SUMS OF GENERALIZED FIBONACCI NUMBERS BY MATRIX METHODS

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ABSTRACT. In this paper, we consider a certain second order linear recurrence and then give generating matrices for the sums of positively and negatively subscripted terms of this recurrence. Further, we use matrix methods and derive explicit formulas for these sums.

1. INTRODUCTION

The Fibonacci sequence is defined by the following equation for $n > 1$

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers have many interesting properties. For example, the sums of the Fibonacci numbers subscripted from 1 to n can be expressed by a formula including Fibonacci numbers. The sums formula is given by

$$\sum_{i=1}^{n} F_i = F_{n+2} - F_1.$$

Matrix methods many times have played an important role stemming from the number theory [1-5]. For instance, let B be an 2×2 companion matrix as follows

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then it is well known that

$$B^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Now we consider a generalization of the Fibonacci numbers. Let A be nonzero integer satisfying $A^2 + 4 \neq 0$. The generalized Fibonacci sequence $\{u_n\}$ is defined by the recurrence relation for $n > 1$

$$u_{n+1} = Au_n + u_{n-1}, \tag{1.1}$$

where $u_0 = 0$ and $u_1 = 1$. For later use, note that $u_2 = A$, $u_3 = A^2 + 1$ and $u_4 = A^3 + 2A$. When $A = 2$, then $u_n = P_n$ (nth Pell number).

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Let α and β be the roots of the equation $x^2 - Ax - 1 = 0$, then the Binet formula of the sequence $\{u_n\}$ has the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Using the recurrence relation of sequence $\{u_n\}$, we can obtain the negatively subscripted terms and these terms satisfy

$$u_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta}.$$

Since $\alpha\beta = -1$, then we have

$$u_{-n} = (-1)^{n+1} u_n \text{ and } u_{-n} = Au_{-(n+1)} + u_{-(n+2)}. \tag{1.2}$$

Thus for later use $u_{-1} = 1$, $u_{-2} = -A$, $u_{-3} = A^2 + 1$ and $u_{-4} = -(A^3 + 2A)$.

Furthermore, by the inductive argument, one can easily verify that the generating matrix for the sequence $\{u_n\}$ is given by

$$W^n = \begin{bmatrix} A & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{bmatrix}. \tag{1.3}$$

In this paper, we construct certain matrices, then we compute the nth powers of these matrices which are the generating matrices for the sums of the positively and negatively subscripted terms of the sequence $\{u_n\}$ from 1 to n.

2. Generating matrix for the sums of the positively subscripted terms of the sequence $\{u_n\}$

In this section we consider the positively subscripted terms of the sequence $\{u_n\}$ and then define a 3×3 matrix C. Further, we compute the nth power of the matrix C and use matrix methods for the explicit formula for the sums of the terms of the sequence $\{u_n\}$.

Define the 3×3 matrix C as follows

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.1}$$

and define the 3×3 matrix E_n as follows

$$E_n = \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ & u_{n+1} & u_n \\ S_{n-1}^+ & u_n & u_{n-1} \end{bmatrix}, \tag{2.2}$$

where S_n^+ denote the sums of the positively subscripted terms of the sequence $\{u_n\}$ from 1 to n, that is

$$S_n^+ = \sum_1^n u_i. \tag{2.3}$$

Then we have the following Lemma.

Lemma 1. *Let the matrices C and E_n have the forms (2.1) and (2.2), respectively. Then for n, $n > 0$*

$$E_n = C^n. \tag{2.4}$$

Proof. We will use the induction method for the proof of Lemma. If $n = 1$, then, by $u_2 = A$, $u_1 = 1$ and $u_0 = 0$, we obtain

$$C^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ S_1^+ & u_2 & u_1 \\ S_0^+ & u_1 & u_0 \end{bmatrix} = E_1.$$

If $n = 2$, then

$$C^2 = \begin{bmatrix} 1 & 0 & 0 \\ A+1 & A^2+1 & A \\ 1 & A & 1 \end{bmatrix}.$$

Since $S_2^+ = A + 1$ and $u_3 = A^2 + 1$, $E_2 = C^2$. Suppose that the claim is true for n. Then we will show that the equation holds for $n + 1$. Thus, by our assumption, we write

$$\begin{aligned} C^{n+1} &= C^n C = E_n C \\ &= \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ & u_{n+1} & u_n \\ S_{n-1}^+ & u_n & u_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

which, by a matrix multiplication, satisfies

$$C^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ + u_{n+1} & Au_{n+1} + u_n & u_{n+1} \\ S_{n-1}^+ + u_n & Au_n + u_{-1} & u_n \end{bmatrix} = E_{n+1}.$$

By the recurrence relation of the sequence $\{u_n\}$ and since $S_n^+ + u_{n+1} = S_{n+1}^+$, we have the conclusion. □

Consequently, we obtain a generating matrix for the sums of the terms of the sequence $\{u_n\}$ from 1 to n.

Also we write the Eq. (2.4) as shown

$$E_{n+1} = E_n E_1 = E_1 E_n. \tag{2.5}$$

In other words, the matrix E_1 is commutative under matrix multiplication. Then we have the Corollary.

Corollary 1. *Let the sum* S_n^+ *have the form (2.3). Then the sum* S_n^+ *satisfies the following nonhomogeneous recurrence relation for* $n > 0$

$$S_{n+1}^+ = AS_n^+ + S_{n-1}^+ + 1.$$

Proof. From (2.5) and since an element of E_{n+1} is the product of a row E_1 and a column of E_n :

$$S_{n+1}^+ = AS_n^+ + S_{n-1}^+ + 1,$$

which is desired. □

Now we are going to derive an explicit formula for the sum S_n^+. Let $K_C(\lambda)$ be the characteristic polynomial of the matrix C. Thus,

$$K_C(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & A-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (\lambda - 1)\left(-\lambda^2 + A\lambda + 1\right).$$

Also it is easily seen that the characteristic polynomial of the matrix W given by (1.3) is $-\lambda^2 + A\lambda + 1$. Therefore the eigenvalues of the matrix C are

$$\lambda_1 = \frac{A + \sqrt{A^2+4}}{2}, \ \lambda_2 = \frac{A - \sqrt{A^2+4}}{2} \text{ and } \lambda_3 = 1.$$

Since $A \neq 0$ and $A^2 + 4 \neq 0$, we have that the eigenvalues of the matrix C are distinct.

Let V be the 3×3 matrix defined as follows:

$$V = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{A} & \lambda_1 & \lambda_2 \\ \frac{-1}{A} & 1 & 1 \end{bmatrix}, \tag{2.6}$$

where λ_1 and λ_2 are the eigenvalues of C. Note that $\det V = \lambda_1 - \lambda_2 \neq 0$.

Then we have the following Theorem.

Theorem 1. *Let* S_n^+ *denote the sums of the terms of the sequence* $\{u_n\}$. *Then*

$$S_n^+ = \frac{u_{n+1} + u_n - 1}{A}.$$

Proof. One can easily verify that

$$CV = VD_1,$$

where C and V are as before, and D_1 is the diagonal matrix such that $D_1 = diag(\lambda_3, \lambda_1, \lambda_2)$. Since $\det V \neq 0$, the matrix V is invertible. So we write that $V^{-1}CV = D_1$. Hence, the matrix C is similar to the diagonal matrix D_1. Thus we obtain $C^n V = VD_1^n$.Since $C^n = E_n$,

$$E_n V = VD_1^n.$$

So by a matrix multiplication, we have the conclusion. □

For example, if we take $A = 2$, then the sequence $\{u_n\}$ is reduced to the usual Pell numbers and by Theorem 1, we have

$$\sum_{1}^{n} P_i = \frac{P_{n+1} + P_n - 1}{2}$$

which is well known from [10].

Now we give a formula for the sum S_n^+ by using a matrix method with the following Corollary.

Corollary 2. *Let S_n^+ denote the sums of the terms u_i from 1 to n. Then for all positive integers n and m*

$$S_{n+m}^+ = u_{n+1} S_m^+ + u_n S_{m-1}^+ + S_n^+$$

where u_n given by (1.1).

Proof. From (2.4), we can write, for all positive integers n and m

$$E_{n+m} = E_n E_m.$$

Clearly

$$\begin{bmatrix} 1 & 0 & 0 \\ S_{n+m}^+ & u_{n+m+1} & u_{n+m} \\ S_{n+m-1}^+ & u_{n+m} & u_{n+m-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ & u_{n+1} & u_n \\ S_{n-1}^+ & u_n & u_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ S_m^+ & u_{m+1} & u_m \\ S_{m-1}^+ & u_m & u_{m-1} \end{bmatrix}.$$

By a matrix multiplication, the proof is easily seen. □

Note that taking by $n = 1$ in Corollary 2, we can obtain the result of Corollary 1.

3. Generating Matrix for the Sums of the Negatively Subscripted Terms u_{-n}

In this section, we consider the negatively subscripted terms of the sequence $\{u_n\}$. First, we give a generating matrix for the negatively subscripted terms. Second, we give a generating matrix for the sums of these terms.

Let the 2×2 matrix T be as follows:

$$T = \begin{bmatrix} -A & 1 \\ 1 & 0 \end{bmatrix} \tag{3.1}$$

and the 2×2 matrix H_n be as follows:

$$H_n = \begin{bmatrix} u_{-(n+1)} & u_{-n} \\ u_{-n} & u_{-(n-1)} \end{bmatrix} \tag{3.2}$$

where u_{-n} is the nth negatively subscripted term of the sequence $\{u_n\}$.

We start with the following Lemma.

Lemma 2. *Let the matrices T and H_n have the form (3.1) and (3.2), respectively. Then for $n > 0$*

$$H_n = T^n.$$

Proof. (Induction on n) If $n = 1$, then, by the identity (1.2), we have

$$T^1 = \begin{bmatrix} -A & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u_{-2} & u_{-1} \\ u_{-1} & u_0 \end{bmatrix}.$$

If $n = 2$, then

$$T^2 = \begin{bmatrix} A^2+1 & -A \\ -A & 1 \end{bmatrix}.$$

Since by (1.2), we have $u_{-3} = u_3 = A^2+1$, $u_{-2} = -u_2 = -A$ and $u_{-1} = 1$, we have

$$T^2 = \begin{bmatrix} A^2+1 & -A \\ -A & 1 \end{bmatrix} = H_2.$$

We suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Thus, by our assumption,

$$\begin{aligned} T^{n+1} &= T^n T^1 \\ &= \begin{bmatrix} u_{-(n+1)} & u_{-n} \\ u_{-n} & u_{-(n-1)} \end{bmatrix} \begin{bmatrix} -A & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Since the negatively subscripted terms of the sequence $\{u_n\}$ satisfy the recurrence relation $u_{-n} = Au_{-(n+1)} + u_{-(n+2)}$, we have $u_{-(n+2)} = -Au_{-(n+1)} + u_{-n}$ and $T^{n+1} = H_{n+1}$. So the proof is complete. □

Let S_n^- denote the sums of the negatively subscripted terms of the sequence $\{u_n\}$, that is

$$S_n^- = \sum_1^n u_{-i}. \tag{3.3}$$

Now we give a matrix method to generate the sum S_n^-. Define the 3×3 matrices R and Q_n as shown

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -A & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q_n = \begin{bmatrix} 1 & 0 & 0 \\ S_n^- & u_{-(n+1)} & u_{-n} \\ S_{n-1}^- & u_{-n} & u_{-(n-1)} \end{bmatrix}. \tag{3.4}$$

Then we have the following Theorem.

Theorem 2. *Let the matrices R and Q_n have the form (3.4). Then for $n > 0$*

$$R^n = Q_n. \tag{3.5}$$

Proof. (Induction on n) If $n = 1$, then we know that $S_1^- = u_{-1} = 1$, $S_n^- = 0$ for $n < 1$, $u_{-2} = -u_2 = -A$, $u_0 = 0$. Thus we obtain $R = Q_1$. If $n = 2$, then we have $S_2^- = u_{-1} + u_{-2} = -A + 1$, $u_{-3} = u_3$ and by a matrix multiplication

$$T^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1-A & A^2+1 & -A \\ 1 & -A & 1 \end{bmatrix} = H_2.$$

Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Thus, by our assumption, we write

$$\begin{aligned} R^{n+1} &= R^n R = Q_n R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ S_n^- & u_{-(n+1)} & u_{-n} \\ S_{n-1}^- & u_{-n} & u_{-(n-1)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -A & 1 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Since $S_{n+1}^- = S_n^- + u_{-(n+1)}$ and by Lemma 2, we obtain $T^{n+1} = Q_{n+1}$. So we have the Theorem. □

In the following Theorem, we give a nonhomogeneous recurrence relation for the sum S_n^-.

Theorem 3. *Let S_n^- denote the sums of the terms u_{-i} for $1 \le i \le n$. Then for $n > 0$*

$$S_{n+1}^- = -AS_n^- + S_{n-1}^- + 1.$$

Proof. Considering (3.5), we write $Q_{n+1} = Q_n Q_1 = Q_1 Q_n$ and say that the matrix Q_1 is commutative under matrix multiplication. By a matrix multiplication, the proof is easy. □

Generalizing $R^n = Q_n$, for all positive integers n and m, we can write that $Q_{n+m} = Q_n Q_m = Q_m Q_n$. Thus we obtain the following Corollary without proof as a generalization of the result of Theorem 3.

Corollary 3. *Let S_n^- denote the sums of the terms u_{-i} for $1 \le i \le n$. Then for all $n, m > 0$*

$$S_{n+m}^- = S_n^- + u_{-(n+1)} S_m^- + u_{-n} S_{m-1}^-.$$

Now we derive an explicit formula for the sums of the negatively subscripted terms u_{-i} for $1 \le i \le n$. For this purpose, we give some results. First, we consider the characteristic polynomial of the matrix T. The characteristic equation of T is $K_T(\lambda) = -(\lambda - 1)(\lambda^2 + A\lambda - 1)$. Thus the eigenvalues of matrix T are

$$\mu_1 = \frac{-A + \sqrt{A^2+4}}{2}, \quad \mu_2 = \frac{-A - \sqrt{A^2+4}}{2} \text{ and } \mu_3 = 1.$$

Note that $A \neq 0$ and $A^2 + 4 \neq 0$, the eigenvalues of T are distinct.

Let Λ be a matrix as follows

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{A} & \mu_1 & \mu_2 \\ \frac{1}{A} & 1 & 1 \end{bmatrix}.$$

Then we have the following Theorem.

Theorem 4. *Let S_n^- denote the sums of the negatively subscripted terms u_{-i} for $1 \leq i \leq n$. Then for $n > 1$*

$$S_n^- = \frac{1 - u_{-(n+1)} - u_{-n}}{A}.$$

Proof. By the characteristic equation of the negatively subscripted terms u_{-i}, we can readily verify that

$$R\Lambda = \Lambda D_2,$$

where D_2 is the 3×3 diagonal matrix such that $D_2 = diag\,(\mu_3, \mu_1, \mu_2)$. Since $\det\Lambda = \mu_1 - \mu_2 \neq 0$, the matrix Λ is invertible. Thus we write $\Lambda^{-1}R\Lambda = D_2$ and so the matrix is similar to the matrix D_2. Therefore, we write $\Lambda^{-1}R^n\Lambda = D_2^n$ or $R^n\Lambda = \Lambda D_2^n$. Since $R^n = Q_n$, we have $Q_n\Lambda = \Lambda D_2^n$.Then we have the conclusion from $Q_n\Lambda = \Lambda D_2^n$ by a matrix multiplication. □

Considering the identity (1.2), we have the following Corollary without proof.

Corollary 4. *Let S_n^- denote the sums of the negatively subscripted terms u_{-i} for $1 \leq i \leq n$. Then for $n > 1$*

$$S_n^- = \begin{cases} (u_n - u_{n+1} + 1)\,/A & \text{if } n \text{ is even,} \\ (u_{n+1} - u_n + 1)\,/A & \text{if } n \text{ is odd.} \end{cases}$$

For example, if take $A = 1$, then the sequence $\{u_n\}$ is reduced to the usual Fibonacci sequence and by Corollary 4, we have the sums of the negatively subscripted terms of the Fibonacci sequence for n is even number

$$\sum_1^n F_{-i} = F_1 - F_2 + F_3 - \ldots + F_{n-1} - F_n = 1 - F_{n-1}$$

and for n is odd number

$$\sum_1^n F_{-i} = F_1 - F_2 + F_3 - \ldots - F_{n-1} + F_n = F_{n-1} + 1.$$

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