On the Three Color Ramsey Numbers $R(C_m, C_4, C_4)$ *

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Abstract

Let G_i be the subgraph of G whose edges are in the i-th color in an r-coloring of the edges of G. If there exists an r-coloring of the edges of G such that $H_i \nsubseteq G_i$ for all $1 \leq i \leq r$, then G is said to be *r-colorable* to $(H_1, H_2, \ldots, H_r)$. The multicolor Ramsey number $R(H_1, H_2, \ldots, H_r)$ is the smallest integer n such that K_n is not r-colorable to $(H_1, H_2, \ldots, H_r)$. It is well known that $R(C_m, C_4, C_4) = m + 2$ for sufficiently large m. In this paper, we determine the values of $R(C_m, C_4, C_4)$ for $m \geq 5$, which show that $R(C_m, C_4, C_4) = m + 2$ for $m \geq 11$.

Keywords: *multicolor Ramsey number, forbidden subgraph, critical graph, cycle*

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex set $V(G)$ and edge set $E(G)$, we denote the order and the size of G by $p(G) = |V(G)|$ and $q(G) = |E(G)|$, respectively.

Let G_i be the subgraph of G whose edges are in the i-th color in an r-coloring of the edges of G. If there exists an r-coloring of the edges of G such that $H_i \nsubseteq G_i$ for all $1 \leq i \leq r$, then G is said to be *r-colorable*

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to $(H_1, H_2, \ldots, H_r)$. The multicolor Ramsey number $R(H_1, H_2, \ldots, H_r)$ is the smallest integer n such that K_n is not r-colorable to $(H_1, H_2, \ldots, H_r)$. In case of $H_1 \cong H_2 \cong \ldots \cong H_r \cong H$, we simply write the multicolor Ramsey number $R(H_1, H_2, \ldots, H_r)$ as $R_r(H)$. Let C_m be a cycle of length m. P_m denotes a path of m vertices. S_m is a star with $m-1$ leaves. Let $\overline{G}$ denote the complement of G.

The only known value of a multicolor classical Ramsey number $R_3(K_3) = R_3(C_3) = 17$, was given by Greenwood and Gleason[6]. Bialostocki and Schönheim proved that $R_3(C_4) = 11$[1]. Using a computer, Yang Yuansheng and Rowlinson determined that $R_3(C_5) = 17$ and $R_3(C_6) = 12$[10, 11]. Faudree, Schelten and Schiermeyer showed that $R_3(C_7) = 25$[5]. Exoo and Reynolds proved that $R(C_4, C_3, C_3) = 17$[4]. Schulte gave $R(C_3, C_4, C_4) = 12$ in his Ph.D. thesis[8]. For $m \geq 5$, it has been shown that $R(C_m, C_3, C_3) = 5m-4$[12].

Erdös, Faudree, Rousseau and Schelp proved the following theorem:

Theorem 1.1.[3] If m is sufficiently large, $R(C_m, C_4, C_4) = m+2$.
In their proof, the lower bounds of $R(C_m, C_4, C_4)$ are obtained by coloring the edges of K_{m+1} with three colors, as follows. Let

$$V(K_{m+1}) = \{v_1, v_2, \ldots, v_{m+1}\},$$

then

$$\begin{aligned} E(G_1) &= \{v_m v_{m+1} \cup v_i v_j;\ 1 \leq i < j \leq m-1\}, \\ E(G_2) &= \{v_m v_i;\ 1 \leq i \leq m-1\}, \\ E(G_3) &= \{v_{m+1} v_i;\ 1 \leq i \leq m-1\}. \end{aligned}$$

Since $G_1 \cong K_{m-1} \cup K_2$ and $G_2 \cong G_3 \cong S_m \cup K_1$, it follows $C_m \not\subseteq G_1$ and $C_4 \not\subseteq G_i (i = 2, 3)$. Hence we have,

Corollary 1.2. $R(C_m, C_4, C_4) \geq m+2$.

For the literature on small Ramsey number we refer to [7] and the relevant references given in it.

In this paper, we prove that

$$R(C_m, C_4, C_4) = m+2, \qquad m \geq 11,$$

and determine the values of $R(C_m, C_4, C_4)$ for $5 \leq m \leq 10$, as shown in Table 1.1.

Table 1.1. The values of $R(C_m, C_4, C_4)$

m	3	4	5	6	7	8	9	10	≥ 11
$R(C_m, C_4, C_4)$	12[8]	11[1]	12	12	12	12	13	13	$m+2$

For the sake of argument, let $f(m)$ be the values of $R(C_m, C_4, C_4)$ in Table 1.1 in the following sections.

2. The lower bounds of $R(C_m, C_4, C_4)$ for $m \geq 5$

A *cutpoint* of a graph is a vertex whose removal increases the number of components. A *nonseparable* graph is connected, nontrivial, and has no cutpoint. A *block* of a graph is a maximal nonseparable subgraph.

If $m \geq 11$, by Corollary 1.2, we have $R(C_m, C_4, C_4) \geq f(m)$. If $5 \leq m \leq 10$, the lower bounds of $R(C_m, C_4, C_4)$ in Corollary 1.2 can be improved as follows.

Lemma 2.1. $R(C_5, C_4, C_4) \geq f(5)$.
Proof. We show a 3-coloring of the edges of K_{11} in Figure 2.1, where all the edges of G_i are in the i-th color. We can find that $G_1 - v_3v_5$ (or $G_1 - v_4v_6$) consists of three blocks. One is isomorphic to $K_{3,4}$ and each of the others has at most 4 vertices, so $C_5 \nsubseteq G_1 - v_3v_5$ (or $C_5 \nsubseteq G_1 - v_4v_6$). The cycles that contain edges v_3v_5 and v_4v_6 have length at least 6 in G_1. So, $C_5 \nsubseteq G_1$. Since G_2 contains cycles of length at least 5 except the three triangles, we have $C_4 \nsubseteq G_2$. Similarly, since G_3 contains cycles of length at least 5 except the four triangles, we have $C_4 \nsubseteq G_3$. Hence, $R(C_5, C_4, C_4) \geq 12$. □

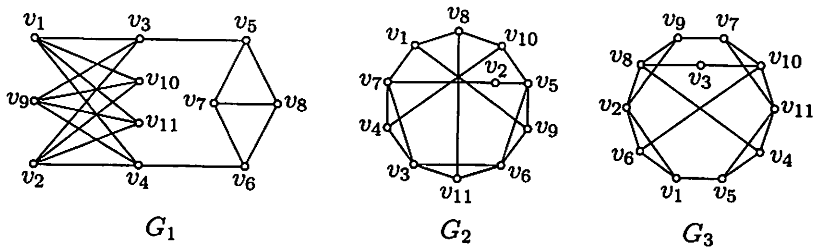

Figure 2.1. A 3-coloring of K_{11} for (C_5, C_4, C_4)

Lemma 2.2. If $m \in \{6, 7, 8\}$, then $R(C_m, C_4, C_4) \geq f(m)$.
Proof. We show a 3-coloring of the edges of K_{11} in Figure 2.2, where all the edges of G_i are in the i-th color. Since G_1 consists of three blocks,

and there are at most 5 vertices in every block, it is forced that $C_m \nsubseteq G_1$ for $6 \leq m \leq 8$; G_2 contains cycles of length at least 5 except the four triangles, we have $C_4 \nsubseteq G_2$. Since $G_3 \cong G_2$, then $C_4 \nsubseteq G_3$. Hence, $R(C_m, C_4, C_4) \geq 12$ for $6 \leq m \leq 8$. □

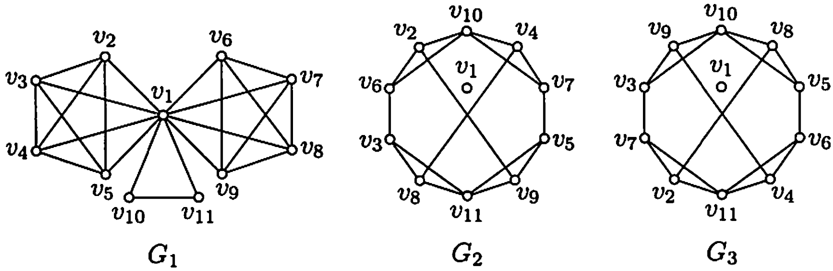

Figure 2.2. A 3-coloring of K_{11} for (C_m, C_4, C_4) for $6 \leq m \leq 8$

Lemma 2.3. If $m \in \{9, 10\}$, then $R(C_m, C_4, C_4) \geq f(m)$.
Proof. We show a 3-coloring of the edges of K_{12} in Figure 2.3, where all the edges of G_i are in the i-th color. Since G_1 consists of three blocks, and there are at most 8 vertices in every block, then $C_m \nsubseteq G_1$ for $9 \leq m \leq 10$; G_2 contains cycles of length at least 5 except the two triangles, we have $C_4 \nsubseteq G_2$. Also G_3 contains cycles of length at least 5 except the two triangles, it follows $C_4 \nsubseteq G_3$ too. Hence, $R(C_m, C_4, C_4) \geq 13$ for $9 \leq m \leq 10$. □

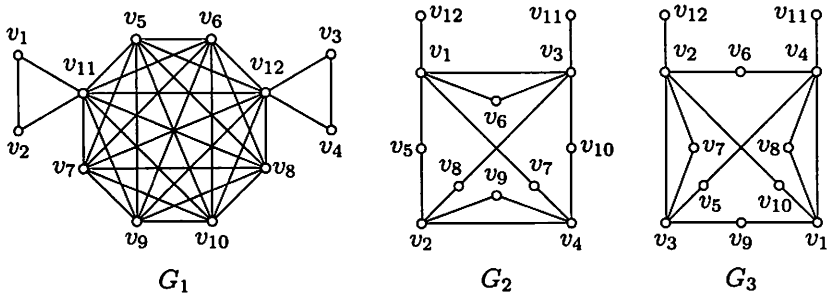

Figure 2.3. A 3-coloring of K_{12} for (C_m, C_4, C_4) for $9 \leq m \leq 10$

3. The values of $R(C_m, C_4, C_4)$ for $m \geq 5$

Lemma 3.1. $K_{3,7}$ is not 2-colorable to (C_4, C_4).
Proof. By contradiction, suppose that $K_{3,7}$ is 2-colorable to (C_4, C_4). Let

$G_i (i = 1, 2)$ be the subgraphs of $K_{3,7}$ whose edges are in the i-th color in a 2-coloring of the edges of $K_{3,7}$ such that $C_4 \nsubseteq G_i$. Let $d_{G_i}(v)$ be the degree of a vertex v in subgraph G_i, and

$$E(K_{3,7}) = \{u_i v_j;\ 1 \le i \le 3,\ 1 \le j \le 7\}.$$

Then one of $d_{G_1}(u_1)$ and $d_{G_2}(u_1)$, say $d_{G_1}(u_1)$ is at least 4. Without loss of generality, we may assume that $u_1 v_i \in E(G_1)$ for $1 \le i \le 4$. Since $C_4 \nsubseteq G_1$, there are at least three vertices of $\{v_1, v_2, v_3, v_4\}$, say v_2, v_3 and v_4 adjacent to u_2 in G_2. Therefore since $C_4 \nsubseteq G_2$, there are at least two vertices of $\{v_2, v_3, v_4\}$, say v_3 and v_4 adjacent to u_3 in G_1. Thus u_1, u_3, v_3 and v_4 would form a C_4 in G_1, a contradiction with $C_4 \nsubseteq G_1$. Hence, the lemma follows. □

To obtain the upper bounds of $R(C_m, C_4, C_4)$, we first define three graph sets $S_m(n)$, $S_m^*(n)$ and $S_m^{**}(n)$. Let $S_m(n)$ denote the set of the graphs of order n and not containing C_m. For a graph G, if $C_m \nsubseteq G$ and $C_m \subseteq G+e$ for any $e \in E(\overline{G})$, then G is said to be a *critical graph.* Let $S_m^*(n)$ be the set of the critical graphs of order n. Let $S_m^{**}(n)$ be the set of the graphs G such that $G \in S_m^*(n)$, $C_{m-1} \subseteq G$, $K_6 \nsubseteq \overline{G}$ and $K_{3,7} \nsubseteq \overline{G}$, that is,

$$\begin{aligned} S_m(n) &= \{G; C_m \nsubseteq G \wedge p(G) = n\}, \\ S_m^*(n) &= \{G; G \in S_m(n) \wedge C_m \subseteq G + e \text{ for any } e \in E(\overline{G})\}, \\ S_m^{**}(n) &= \{G; G \in S_m^*(n) \wedge C_{m-1} \subseteq G \wedge K_6 \nsubseteq \overline{G} \wedge K_{3,7} \nsubseteq \overline{G}\}. \end{aligned}$$

Lemma 3.2. If $R(C_{m-1}, C_4, C_4) \le f(m-1) \le f(m)$ and $\overline{G}$ is not 2-colorable to (C_4, C_4) for every $G \in S_m^{**}(f(m))$, then $R(C_m, C_4, C_4) \le f(m)$.
Proof. By contradiction, suppose that $R(C_m, C_4, C_4) > f(m) = n$. Let $G_i (i = 1, 2, 3)$ be the subgraphs of K_n whose edges are in the i-th color in a 3-coloring of the edges of K_n such that $G_1 \in S_m(n)$, $G_2 \in S_4(n)$ and $G_3 \in S_4(n)$. Then $\overline{G_1}$ is 2-colorable to (C_4, C_4). Since $R(C_{m-1}, C_4, C_4) \le n$, K_n is not 3-colorable to (C_{m-1}, C_4, C_4). Therefore since $C_4 \nsubseteq G_2$ and $C_4 \nsubseteq G_3$, it is forced that $C_{m-1} \subseteq G_1$. While there exists an edge $e \in E(\overline{G_1})$ such that $C_m \nsubseteq G_1 + e$, it will be transformed to G_1. We continue on this way until $G_1 \in S_m^*(n)$. Since $\overline{G_1}$ is 2-colorable to (C_4, C_4), by $R_2(C_4) = 6$[2] and Lemma 3.1, we have $K_6 \nsubseteq \overline{G_1}$ and $K_{3,7} \nsubseteq \overline{G_1}$. Now, we still have $C_{m-1} \subseteq G_1$. By the definition of $S_m^{**}(n)$, it follows $G_1 \in S_m^{**}(n)$. However now $\overline{G_1}$ is 2-colorable to (C_4, C_4), a contradiction with the hypothesis. □

We employ an algorithm CCG(Construct Critical Graphs) to construct all graphs in $S_m^{**}(n)$ by adding some edges to $(n - m + 1)K_1 \cup C_{m-1}$, where $n = f(m)$ and the isomorph program is same as the one used in [9, 10, 11, 12].

Algorithm CCG:

```
S**_m(n) = ∅; S_1 = { (n − m + 1)K_1 ∪ C_{m−1} };
while S_1 ≠ ∅ do
  S_0 = S_1; S_1 = ∅;
  for every G ∈ S_0 do
    if K_{3,7} ⊄ Ḡ, K_6 ⊄ Ḡ and G is a critical graph
      if G is not isomorphic to any graphs in S**_m(n)
        S**_m(n) = S**_m(n) ∪ {G};
      endif
    else
      if K_{3,7} ⊆ Ḡ or K_6 ⊆ Ḡ
        Let H be the forbidden subgraph K_{3,7}(or K_6) in Ḡ;
      else Let H = Ḡ;
      endif
      for every edge e ∈ E(H) do
        if C_m ⊄ G + e and G + e is not isomorphic to any
          graphs in S_1
          S_1 = S_1 ∪ {G + e};
        endif
      endfor
    endif
  endfor
endwhile
```

First, We construct all graphs in $S_5^{**}(12)$ using algorithm CCG. The value of $|S_5^{**}(12)|$ is shown in Table 3.1. Then, in order to determine whether $\overline{G}$ is 2-colorable to (C_4, C_4), we color the edges of $\overline{G}$ with two colors using a standard backtrack search algorithm. With the help of computer, we show that $\overline{G}$ is not 2-colorable to (C_4, C_4) for every graph $G \in S_5^{**}(12)$. Since $R(C_4, C_4, C_4) = 11 < f(5)$, by Lemma 3.2, $R(C_5, C_4, C_4) \le f(5) = 12$.

Consider the Ramsey number $R(C_6, C_4, C_4)$ similarly. We construct all graphs in $S_6^{**}(12)$ using algorithm CCG. Then we show that $\overline{G}$ is not 2-colorable to (C_4, C_4) for every graph $G \in S_6^{**}(12)$. Since $R(C_5, C_4, C_4) \le 12 \le f(6)$, by Lemma 3.2, $R(C_6, C_4, C_4) \le f(6)$.

We can construct all graphs in $S_m^{**}(f(m))$ for $7 \le m \le 19$. The values of $|S_m^{**}(f(m))|$ are shown in Table 3.1. Using the standard backtrack search algorithm, we show that $\overline{G}$ is not 2-colorable to (C_4, C_4) for every graph $G \in S_m^{**}(f(m))$. Since $f(m-1) \le f(m)$, by Lemma 3.2, we can prove $R(C_m, C_4, C_4) \le f(m)$ sequentially, for $m = 7, \ldots, 19$. Hence, we have,

Lemma 3.3. For $5 \le m \le 19$, $R(C_m, C_4, C_4) \le f(m)$. □

Table 3.1. The values of $|S_m^{**}(f(m))|$

m	5	6	7	8	9	10	11	12
$f(m)$	12	12	12	12	13	13	13	14
$\|S_m^{**}(f(m))\|$	100	60	70	79	0	5	26	1
m	13	14	15	16	17	18	19	
$f(m)$	15	16	17	18	19	20	21	
$\|S_m^{**}(f(m))\|$	0	0	0	0	0	0	0	

In [3], the following lemma is also established:

Lemma 3.4. Let G be a graph that contains a cycle C_m, but no C_{m+1}. If $K_r \not\subseteq \overline{G}$, then each vertex in $V(G) - V(C_m)$ is adjacent to at most $r-2$ vertices of $V(C_m)$ in G.

Lemma 3.5. If $m \geq 20$, then $R(C_m, C_4, C_4) \leq f(m)$.
Proof. We will prove that $R(C_m, C_4, C_4) \leq f(m)$ for $m \geq 19$ by induction.

(1) For $m = 19$, by Lemma 3.3, we have $R(C_{19}, C_4, C_4) \leq f(m) = 21$.

(2) Suppose that $R(C_k, C_4, C_4) \leq k+2$ for $k \geq 19$. We will show that $R(C_{k+1}, C_4, C_4) \leq k+3$, as follows.

Assume that $R(C_{k+1}, C_4, C_4) > k+3$, then K_{k+3} is 3-colorable to (C_{k+1}, C_4, C_4). Let $G_i (i = 1, 2, 3)$ be the subgraphs of K_{k+3} whose edges are in the i-th color in a 3-coloring of the edges of K_{k+3} such that $C_{k+1} \not\subseteq G_1$, $C_4 \not\subseteq G_2$ and $C_4 \not\subseteq G_3$. Then $\overline{G_1}$ is 2-colorable to (C_4, C_4). By the induction hypothesis, $R(C_k, C_4, C_4) \leq k+2$, there exists a cycle of length k in G_1, denoted by C_k. Let $v_i \in (V(G_1) - V(C_k))$ for $1 \leq i \leq 3$. Since $R_2(C_4) = 6$, we have $K_6 \not\subseteq \overline{G_1}$. By Lemma 3.4, each v_i is adjacent to at most 4 vertices of $V(C_k)$ in G_1. So, there are at least $k - 12 \geq 7$ vertices of $V(C_k)$ that are nonadjacent to any vertices of $\{v_1, v_2, v_3\}$ in G_1. Hence $K_{3,7} \subseteq \overline{G_1}$. By Lemma 3.1, $\overline{G_1}$ is not 2-colorable to (C_4, C_4), a contradiction. Hence, the assumption that $R(C_{k+1}, C_4, C_4) > k+3$ does not hold. So, $R(C_{k+1}, C_4, C_4) \leq k+3$. This completes the induction step, and the proof is finished. □

By results in [1] and [8], Corollary 1.2, Lemmas 2.1-2.3, Lemma 3.3 and Lemma 3.5, we obtain the values of $R(C_m, C_4, C_4)$ for $m \geq 3$, as given in Table 1.1. So, we have

Theorem 3.6.

$$R(C_m, C_4, C_4) = \begin{cases} 11, & m = 4, \\ 12, & m = 3, 5, 6, 7, 8, \\ 13, & m = 9, 10, \\ m + 2, & m \geq 11. \end{cases}$$

□

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