textit{coloured } \psi\}$.

A vertex $x^c \in V(H)$ (respectively a subset $S^c \subseteq V(H)$) we will call the copy of the vertex $x \in X$ (respectively the copy of the subset $S \subseteq X$). The vertex $x \in X$ (respectively the subset $S \subseteq X$) will be named as the original of the vertex x^c (respectively the original of the subset S^c) and if it is necessary the original of the vertex x^c (respectively of the subset S^c) we will denote by x^0 (respectively S^0).

The duplication of a vertex of a graph was introduced in [2]. In [9] the definition of the duplication of a subset of vertices of a graph was given as a generalization of the duplication of a vertex of a graph. The existence of kernels and (k, l)-kernels (i.e. kernels generalized in distance sense) in duplication was studied in [1] and [9], [10]. In [6] this definition was applied to edge coloured digraphs and the existence of kernels by monochromatic paths of duplication were studied.

It has been proved:

Theorem 1 *[6] Let D be an edge coloured digraph which has no monochromatic circuits and X be a proper subset of $V(D)$. A digraph D has a kernel by monochromatic paths if and only if D^X has a kernel by monochromatic paths.*

The main results of this paper is generalization of Theorem 1 for an arbitrary edge coloured digraph.

Let $X \subset V(D)$ and D^X be the duplication of the edge coloured digraph D. Let $x, y \in X$, $x^c, y^c \in X^c$ and $w, z \in V(D) \setminus X$. Then from the definition of D^X directly follows dependencies
(1) the following statements are equivalent
(1.1) there exists a monochromatic path $x ... y$ in D
(1.2) there exists a monochromatic path $x ... y$ in D^X
(1.3) there exists a monochromatic path $x ... y^c$ in D^X
(1.4) there exists a monochromatic path $x^c ... y$ in D^X

(1.5) there exists a monochromatic path $x^c...y^c$ in D^X
(2) the following statements are equivalent
(2.1) there exists a monochromatic path $w...z$ in D
(2.2) there exists a monochromatic path $w...z$ in D^X
(3) the following statements are equivalent
(3.1) there exists a monochromatic path $w...x$ in D
(3.2) there exists a monochromatic path $w...x^c$ in D
(3.3) there exists a monochromatic path $w...x$ in D^X
(4) the following statements are equivalent
(4.1) there exists a monochromatic path $x...w$ in D
(4.2) there exists a monochromatic path $x...w$ in D^X
(4.3) there exists a monochromatic path $x^c...w$ in D^X
From these dependencies immediately follows:

Corollary 1 *Let D^X be the duplication of the edge coloured digraph D and let $X \subset V(D)$. Let $x, y \in V(D)$. Then there exists a monochromatic path $x...y$ in D if and only if there exists a monochromatic path $x...y$ in D^X.*

2 Imp-sets of the duplication

In this section we study imp-sets of the duplication of a subset of vertices of an arbitrary edge coloured digraph.

Theorem 2 *Let D be an edge coloured digraph and $X \subset V(D)$. Let S be an arbitrary imp-set of D^X. For an arbitrary $x \in X$ such that $C_D(x) \neq \emptyset$ exactly one condition is fulfilled:*
(1) $x \notin S$ and $x^c \notin S$ or
(2) either $x \in S$ or $x^c \in S$.

P R O O F: Let $S \subset V(D^X)$ be an imp-set of D^X and assume at the contrary that there is $x \in X$ and $C_D(x) \neq \emptyset$ such that $x \in S$ and $x^c \in S$. Because $C_D(x) \neq \emptyset$ so there is a monochromatic circuit including the vertex x in a digraph D. From the definition of the duplication there exists the monochromatic path $x...x^c$ in D^X , a contradiction with the assumption of S. □

Theorem 3 *Let D be an edge coloured digraph and $X \subset V(D)$. If $S^* \subset V(D^X)$ is an imp-set of D^X, then $(S^* \cap V(D)) \cup S$ is an imp-set of D, where S is the original of the set $S^* \cap X^c$.*

P R O O F: Assume that $S^* \subseteq V(D^X)$ is an imp-set of the duplication D^X and S is the original of the set $S^* \cap X^c$, i.e. $S^* \cap X^c = S^c$. Let

$S' = S^* \cap V(D)$. Evidently S', S^c and S are imp-sets of D^X, hence S and S' are imp-sets of D. To show that $S \cup S'$ is an imp-set of the digraph D it is enough to prove that there is no monochromatic path $x...y$ and $y...x$ for every $x \in S'$ and $y \in S$. Let $x \in (S' \setminus S)$ and $y \in (S \setminus S')$. From (1),(3) and (4) we obtain that the existence of monochromatic path $x...y$ in D is equivalent to existence of monochromatic path $x...y^c$ in D^X. Moreover the existence of monochromatic path $y...x$ in D is equivalent the existence of monochromatic path $y^c...x$ in D^X where $y^c \in S^* \cap X^c$ is the copy of the vertex y. Since S^* is an imp-set of the duplication D^X, then there is no monochromatic path $x...y^c$ and $y^c...x$ in D^X. Hence there is no monochromatic paths $x...y$ and $y...x$ in D, which means that there is no monochromatic path between vertices from S' and S in D. Thus the Theorem is proved. □

Theorem 4 *Let D be an edge coloured digraph and $X \subset V(D)$. Let X' be the subset of X such that for every $x \in X'$, $C_D(x) \neq \emptyset$. If S is an imp-set of D, then $S \cup (S \cap (X \setminus X'))^c$ is an imp-set of D^X.*

P R O O F: Assume that $X \subset V(D)$ and X' be a subset of X such that for every $x \in X'$, $C_D(x) \neq \emptyset$. Let S be an arbitrary imp-set of D. We proceed by contradiction. Assume that $S \cup (S \cap (X \setminus X'))^c$ is not imp-set of D^X and consider following cases:

(1) $S \cap (X \setminus X') = \emptyset$

Then $S \cup (S \cap (X \setminus X'))^c = S$ and from the assumption we have that the set S is not imp-set of D^X. Hence by the definition of D^X the set S is not imp-set of D, a contradiction with the assumption.

(2) $S \cap (X \setminus X') \neq \emptyset$

Because $S \cup (S \cap (X \setminus X'))^c$ is not imp-set of D^X hence there exist two distinct vertices $x, y \in S \cup (S \cap (X \setminus X'))^c$ and a monochromatic path $x...y$ in D^X. We distinguish following possibilities:

(2.1) $x, y \in S$

Then by Corollary 1 immediately follows that there exists a monochromatic path $x...y$ in D, a contradiction that S is an imp-set of D.

(2.2) $x, y \in (S \cap (X \setminus X'))^c$

Then by (1) we obtain that there exists a monochromatic path between originals of x, y in D, i.e. $x^0...y^0$ is monochromatic in D. Because $x^0, y^0 \in S$ hence we have a contradiction with the assumption of S.

(2.3) $x \in S$ and $y \in (S \cap (X \setminus X'))^c$

In this case by (3) we have that there exists a monochromatic path $x...y^0$ in the digraph D. Of course $y^0 \neq x$ i.e. $y \neq x^c$ in otherwise $x \in X'$, so $C_D(x) \neq \emptyset$ and by Theorem 2 a contradiction that $x^c \in (S \cap (X \setminus X'))^c$. Because $y \in (S \cap (X \setminus X'))^c$ hence $y^0 \in S$, a contradiction with the assumption of S.

(2.4) $x \in (S \cap (X \setminus X'))^c$ and $y \in S$

In this case we prove similarly as in the case (2.3).

All this together completes the proof. □

In this same manner we can prove

Theorem 5 *Let D be an edge coloured digraph and $X \subset V(D)$. Let X' be the subset of X such that for every $x \in X'$, $C_D(x) \neq \emptyset$. If S is an imp-set of D then $(S \setminus X') \cup (S \cap X)^c$ is an imp-set of D^X.*

3 Kernels by monochromatic paths of D^X

Theorem 6 *Let D be an edge coloured digraph and $X \subset V(D)$. If J^* is a kernel by monochromatic paths of the duplication D^X, then $(J^* \cap V(D)) \cup J$ is a kernel by monochromatic paths of the digraph D where J is the original of $J^* \cap X^c$.*

P R O O F: Assume that $J^* \subset V(D^X)$ is a kernel by monochromatic paths of D^X. Theorem 3 implies that $(J^* \cap V(D)) \cup J$ is an imp-set of D. We shall show that $(J^* \cap V(D)) \cup J$ is a dmp-set of D. Let $x \in V(D) \setminus (J^* \cup J)$. Since J^* is a dmp-set of D^X hence there is a monochromatic path $x...y$ for some $y \in J^*$ in D^X. Consider the possible cases:

(1) $x \in X$

If $y \in J^* \cap V(D)$, then by Corollary 1 the existence of a monochromatic path $x...y$ in D^X is equivalent the existence of a monochromatic path $x...y$ in D.

If $y \in J^* \cap X^c$, then from (1) we obtain that there exists a monochromatic path $x...y^0$ where $y^0 \in J$ is the original of the vertex y.

(2) $x \in V(D) \setminus X$

If $y \in J^* \cap V(D)$, then by (1) we obtain that there is a monochromatic path $x...y$ in D. If $y \in J^* \cap X^c$, then by (3) we obtain that there is in D a monochromatic path $x...y^0$, where $y^0 \in J$ is the original of the vertex y.

Finally for every $x \in V(D) \setminus (J^* \cup J)$ there is a vertex $y \in (J^* \cap V(D)) \cup J$ and a monochromatic path $x...y$ which gives that $(J^* \cap V(D)) \cup J$ is a dmp-set of D and this completes the proof. □

From the above Theorem immediately follows:

Corollary 2 *Let D be an edge coloured digraph and $X \subset V(D)$. If D^X has a kernel by monochromatic paths then D has a kernel by monochromatic paths.*

Theorem 7 *Let D be an edge coloured digraph and $X \subset V(D)$. Let X' be the subset of X such that for every $x \in X$, $C_D(x) \neq \emptyset$. If J is a kernel by monochromatic paths of D such that $J \cap X' = \emptyset$, then $J \cup (J \cap X)^c$ is a kernel by monochromatic paths of D^X.*

P R O O F: Let $X' \subseteq X$ be a subset of $V(D)$ such that for every $x \in X'$, $C_D(x) \neq \emptyset$. Assume that J is a kernel by monochromatic paths of D such that $J \cap X' = \emptyset$ and $(J \cap X)^c$ is the copy of $J \cap X$ in D^X. We will show that $J \cup (J \cap X)^c$ is a kernel by monochromatic paths of D^X. If $J \cap X = \emptyset$, then $(J \cap X)^c = \emptyset$. Hence $J \cup (J \cap X)^c = J$. Since J is a kernel by monochromatic paths of the digraph D, then for every $x, y \in J$ there is no monochromatic path in the digraph D between them. By Corollary 1 we obtain that there is no monochromatic path $x...y$ in D^X. Hence J is an imp-set of D^X. We shall prove that J is a dmp-set of D^X. From the assumption J is a dmp-set of D. Let $z \in V(D^X) \setminus J$. If $z \in V(D)$ then by assumption of J there exists $x \in J$ and a monochromatic path $z...x$ in a digraph D. Then by (1) there is a monochromatic path $z...x$ in a digraph D^X. If $z \in X^c$ then $z^0 \notin J$ and there is a monochromatic path $z^0...x$ in D. Hence by (4) there is a monochromatic path $z...x$ in D^X. This means that $J \cup (J \cup X)^c$ is a kernel by monochromatic paths in D^X in case when $J \cap X = \emptyset$. Thus assume that $J \cap X \neq \emptyset$. Because $J \cap X' = \emptyset$ hence $J \cap (X \setminus X') = J \cap X \neq \emptyset$. Then from Theorem 4 we get that $J \cup (J \cap X)^c$ is an imp-set of D^X. So we need only to prove that this set is a dmp-set of D^X. Since $V(D^X) \setminus (J \cup (J \cap X)^c) = (V(D) \setminus J) \cup (X^c \setminus (J \cap X)^c)$ so let us consider two cases:

(1) $x \in V(D) \setminus J$

Then there is a monochromatic path $x...y$ in D, for some $y \in J$ because J is a kernel by monochromatic paths of D. Thus by Corollary 1 there is a monochromatic path $x...y$ in D^X.

(2) $x \in X^c \setminus (J \cap X^c)$

Because J is a kernel by monochromatic paths of D, so for the original $x^0 \in X \setminus J$ of the vertex x there exists a monochromatic path $x^0...y$, for some $y \in J$. If $y \in X$, then (1) implies that there is a monochromatic path $x...y$ in the digraph D^X, where $y \in J \cap X$. If $y \in V(D) \setminus X$, then from (3) we obtain that there is a monochromatic path $x...y$ in D^X.

Therefore the set $J \cup (J \cap X)^c$ is a dmp-set of D^X. Hence by previous considerations the set $J \cup (J \cap X)^c$ is a kernel by monochromatic paths of the duplication D^X.

Thus the Theorem is proved. □

Corollary 3 *Let D be an edge coloured digraph and $X \subset V(D)$. Let X' be the subset of X such that for every $x \in X'$, $C_D(x) \neq \emptyset$. If D has a kernel by monochromatic paths J such that $J \cap X' = \emptyset$, then D^X has a kernel by monochromatic paths.*

From the Corollary 2 and Corollary 3 immediately follows Theorem 1.

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