$2n$-cyclic Blended Labeling of Graphs

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Abstract

We develop a new type of a vertex labeling of graphs, namely $2n$-cyclic blended labeling, which is a generalization of some previously known labelings. We prove that a graph with this labeling factorizes the complete graph on $2nk$ vertices, where k is odd and $n, k > 1$.

1 Introduction

Let H and G be simple graphs. By a *decomposition* of a graph H on n vertices we mean a family of pairwise edge disjoint subgraphs $\mathcal{G} = \{G_0, G_1, \ldots, G_s\}$ such that every edge of H belongs to a member of $\mathcal{G}$. If each subgraph G_r is isomorphic to a graph G we speak about a *G-decomposition of H*. If G has exactly n vertices and none of them is isolated, then G is a *factor* of H and such a G-decomposition is called a *G-factorization*. The decomposition is *cyclic* if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of H and isomorphisms $\phi_r : G_0 \longrightarrow G_r$, $r = 0, 1, 2, \ldots, s$, such that $\phi_r(x_i) = x_{i+r}$ for each $i = 1, 2, \ldots, n$. Subscripts are taken modulo n.

Many papers were written on graph decompositions. Decompositions of complete graphs and complete bipartite graphs received special attention. However, most of these papers deal with decompositions into isomorphic subgraphs of smaller order.

Not that much is known about decompositions of complete graphs into isomorphic spanning trees. An obvious necessary condition for the existence

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of a G-decomposition of K_n is that the number of edges of G divides the number of edges of K_n. It follows that a factorization of a complete graph with an odd number of vertices into spanning trees is impossible. Therefore, we deal only with complete graphs K_{2n} with an even number of vertices. Another easily observed necessary condition for the existence of a spanning tree factorization of K_{2n} is that the largest degree of a vertex of the spanning tree is at most n. This condition is called the *degree condition* in [6].

It is a well known fact that K_{2n} can be factorized into hamiltonian paths P_{2n}. It is also easy to observe that a cyclic factorization of K_{2n} into symmetric double stars is possible. By a symmetric double star we mean two stars $K_{1,n-1}$ with central vertices connected by an edge. Until recently, almost nothing was published about other classes of spanning trees.

The first general result is due to P. Eldergill [1]. He introduced a method for a cyclic decomposition of K_{2n} into symmetric trees. By a *symmetric tree* he means a tree symmetric with respect to an edge. That is, a tree with an automorphism ψ and an edge (x, y) such that $\psi(x) = y$ and $\psi(y) = x$.

Eldergill's method is, similarly to many other methods of decomposition, based on a graph labeling. A *labeling* of a graph G with at most n vertices is usually defined as an injection λ from the vertex set $V(G)$ into a subset of the set $\{0, 1, 2, \ldots, 2n\}$. The vertex labels then induce the edge labels in some way. There are many different ways how such an edge label can be defined. However, for the purpose of a graph decomposition the edge label is usually defined naturally as the "length" of the edge. An exact definition is given below.

Two important types of vertex labelings were introduced in 1960's by A. Rosa. In [7] he defined a ρ-labeling and a graceful labeling, which he used for decompositions of K_{2n+1} into $2n+1$ copies of a graph with n edges. Graceful or ρ-labelings were often used to construct new types of labelings which in some sense generalize their properties. Among them are: ρ-symmetric graceful labeling introduced in [1] by P. Eldergill, allowing decomposition of K_{2n} into symmetric graphs, or a blended ρ-labeling introduced by the first author [3]. A blended ρ-labeling exists for a wider class of graphs than symmetric trees and guarantees a decomposition of K_{4k+2}. We will develop their further generalization.

The main goal of this paper is to investigate methods for factorizations of K_{4k}, where k is not a power of 2. We will show that these methods allow factorizations of K_{4k} into spanning trees with diameter 4, as opposed to the methods known so far.

2 Known methods and results

As we already mentioned, two fundamental types of labelings are the ρ-labeling and the graceful labeling (also called β-labeling) defined by A. Rosa.

Definition 2.1 *Let G be a graph with n edges and the vertex set $V(G)$ and let λ be an injection $\lambda : V(G) \longrightarrow S$ where S is a subset of the set $\{0, 1, 2, \ldots, 2n\}$. The* length *of an edge (x, y) is defined as $l(x, y) = \min\{|\lambda(x) - \lambda(y)|, 2n + 1 - |\lambda(x) - \lambda(y)|\}$. If the set of all lengths of n edges is equal to $\{1, 2, \ldots, n\}$ and $S \subseteq \{0, 1, 2, \ldots, 2n\}$, then λ is a* ρ-labeling; *if $S \subseteq \{0, 1, 2, \ldots, n\}$ instead, then λ is a* graceful labeling.

Every graceful labeling is indeed also a ρ-labeling, and a graph which admits a graceful labeling is called *graceful*.

In 1967 Rosa proved the following theorem.

Theorem 2.2 (A. Rosa) *If a graph G with n edges has a ρ-labeling, then there is a cyclic G-decomposition of K_{2n+1} into $2n + 1$ copies of G.*

The idea of the proof is the following. We can unify the vertices of K_{2n+1} with the elements of Z_{2n+1} (the additive group of integers modulo $2n + 1$). Computing the lengths of the edges as in the definition of the ρ-labeling, we have $2n + 1$ edges of each length i for $i = 1, 2, \ldots, n$. Since in T there is exactly one edge of each length, we can rotate T in K_{2n+1} to obtain a T-decomposition.

We state here the notions related to decomposition of K_{2n} into symmetric graphs. To simplify our notation we will from now on occasionally unify a vertex with its label. It means that rather than saying "the vertex x such that $\lambda(x) = i$", we will say just "the vertex i".

Definition 2.3 *A connected graph G with an edge (x, y) (called a* bridge*) is* symmetric *if there is an automorphism ψ of G such that $\psi(x) = y$ and $\psi(y) = x$. The isomorphic components of $G - (x, y)$ are called* banks *and denoted by H, H', respectively. A labeling of a symmetric graph G with $2n + 1$ edges and banks H, H' is* ρ-symmetric graceful *if H has a ρ-labeling and $\psi(i) = i + n \pmod{2n}$ for each vertex i in H. A labeling of a symmetric graph G with $2n - 1$ edges is* symmetric graceful *if it is ρ-symmetric graceful and the bank H is moreover graceful. A graph which admits a ρ-symmetric graceful labeling or a symmetric graceful labeling is called* ρ-symmetric graceful *or* symmetric graceful, *respectively.*

The following theorem was proved by Eldergill for symmetric trees. Since the assumption that the graph must be acyclic was never used, the theorem is true for symmetric graphs in general.

Theorem 2.4 (Eldergill) *Let G be a symmetric graph with $2n-1$ edges. Then there exists a cyclic G-decomposition of K_{2n} if and only if G is ρ-symmetric graceful.*

One can easily observe that the construction of a ρ-symmetric graceful labeling is based on the ρ-labeling or graceful labeling defined by A. Rosa. In a graph with $n-1$ edges that has either a graceful or a ρ-labeling there is one edge of each length $1, 2, \ldots, n-1$, while in a graph with $2n-1$ edges, which is symmetric graceful or ρ-symmetric graceful there are two edges of each length $1, 2, \ldots, n-1$ and one symmetric edge of the maximum length n. Since any graceful graph with $n-1$ edges yields a symmetric graceful graph with $2n-1$ edges, one can find an infinite class of symmetric graceful graphs whenever an infinite class of graceful graphs is known. Eldergill's method is very restrictive, allowing decompositions only into symmetric graphs, which all have an odd diameter. To answer the question about spanning trees with more general structure we need a more powerful decomposition method.

To find such a method, the first author defined a blended ρ-labeling [3]. The existence of this labeling for a graph G guarantees a G-decomposition of K_{4k+2}. Such a decomposition is called *bi-cyclic*. He also constructed several classes of non-symmetric trees that admit a blended ρ-labeling. As our method is just a straightforward extension of the blended ρ-labeling, we state its definition here.

Definition 2.5 *Let G be a graph with $4k+1$ edges, $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2k+1$. Let λ be an injection, $\lambda : V_i \longrightarrow \{0_i, 1_i, \ldots, (2k)_i\}$, $i = 0, 1$.*

The pure length *of an edge (x_i, y_i) with $x_i, y_i \in V_i, i \in \{0,1\}$ is defined as follows: If $\lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$, then $l_{ii}(x_i, y_i) = \min\{|a-b|, 2k+1-|a-b|\}$ for $i = 0, 1$ and the* mixed length *of an edge (x_0, y_1) with $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$, as $l_{01}(x_0, y_1) = b - a \bmod (2k+1)$ for $x_0 \in V_0, y_1 \in V_1$.*

Then G has a blended ρ-labeling *or just* blended labeling *for short if*

(*i*) $\{l_{ii}(x_i, y_i) | (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, k\}$ for $i = 0, 1$

(*ii*) $\{l_{01}(x_0, y_1) | (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2k\}$.

A graph G with a blended labeling can be split into three subgraphs. Two subgraphs, H_0 and H_1, are induced on the vertices of the sets V_0 and V_1, respectively, and contain the edges of the pure lengths l_{00} and l_{11}. We call these edges *pure edges*. The third subgraph is the bipartite graph H_{01} with the partite sets V_0, V_1 and edges of mixed length l_{01}. We call these edges *mixed edges*. If a blended labeling is restricted to these three subgraphs, we can observe that the labeling on H_0 and H_1 is nothing else than the usual ρ-labeling, which guarantees cyclic decompositions of the complete graphs K_{2k+1} on the vertices of V_0 and V_1, respectively. The labeling of H_{01} is called in [3] and elsewhere a *bipartite ρ-labeling.* A bipartite ρ-labeling of a graph H_{01} with $2k+1$ edges, one of each mixed length, allows a decomposition of the complete bipartite graph $K_{2k+1,2k+1}$ into $2k+1$ isomorphic copies of H_{01}, when the vertices in each of the partite sets V_0, V_1 rotate concurrently.

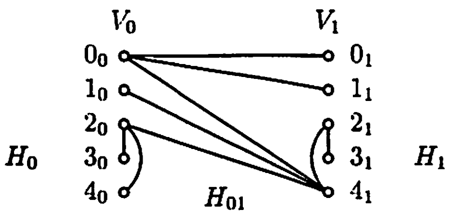


Figure 2.1: *Blended labeling of a tree on* 10 *vertices*

However, a blended ρ-labeling can only be used for graphs with $4k+2$ vertices. The method which allows decompositions of K_{4k} into non-symmetric graphs is based on a *switching blended labeling.* This is a modification of the blended labeling and was defined by the first author and M. Kubesa in [4]. Switching blended labeling is still too restrictive, since it requires a specific "strong" type of automorphism, which does not exist for certain classes of trees. Later we show that trees with diameter 4 do not allow a switching blended labeling at all.

Therefore, we develop a new technique for decompositions of complete graphs with an even number of vertices. This technique allows decompositions of the complete graphs K_{2nk} where $n, k > 1$ and k is odd.

3 Decomposition of K_{2nk}

Here we give a method of factorization of the complete graph on $2nk$ vertices into n isomorphic "locally dense" factors. The method is based on Eldergill's cyclic factorization of K_{2n} into symmetric trees. First we take a tree T on $2n$ vertices with a symmetric graceful labeling, which allows a factorization of K_{2n}. Then we blow up this tree to construct a bigger graph U on $2nk$ vertices (for any $k > 1$), which is a connected factor of K_{2nk} and show that there is a U-factorization of K_{2nk}. In the next section we develop a method of decomposition of U into k isomorphic copies of a graph G (for k odd). Finally, by decomposing each copy of the graph U into k isomorphic copies of G we obtain a G-decomposition of K_{2nk} into nk isomorphic copies of G.

The construction of the graph $U = U(T, s; k)$ can be described in two steps. First we obtain the graph $T[\overline{K}_k]$ by blowing up each vertex i of the tree T into the set V_i with k vertices and each edge (i,j) of T into all k^2 edges between the vertices of the sets V_i and V_j. Then we choose a vertex s in T and its symmetric image $\psi(s) = s + n$ and add all edges into the corresponding sets V_s and V_{s+n} so that we have two complete graphs K_k in addition to the edges of $T[\overline{K}_k]$. For convenience we use the following notation: K_{V_i} denotes the complete graph on the vertices of the vertex set V_i and K_{V_i,V_j} denotes the complete bipartite graph on the vertices of the partite sets V_i, V_j.

Definition 3.1 *Let T be a symmetric tree on $2n$ vertices with a ρ-symmetric graceful labeling. We define the graph $U(T,s;k)$ with the underlying tree T, where s is the label of any vertex of T, $0 \leq s \leq n-1$, to have the vertex set*

$$V(U(T,s;k)) = \bigcup_{i=0}^{2n-1} V_i, \ |V_i| = k, \ V_i \cap V_j = \emptyset \text{ for } i \neq j,$$

and the edge set

$$\begin{aligned} E(U(T,s;k)) &= \{(x,y) | x \in V_i, y \in V_j \wedge (i,j) \in E(T)\} \\ &\cup \{(x,y) | x,y \in V_s\} \cup \{(x,y) | x,y \in V_{s+n}\}. \end{aligned}$$

In other words, the graph $U(T,s;k)$ is a union of $2n-1$ bipartite graphs K_{V_i,V_j} on the vertices of the sets V_i, V_j whenever i is adjacent to j in T and two complete graphs K_{V_s} and $K_{V_{s+n}}$ on the vertices of the vertex sets V_s, V_{s+n} for the chosen vertex with label s in T. Each vertex set V_i is of size k and the subscript i is the label of the corresponding vertex in T.

It is easy to observe that K_{2nk} can be decomposed into n isomorphic copies of $U(T,s;k)$ (see Figure 3.1) and we will give a proof of this fact. One can also notice that similar approach can be used for other G-decompositions of K_{2n}. For instance, we can use similar approach whenever there is a bi-cyclic G-decomposition of the complete graph K_{2n} into n copies of G. Recall that bi-cyclic decompositions are based on blended labelings. Even more general types of decomposition can be probably used—one must be just careful about the choice of the two particular vertices in G that correspond to the complete graphs K_k in U.

Lemma 3.2 *Let T be a tree on $2n$ vertices with a ρ-symmetric graceful labeling. Then there is a $U(T,s;k)$-factorization of K_{2nk} into n isomorphic copies of $U(T,s;k)$ for any $k \geq 1$.*

Proof. When T is a ρ-symmetric graceful tree on $2n$ vertices, then according to Theorem 2.4 there is a cyclic T-factorization of K_{2n} with the factors $T_0, T_1, \ldots, T_{n-1}$. By Definition 3.1, the graph $U(T,s;k)$ with the underlying tree T is a connected factor of K_{2nk}. From each copy of T we obtain an isomorphic copy of $U(T, s; k)$. We may assume that $T = T_0$ and construct the graph $U(T_0, s; k)$. Every other factor T_r for $r = 1, 2, \ldots, n-1$ is obtained by a cyclic permutation of the labels on the vertices of T_0. Using the same permutations for the subscripts of the sets V_i of $U(T_0, s; k)$, we get the remaining factors $U(T_r, s+r; k)$. Together $U(T_0, s; k), U(T_1, s+1; k), \ldots, U(T_{n-1}, s+n-1; k)$ form a $U(T, s; k)$-factorization. We just need to check that each edge of K_{2nk} belongs to exactly one copy of $U(T, s; k)$.

The vertices of the complete graph K_{2nk} can be split into $2n$ sets V_i for $i = 0, 1, \ldots, 2n-1$ with k vertices in each of them. Then we can view the edge set of K_{2nk} as a union of the edge sets of $n(2n-1)$ complete bipartite graphs K_{V_i, V_j}, $i \neq j$ and $2n$ complete graphs K_{V_i} on k vertices of each of the sets V_i.

Since there is a T-factorization of K_{2n}, each edge (i, j) of K_{2n} belongs to exactly one factor T_r. By the definition of $U(T, s; k)$, the edge $(i, j) \in E(T_r)$ corresponds to the complete bipartite graph K_{V_i, V_j} in $U(T_r, s+r; k)$. Then each complete bipartite graph K_{V_i, V_j} also belongs to exactly one factor of K_{2nk}, in particular, to $U(T_r, s+r; k)$.

Now we check the complete graphs K_{V_i} for $i = 0, 1, \ldots, 2n-1$. In T_0 the vertex s and its symmetric image $s+n \pmod{2n}$ are chosen to add K_{V_s} and $K_{V_{s+n}}$ into $U(T_0, s; k)$. In T_r the corresponding vertices are $\phi_r(s) = s + r \pmod{2n}$ and $\phi_r(s+n) = s+n+r \pmod{2n}$. So we have two different vertices in each T_r for $r = 0, 1, \ldots, n-1$.

Suppose now that while making copies of T we obtain the same image of the vertex s or $s+n$ in two different factors T_r and T_t. Because our T-factorization is cyclic, we can assume that $r = 0$ and $t \in \{1, 2, \ldots, n-1\}$. Firstly, let

(i)

$$\begin{aligned} \phi_t(s) &= \phi_0(s) = s, && \text{then} \\ s+t &= s \pmod{2n}, && \text{and} \\ t &= 0, \end{aligned}$$

which contradicts our assumption that $t \neq 0$.

(ii) Secondly, if

$$\begin{aligned} \phi_t(s+n) &= \phi_0(s+n) = s+n, && \text{then} \\ s+n+t &= s+n \pmod{2n}, && \text{and} \\ t &= 0, \end{aligned}$$

we again get the same contradiction.

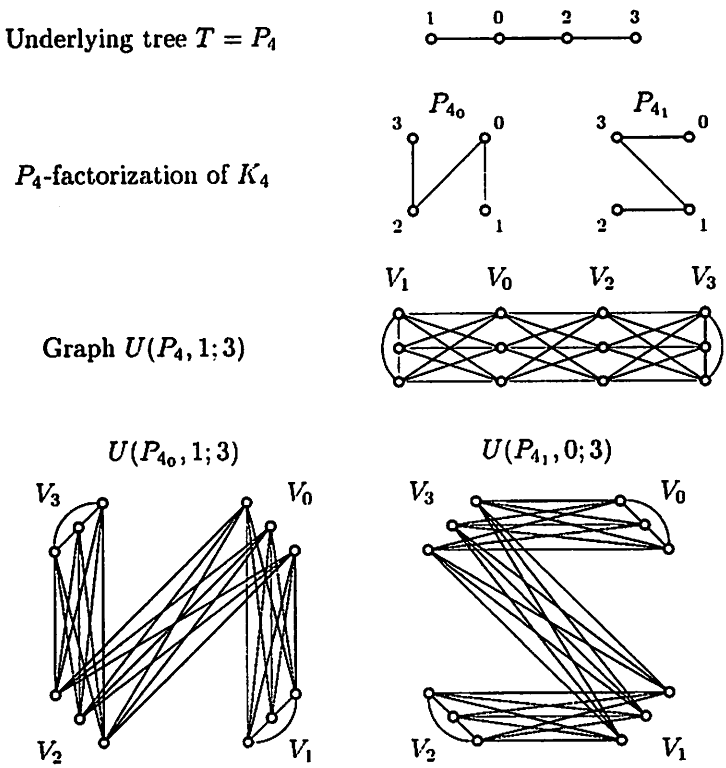
(iii) Finally, if

$$\begin{aligned} \phi_t(s+n) &= \phi_0(s) = s, && \text{then} \\ s+n+t &= s \pmod{2n}, && \text{and} \\ n+t &= 0 \pmod{2n}, && \text{or} \\ t &= n, \end{aligned}$$

which is impossible, since we have assumed that $t \in \{1, 2, \ldots, n-1\}$.

Therefore, the images of the vertices s and $s+n$ appear in $2n$ different vertices of K_{2n} and each of them is in exactly one factor T_r. This means that also each corresponding complete graph K_{V_i} for $i = 0, 1, \ldots, 2n-1$ is in exactly one factor $U(T_r, s; k)$.

Since all complete graphs K_{V_i} and all complete bipartite graphs K_{V_i,V_j} are pairwise edge disjoint, then also each edge of K_{2nk} is in exactly one $U(T_r, s; k)$, and so $U(T_0, s; k), U(T_1, s; k), \ldots, U(T_{n-1}, s; k)$ give a $U(T, s; k)$-factorization of K_{2nk}. □



$U(P_4, 1; 3)$-factorization of K_{12}

Figure 3.1: *$U(T, s; k)$-factorization of K_{2nk}*

4 $2n$-cyclic blended labeling

Now we show a decomposition of the graph $U = U(T, s; k)$ into k isomorphic copies of a graph G with $2nk - 1$ edges, and consequently we obtain also a G-decomposition of K_{2nk} into nk isomorphic copies of G. Hence, we need to explore the properties of a graph G that would decompose U. To characterize such a graph G we introduce a new type of labeling.

The labeling is in fact a generalization of the blended ρ-labeling. The main idea is that we split the graph U into two copies of the complete graph K_k and $2n-1$ copies of the complete bipartite graph $K_{k,k}$. Each of these graphs is then decomposed separately using known methods based on vertex labelings. The complete graphs K_k are both decomposed cyclically into k copies of a graph with $(k-1)/2$ edges, which requires k to be odd. Each complete bipartite graph $K_{k,k}$ is then decomposed bi-cyclically into k copies of a graph with k edges.

Definition 4.1 *Let G be a graph with $2nk - 1$ edges, for k odd and $k, n > 1$, and the vertex set $V(G) = \bigcup_{i=0}^{2n-1} V_i$, where $|V_i| = k$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, for $i = 0, 1, \ldots, 2n-1$.*

The mixed length *of an edge (x_i, y_j) with $\lambda(x_i) = a_i$ and $\lambda(y_j) = b_j$ for $i < j$ is defined as*

$$l_{i,j}(x_i, y_j) = b - a \pmod k$$

for $x_i \in V_i$, $y_j \in V_j$ and the pure length *of an edge (x_i, y_i) with $x_i, y_i \in V_i, \lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$ as*

$$l_{ii}(x_i, y_i) = \min\{|a - b|, k - |a - b|\}.$$

We say that G has a $2n$-cyclic blended labeling *(or for short just $2n$-cyclic labeling) if there exists an underlying tree T on $2n$ vertices with a ρ-symmetric graceful labeling such that*

(i) for each edge $(i, j) \in E(T)$
$\{l_{i,j}(x_i, y_j) | (x_i, y_j) \in E(G)\} = \{0, 1, 2, \ldots, k-1\}$

(ii) and for some vertex $s \in T$ and its symmetric image $t = s + n \pmod{2n}$
$\{l_{ss}(x_s, y_s) | (x_s, y_s) \in E(G)\} = \{1, 2, \ldots, (k-1)/2\}$, *and*
$\{l_{tt}(x_t, y_t) | (x_t, y_t) \in E(G)\} = \{1, 2, \ldots, (k-1)/2\}$.

We notice that similarly as a graph with a blended ρ-labeling, a graph G with a $2n$-cyclic blended labeling can be also split into subgraphs H_s and H_t on the vertices of the sets V_s and V_t with pure edges, and $2n-1$ subgraphs H_{ij} for each $(i, j) \in E(T)$ with mixed edges. When a $2n$-cyclic labeling is restricted to H_s and H_t, we have just the usual ρ-labeling, while when restricted to H_{ij} we obtain a bipartite ρ-labeling.

We will show that we can use a graph G with a $2n$-cyclic labeling to find a $2n$-cyclic decomposition of the graph U. The decomposition is obtained by permuting the vertices of U under a permutation with $2n$ cycles, each of them of length k, so that the vertices of each of the sets V_i rotate separately. Then the ρ-labelings of the subgraphs H_s and H_t guarantee, according to Theorem 2.2, a cyclic decomposition of K_{V_s} and K_{V_t} into k copies of H_s and H_t, respectively. Similarly, the bipartite ρ-labelings of subgraphs H_{ij} guarantee bi-cyclic decompositions of each of the complete bipartite graphs K_{V_i,V_j} into k copies of H_{ij}.

Before we state the lemma we need to define the notion of an $2n$-cyclic decomposition. We again unify a vertex with its label in our notation.

Definition 4.2 *Let G be a graph with at most $2nk$ vertices such that there exits a G-decomposition $G_0, G_1, \ldots, G_s$ of a graph U on $2nk$ vertices. We say that the G-decomposition is $2n$-cyclic if there exists an ordering*
$(0_0, 1_0, 2_0, \ldots, (k-1)_0, 0_1, 1_1, 2_1, \ldots, (k-1)_1, \ldots\ldots, 0_{2n-1}, 1_{2n-1}, 2_{2n-1}, \ldots, (k-1)_{2n-1})$
of the vertices of U and isomorphisms $\phi_r : G_0 \longrightarrow G_r$ where $r = 1, 2, \ldots, s$, such that $\phi_r(x_i) = (x+r)_i \pmod k$ for every $x = 0, 1, 2, \ldots, k-1$ and $i = 0, 1, 2, \ldots, 2n-1$.

Now we are ready to prove the following lemma.

Lemma 4.3 *Let a graph G with $2nk-1$ edges, for k odd and $k, n > 1$, have a $2n$-cyclic blended labeling. Then there exists a $2n$-cyclic G-decomposition of $U(T, s; k)$ into k copies of G.*

Proof. Let a graph $U = U(T, s; k)$ have the vertex set $V(U) = \bigcup_{i=0}^{2n-1} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $i = 0, 1, 2, \ldots, 2n-1$.

Suppose that $G = G_0$ and define the graphs $G_1, G_2, \ldots, G_{k-1}$, all with the vertex sets $V(G_i) = V(U(T, s; k))$ to be isomorphic to G. For each G_r, where $r = 1, 2, \ldots, k-1$, let there be an isomorphism $\phi_r : G_0 \longrightarrow G_r$ such that $\phi_r(x_i) = (x+r)_i \pmod k$ for any $x_i \in G_0$.

The lengths of the edges are preserved by the automorphisms ϕ_r. In particular, if $t \in \{s, s+n\}$ and $(x_t, (x+a)_t)$ is an edge of a pure length a, $1 \leq a \leq \frac{k-1}{2}$, in G_0, then $((x+r)_t, (x+a+r)_t)$ is the edge of the pure length a in G_r, and if $(x_i, (x+b)_j)$ is an edge of a mixed length b, $0 \leq b \leq k-1$, in G_0, then $((x+r)_i, (x+b+r)_j)$ is the edge of the mixed length b in G_r.

In U we have k edges of each pure length $l_{tt} \in \{1, 2, \ldots \frac{k-1}{2}\}$, where $t \in \{s, s+n\}$, and k edges of each mixed length $l_{ij} \in \{0, 1, 2, \ldots, k-1\}$ for each $(i,j) \in T$. In G we have one edge of each pure length l_{tt}, where $t \in \{s, s+n\}$, and one edge of each mixed length l_{ij} for each $(i,j) \in T$. While making k isomorphic copies of G we obtain k copies of edges of each mixed and pure length.

Suppose now that the same edge $(x_t, (x+a)_t)$ of the pure length $l_{tt} = a$ is in two different copies of G, G_r and G_p. We can again without loss of generality assume that $r = 0$. But if $(x_t, (x+a)_t) \in G_p$, then $(x_t, (x+a)_t) = ((y+p)_t, (y+p+a)_t)$ for some y since each edge of G_p arises from an edge of G_0 by adding p to both its endvertices. Hence, $(y_t, (y+a)_t) \in G_0$. However, $(x_t, (x+a)_t)$ is the only edge of the pure length $l_{tt} = a$ in G, which yields $x = y$ and therefore $p = 0$. This contradicts our assumption that G_p is different from G_0. Similarly we suppose that an edge $(x_i, (x+b)_j)$ of a mixed length $l_{ij} = b$ is in two different copies of G, G_0 and G_p, where $p \in \{1, 2, \dots, k-1\}$. If $(x_i, (x+b)_j) \in G_p$, then $(x_i, (x+b)_j) = ((y+p)_i, (y+p+b)_j)$ for some y for the same reasons as above, and $(y_i, (y+b)_j) \in G_0$. From the uniqueness of the edge of the mixed length $l_{ij} = b$ in G_0 we again get $x = y$ and $p = 0$, which is a contradiction.

Thus in k copies of G we have all $k(2nk-1)$ different edges of U, and so $G_0, G_1, G_2, \dots, G_{k-1}$ form a $2n$-cyclic decomposition of U. □

Finally we can state the main theorem of this paper, which is a direct consequence of the previous two lemmas.

Theorem 4.4 *Let G with $2nk-1$ edges be a graph that allows a $2n$-cyclic blended labeling for k odd and $k, n > 1$. Then there exists a G-decomposition of K_{2nk} into nk copies of G.*

Proof. By Lemma 3.2 the complete graph K_{2nk} can be factorized into n copies of $U(T, s; k)$, and by Lemma 4.4 the graph $U(T, s; k)$ can be factorized into k copies of G if G has $2n$-cyclic blended labeling. Therefore, K_{2nk} is decomposable into nk isomorphic copies of G. □

5 $2n$-cyclic labeling of lobsters with $d = 4$

Here we give an example of an infinite class of trees which have a $2n$-cyclic labeling and the diameter $d = 4$. As we know, the blended labeling can be used only for factorizations of K_{4k+2}, while for K_{4k} the only method known so far was the switching blended labeling. We will show that no trees with diameter 4 and $4k$ vertices allow a switching blended labeling. Before we give a proof of this fact, we need to state the definition of the switching blended labeling.

Definition 5.1 *Let T be a tree on $2n = 4k$ vertices such that $V(T) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$ with $|V_0| = |V_1| = 2k$. Let λ be an injection, $\lambda : V_m \longrightarrow \{0_m, 1_m, 2_m, \dots, (2k-1)_m\}$, $m = 0, 1$. The pure length and the mixed length of an edge are defined as for the blended labeling. The tree T has a* switching blended labeling *(or just* switching labeling *for short) if*

(1) $\{l_{00}(x_0, y_0) | (x_0, y_0) \in E(T)\} = \{1, 2, \dots, k\}$,

(2) $\{l_{11}(x_1, y_1) | (x_1, y_1) \in E(T)\} = \{1, 2, \ldots, k-1\}$,

(3) $\{l_{01}(x_0, y_1) | (x_0, y_1) \in E(T)\} = \{0, 1, 2, \ldots, 2k-1\}$, *and*

(4) *there exists an automorphism* φ *of* $T - (i_0, (i+k)_0)$, *where* $(i_0, (i+k)_0)$ *is the unique edge of the pure length* k *in* T, *such that* $\varphi(i_0) = j_1$ *and* $\varphi((i+k)_0) = (j+k)_1$ *for some* j.

Theorem 5.2 *If a tree* T *on* $4k$ *vertices, where* $k \geq 2$, *allows a switching blended labeling, then* $\operatorname{diam} T > 4$.

Proof. Suppose to the contrary that a tree T with $4k$ vertices has a switching labeling, and $\operatorname{diam} T = d \leq 4$. Then there is the edge $e_0 = (i_0, (i+k)_0)$ of the maximum pure length $l_{00}(e_0) = k$ in T. Let e_1 be the edge of the same pure length $l_{11}(e_1) = k$ in V_1, such that $e_1 = (j_1, (j+k)_1) \notin T$ and $\varphi(i_0) = j_1$, $\varphi((i+k)_0) = (j+k)_1$.

By G we denote the graph $G = T + e_1$. Then the graph $G - e_0$ is isomorphic to T and in G there is a cycle C_p, which contains both edges e_0, e_1. Since the endvertices of e_0 are both in V_0 and the endvertices of e_1 are both in V_1, the minimum length of the cycle C_p is $p = 4$.

Suppose first that $p = 4$. It means that $C_4 = i_0, (i+k)_0, (j+k)_1, j_1$ or $C_4 = i_0, (i+k)_0, j_1, (j+k)_1$. Notice that these cases are equivalent, since $j = j + k + k \pmod{2k}$. Hence we investigate just the former case. Then the edges (i_0, j_1) and $((i+k)_0, (j+k)_1)$ must be in T. But this is not possible, because they are both of the same mixed length $l_{01}((i_0, j_1)) = j - i \pmod{2k}$, and $l_{01}(((i+k)_0, (j+k)_1)) = j + k - (i+k) = j - i \pmod{2k}$, which contradicts property (3) of the switching labeling. Therefore the length of the cycle C_p is at least $p = 5$ and the diameter d of T is at least 4.

Now suppose that $p = 5$. Then there is a cycle $C_5 = i_0, (i+k)_0, (j+k)_1, j_1, v$ (again the case $C_5 = i_0, (i+k)_0, j_1, (j+k)_1, v$ is equivalent). Because we assumed that the diameter d of the tree T (or equivalently of $G - e_1$ or $G - e_0$) is equal to 4, all other edges in T must be incident to the vertex v. This is true because if there is an edge xi_0, where $x \neq (i+k)_0, v$, then from (4) it follows that there must be also an edge $yj_1, y \neq (j+k)_1, v$, and vice versa. But then there is the path $x, i_0, v, j_1, (j+k)_1, (i+k)_0$ in $G - e_0$ or $y, j_1, v, i_0, (i+k)_0, (j+k)_1$ in $G - e_1$, both of them of length 5, which contradicts our assumption that $d \leq 4$. Similarly, if there is one of edges $x(i+k)_0, y(j+k)_1$, where $x \neq i_0, (j+k)_1$ and $y \neq j_1, (i+k)_0$, then there must be the other one, too. Then again there is the path $x, (i+k)_0, (j+k)_1, j_1, v, i_0$ in $G - e_0$ or $y, (j+k)_1, j_1, v, i_0, (i+k)_0$ in $G - e_1$, giving the same contradiction.

But now if the vertex v belongs to V_0, then all the edges incident to v are either pure edges with both endvertices in V_0 or mixed edges with one endvertex v in V_0 and the other endvertex in V_1. It means there is only one pure edge in V_1, namely $(j_1, (j+k)_1)$ of the length $k \geq 2$. This is impossible, since the tree T must contain edges of all pure lengths $l_{11} = 1, 2, \ldots, k$. The same argument holds when $v \in V_1$ and the proof is complete. □

Further we give a $2n$-cyclic labeling for certain class of lobsters. A *lobster* is a tree from which by deleting all vertices of degree one we obtain a caterpillar. A *caterpillar* is a tree from which by deleting all vertices of degree one we obtain a path or a single vertex. In the latter case the caterpillar is isomorphic to a star.

Any lobster with diameter 4 can be obtained from a star $K_{1,p}$. If $p = 1$ or 2, then the lobster is indeed a caterpillar. Therefore, we will assume that $p \geq 3$. The vertex c of degree p will be called the *central vertex*, the remaining vertices will be denoted $t_0, t_1, \ldots, t_{p-1}$ and called *secondary vertices*. To obtain our lobster, we join new vertices of degree 1 to at least three secondary vertices t_i.

We denote such a lobster by $L(p; d_0, d_1, \ldots, d_{p-1})$, where d_q is the number of neighbors of degree one of the secondary vertex t_q and p is the degree of the central vertex c. See example in Figure 5.1.

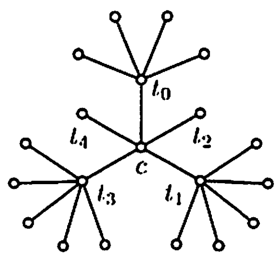


Figure 5.1: *Lobster* $L(5; 4, 5, 0, 5, 0)$.

Now we consider lobsters $L(k; d_0, d_1, \ldots, d_{k-1})$ on $2nk$ vertices where $n, k > 1$, and k is odd, so that the necessary condition for the existence of a $2n$-cyclic labeling is satisfied. Let $k = 2m + 1$. The vertices of degree one are distributed almost regularly in multiples of k. Particularly, the degree of each secondary vertex t_q is determined as follows. Always is $d_0 = k - 1$. Let $n - 1 = am + b$, where $0 \leq b < m$. If $b = 0$, then $d_q = ak$ for $q = 1, 2, \ldots, 2m$. If $0 < b < m$, then $d_q = d_{m+q} = (a + 1)k$ for $q = 1, 2, \ldots, b$, and $d_q = d_{m+q} = ak$ for $q = b + 1, b + 2, \ldots, m$.

Notice the case when $m = n - 1$. Then $d_0 = k - 1$, and $d_1 = d_2 = \ldots = d_{2m} = k$. This means we obtain a *k-balanced lobster*, which is a lobster such that the degrees of the secondary vertices of joined stars K_{1,d_q} differ at most by 1 and the central vertex has degree k. For certain values of k a blended labeling of k-balanced lobsters on $2n$ vertices, n odd, was found by Kubesa [6]. He also gives a blended labeling for two infinite classes of certain "totally unbalanced" lobsters of diameter 4.

Construction For each graph with a $2n$-cyclic labeling there must also exist an underlying tree T with a ρ-symmetric graceful labeling. The tree we use in our construction is always a double star S on $2n$ vertices. In order to obtain a ρ-symmetric graceful labeling of S, we assign labels 0 and n to the central vertices of the stars $K_{1,n-1}$, which are then connected by the symmetric edge $(0, n)$ of the maximum length n. The labels of the vertices of degree 1 joined to the central vertex 0 are $1, 2, \dots, n-1$, thus the edges have the lengths $1, 2, \dots, n-1$. The labels of the vertices of degree 1 joined to the central vertex n are $n+1, n+2, \dots, 2n-1$, and the edges have again the lengths $1, 2, \dots, n-1$.

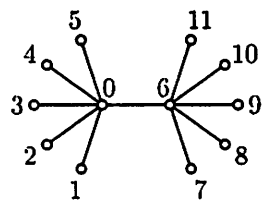


Figure 5.2: *Symmetric graceful labeling of a double star S on 12 vertices.*

Now we give a $2n$-cyclic labeling of the lobster $L(k; d_0, d_1, \dots, d_{k-1})$ with the vertex set $V(L) = \bigcup_{i=0}^{2n-1} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i, \dots, (k-1)_i\}$, $i = 0, 1, 2, \dots, 2n-1$. Let $k = 2m+1$.

For each edge $(i, j) \in S$ there is the corresponding subgraph H_{ij} with k mixed edges. Also for the vertex $0 \in V(S)$ and the symmetric vertex n there are subgraphs H_0 and H_n with m pure edges in each of them. The subgraph H_{0n} corresponding to the symmetric edge $(0, n)$ in S has the mixed edges $(0_0, 0_n), (0_0, 1_n), (0_0, 2_n), \dots, (0_0, m_n)$ of lengths $l_{0n} = 0, 1, 2, \dots, m$ and $(1_0, 0_n), (2_0, 0_n), \dots, (m_0, 0_n)$ of lengths $l_{0n} = m+1, m+2, \dots, 2m$. We add m pure edges on vertices of sets V_i, for $i = 0, n$. The edges are $(0_i, m_i), (0_i, (m+1)_i), \dots, (0_i, (2m)_i)$ of pure lengths $l_{ii} = m, m-1, \dots, 1$.

The central vertex c of the lobster is the vertex with the label 0_n. The secondary vertices $t_0, t_1, \dots, t_m$ have the labels $0_0, 1_0, \dots, m_0$, and the secondary vertices $t_{m+1}, t_{m+2}, \dots, t_{2m}$ receive the labels $(m+1)_n, (m+2)_n, \dots, (2m)_n$. So far all the vertices t_q are of degree one for $q = 1, 2, \dots, 2m$, only the vertex t_0 has k neighbors, so that $d_0 = k-1$. Now we increase degrees of the other vertices t_q.

Since there are edges $(0, j)$ and $(n, n+j)$ for $j = 1, 2, \ldots, n-1$ in the underlying double star S, we must construct subgraphs H_{0j} and $H_{n,n+j}$. We always add the edges (q_0, r_j) for $q \equiv j \pmod{m}$, where $q = 1, 2, \ldots, m$ and $r = 0, 1, \ldots, k-1$. It means that each H_{0j} is the star $K_{1,k}$ with the central vertex t_q, which receives the label q_0. Obviously, if $n-1 = am$, each of the vertices t_q is connected to exactly ak vertices of degree 1 ($d_q = ak$), and for $n-1 = am+b$, where $1 \le b < m$, there are k more neighbors of degree 1 added to each of the first b vertices t_q so that $d_q = (a+1)k$. Similarly, we add the edges $((m+q)_n, r_{m+j})$ for $q \equiv j \pmod{m}$, where $q = 1, 2, \ldots, m$ and $r = 0, 1, \ldots, k-1$. Then each subgraph $H_{n(m+j)}$ is again the star $K_{1,k}$ with the central vertex $(m+q)_n$, which means $d_{m+q} = ak$ for $q = b+1, b+2, \ldots, m$, or $d_{m+q} = (a+1)k$ for $q = 1, 2, \ldots, b$. See example in Figure 5.3.

There is a $2n$-cyclic factorization of the graph $U(S, 0; k)$ into k copies of the lobster $L(k; d_0, d_1, \ldots, d_{k-1})$ on $2nk$ vertices with the $2n$-cyclic labeling given above. Consequently, according to Theorem 4.4 there is also the $L(k; d_0, d_1, \ldots, d_k - 1)$-factorization of K_{2nk}.

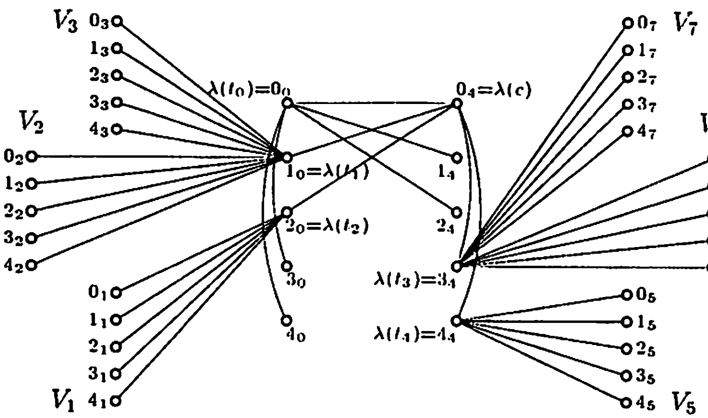


Figure 5.3: *$2n$-cyclic blended labeling of the lobster* $L(5; 4, 10, 5, 10, 5)$

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