Some Properties of Prime Graphs

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Abstract

In this paper, we give some necessary conditions for a prime graph. We also present some new families of prime graphs such as $K_n \odot K_1$ is prime if and only if $n \leq 7$, $K_n \odot \overline{K_2}$ is prime if and only if $n \leq 16$ and $K_m \bigcup S_n$ is prime if and only if $\pi(m+n-1) \geq m$. We also show that a prime graph of order greater than or equal to 20 has a nonprime complement.

1. Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notations and terminology of graph theory as in [1] and [4].

A graph G of order n is said to have a prime labeling (or simply G is called prime) if there is an injection

$$f : V(G) \to \{1, 2, ..., n\}$$

such that for each edge $xy \in E(G)$, $f(x)$ and $f(y)$ are relatively prime.

The notion of a prime labeling originated by Entringer and was introduced in a paper by Tout, Dabboucy and Howalla [9] (see also [4] , [5]) . Entringer conjectured that all trees have a prime labelings. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders and all trees of order less than 35 (see [4]). Other graphs with prime labelings include all cycles and the disjoint union of C_{2k} and C_n [3]

The complete graph K_n does not have a prime labeling for $n \geq 4$ and W_n is prime if and only if n is even (see [5]).

Seoud, Diab and Elsakhawi [7] have shown the following graphs are prime: fans, helms, flowers, stars, $K_{2,n}$ and $K_{3,n}$ unless $n = 3$ or 7. They also shown that $P_n + \overline{K_m}$ $(m \geq 3)$ is not prime. Seoud and Youssef [8] proved the following graphs are not prime $C_m + C_n$, C_n^2 for $n \geq 4$, P_n^2 for $n = 6$ and for $n \geq 8$. They also give an exact formula for the maximum number of edges in a prime graph of order n and give an upper bound for the chromatic number of a prime graph. For more details of known results of prime graphs, see [4].

Recall that the union $G \bigcup H$ of two disjoint graphs G and H is the graph having vertex set $V(G) \bigcup V(H)$ while the corona $G \odot H$ of G and H is the graph obtained by taking one copy of G (which has n_1 vertices) and n_1 copies of G_2, and then joining the $i\ th$ vertex of G_1 to every vertex in the $i\ th$ copy of G_2.

The chromatic number $\chi(G)$ of a graph G is the least number of colours needed to colour the vertices of G so that no two adjacent vertices receive the same colour.

The clique number $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. If $K_n \subseteq G$, for some n, then $\chi(G) \geq n$, so in general $\chi(G) \geq \omega(G)$. The maximum (resp. minimum) of the vertex degrees of a graph G is called the maximum (resp. minimum) degree of G and is denoted by $\Delta(G)$ (resp. $\delta(G)$).

In section 2, we give some necessary conditions for a prime graph and we show that a prime graph of order greater than or equal 20 has a non-prime complement. In section 3, we give some new familes of prime disconnected graphs.

2. Some properties of prime graphs

In this section, we follow the basic notation and terminology of number theory as in [6]. In particular, we let p_r be the r th prime number, where $p_1 = 2$, $\pi(n)$ is the number of primes less than or equal to n and Euler's $\phi-$ function, $\phi(n)$, defined as the number of positive integers less than or equal to n that are relatively prime to n .

We denote by (a,b) the greatest common divisor of the integers a and b .

As in [8], a graph G of order n is prime if and only if G is isomorphic to a spanning subgraph of the graph R_n of order n with vertex set $V(R_n) = \{v_1, v_2, \ldots, v_n\}$ and whose edge set is defined as $E(R_n) = \{v_i v_j : (i,j) = 1\}$.

A graph R_n is called the maximal prime graph and $|E(R_n)|$ is the maximum number of edges in a prime graph of order n , Seoud and Youssef [8] showed that if G is a prime graph of order n , then

(i) $|E(G)| \le \sum_{i=1}^{n} \phi(i) - 1$,

(ii) $\chi(G) \le \pi(n) + 1$,

(iii) $\beta(G) \ge \left\lfloor \frac{n}{2} \right\rfloor$, where $\beta(G)$ is the vertex independence number of G .

In the following theorem, we study some basic properties of the maximal prime graph R_n .

Theorem 2.1

(a) $\omega(R_n) = \pi(n) + 1$

(b) $\delta(R_n) = n + \sum_{s=1}^{m}(-1)^s \sum_{1\le j_1<\ldots<j_s\le m} \left\lfloor \frac{n}{p_{j_1}\cdots p_{j_s}} \right\rfloor$

where $p_1 < p_2 < \ldots < p_{\pi(n)}$ is the list of all prime numbers $\le n$ and $m = \max\{1 \le k \le \pi(n) : p_1 \ldots p_k \le n\}$

(c) $\Delta(R_n) = n-1$ and

$$|\{v \in V(R_n) : \deg(v) = \Delta(R_n)\}| = \pi(n) - \pi\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$

Proof

(a) Let $p_1 < \ldots < p_{\pi(n)}$ be the list of all prime numbers $\le n$, then the induced subgraph of R_n generated by $U = \{v_i \in V(R_n) : i = 1 \text{ or } i = p_j \text{ for some } 1 \le j \le \pi(n)\}$ is the graph $K_{\pi(n)+1}$, hence $\omega(R_n) \ge \pi(n) + 1$. Conversely, for $1 \le i \le \pi(n)$ define,

$$V_i = \{v_j \in V(R_n) : p_i \mid j\}, \text{ then}$$

$$V(R_n) = \{v_1\} \cup \bigcup_{i=1}^{\pi(n)} V_i$$

$$= \{v_1\} \cup \bigcup_{i=1}^{\pi(n)} (V_i \setminus \bigcup_{j<i} V_j)$$

is a partition of $V(R_n)$ into independent sets and let $K_{\omega(R_n)}$ be a maximal complete subgraph of R_n, then for $1 \le i \le \pi(n)$, $|V(K_{\omega(R_n)}) \cap (V_i \setminus \bigcup_{j<i} V_j)| \le 1$, hence $|V(K_{\omega(R_n)})| \le \pi(n) + 1$, that is, $\omega(R_n) \le \pi(n) + 1$.

(b) For $1 \le j \le \pi(n)$, let $B_j = \{1 \le k \le n : p_j \mid k\}$. We have

$$\delta(R_n) = \min_{1<i\le n} |\{1 \le j \le n : (i, j) = 1\}|$$

$$= n - \max_{1<i\le n} |\bigcup_{p_j \mid i} B_j|$$

$$= n - \max \{ |\bigcup_{s=1}^{k} B_{j_s}| : p_{j_1} \cdots p_{j_k} \le n,\ 1 \le j_1 < j_2 < \ldots < j_k \le \pi(n),\ k \ge 1 \} .$$

We show that if $p_{j_1} \cdots p_{j_k} \le n$, $1 \le j_1 < j_2 < \ldots < j_k \le \pi(n)$, $k \ge 1$, then

$$|\bigcup_{s=1}^{k} B_{j_s}| \le |\bigcup_{j=1}^{k} B_j| .$$

For this purpose, we may assume that $j_k > k$ and choose $i \in \{1, 2, \ldots, j_k\} \setminus \{j_1, j_2, \ldots, j_k\}$, then the function

$$g : B_{j_k} \setminus \bigcup_{s=1}^{k-1} B_{j_s} \longrightarrow B_i \setminus \bigcup_{s=1}^{k-1} B_{j_s}$$

defined by

$$g(q) = p_i \frac{q}{p_{j_k}}$$

is injective and hence

$$\begin{aligned} |\bigcup_{s=1}^{k} B_{j_s}| &= |\bigcup_{s=1}^{k-1} B_{j_s} \cup (B_{j_k} \setminus \bigcup_{s=1}^{k-1} B_{j_s})| \\ &\le |\bigcup_{s=1}^{k-1} B_{j_s}| + |(B_i \setminus \bigcup_{s=1}^{k-1} B_{j_s}| \\ &= |\bigcup_{s=1}^{k-1} B_{j_s} \cup (B_i \setminus \bigcup_{s=1}^{k-1} B_{j_s})| \\ &= |\bigcup_{s=1}^{k-1} B_{j_s} \cup B_i| \end{aligned}$$

and our assertion follows by induction on j_k. Then we get $\delta(R_n) = n - |\bigcup_{j=1}^{m} B_j|$, where

$$m = \max \{ 1 \le k \le \pi(n) : p_1 \dots p_k \le n \}$$

and the desired formula follows from the inclusion-exclusion principle [2, p. 177].

(c) Since, for every $v_k \in V(R_n)$, $1 \le k \le n$, $\deg(v_k) = n-1$ if and only if k = 1 or k is prime such that $\left\lfloor \frac{n}{2} \right\rfloor < k \le n$.

Then $\Delta(R_n) = n-1$ and

$$|\{ v \in V(R_n) : \deg(v) = \Delta(R_n) \}| = \pi(n) - \pi(\left\lfloor \frac{n}{2} \right\rfloor) + 1 . \square$$

The following corollary gives some necessary conditions for a prime graph.

Corollary 2.2

If G is a prime graph of order n, then,

(a) $\omega(G) \le \pi(n) + 1$

(b) $\delta(G) \le \delta(R_n)$ and

$$|\{v \in V(G) : \deg(v) = n-1\}| \le \pi(n) - \pi(\left\lfloor \frac{n}{2} \right\rfloor) + 1 .$$

Proof.

If G is a prime graph of order n, then G is a spanning subgraph of R_n.

(a) Since $G \subseteq R_n$, then $\omega(G) \le \omega(R_n) = \pi(n) + 1$.

(b) Since $G \subseteq R_n$, then $\delta(G) \le \delta(R_n)$ and

$$|\{v \in V(G) : \deg(v) = n-1\}| \le |\{v \in V(R_n) : \deg(v) = n-1\}|$$
$$= \pi(n) - \pi(\left\lfloor \frac{n}{2} \right\rfloor) + 1 . \square$$

By using the necessary conditions for a prime graph, mentioned in Corollary 2.2, we prove the following results.

Theorem 2.3

$K_n \odot K_1$ is prime if and only if $n \le 7$.

Proof.

If $n \ge 8$, then $\pi(2n) \le n-2$ and since $\omega(K_n \odot K_1) = n$, then $\omega(K_n \odot K_1) > \pi(2n) + 1$, hence by Corollary 2.2, $K_n \odot K_1$ is not prime.

Conversely, if $n \le 7$, then $K_1 \odot K_1 = P_2$ and $K_2 \odot K_1 = P_4$ are trivially prime and $K_3 \odot K_1$ is simply prime. For $4 \le n \le 7$, let

$V(K_n \odot K_1) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ where

$E(K_n \odot K_1) = \{u_i u_j : 1 \le i < j \le n\} \cup \{u_i v_i : 1 \le i \le n\}$, then $(K_n \odot K_1)$, $4 \le n \le 7$, are prime with the following prime labeling functions :

$f : V(K_4 \odot K_1) \longrightarrow \{1, 2, \ldots, 8\}$

$f(u_i) = 2i - 1$, $1 \le i \le 4$

$f(v_i) = 2i$, $1 \le i \le 4$

$f : V(K_5 \odot K_1) \longrightarrow \{1, 2, \ldots, 10\}$

$f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(u_4) = 5$, $f(u_5) = 7$,

$f(v_1) = 10$, $f(v_2) = 9$, $f(v_3) = 4$, $f(v_4) = 6$ and $f(v_5) = 8$

$f : V(K_6 \odot K_1) \longrightarrow \{1, 2, \ldots, 12\}$

$f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(u_4) = 5$, $f(u_5) = 7$,

$f(u_6) = 11$, $f(v_1) = 10$, $f(v_2) = 9$, $f(v_3) = 4$,

$f(v_4) = 6$, $f(v_5) = 8$ and $f(v_6) = 12$

$f : V(K_7 \odot K_1) \longrightarrow \{1, 2, \ldots, 14\}$

$f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(u_4) = 5$, $f(u_5) = 7$,

$f(u_6) = 11$, $f(u_7) = 13$, $f(v_1) = 10$, $f(v_2) = 9$, $f(v_3) = 4$,

$f(v_4) = 6$, $f(v_5) = 8$, $f(v_6) = 12$ and $f(v_7) = 14$. □

In the following theorem we show that every prime graph of order 16 or 18 or 20 or greater than or equal to 21, has a non prime complement.

Theorem 2.4

$K_n \odot \overline{K_2}$ is prime if and only if $n \le 16$.

Proof

If $n \ge 17$, then $\pi(3n) \le n-2$ and since $\omega(K_n \odot \overline{K_2}) = n$, then $\omega(K_n \odot \overline{K_2}) > \pi(3n)+1$, hence by Corollary $2\cdot 2$, $K_n \odot \overline{K_2}$ is not prime. Conversely, if $n \le 16$, let
$V(K_n \odot \overline{K_2}) = \{u_i : 1 \le i \le n\} \cup \{v_{1i}, v_{2i} : 1 \le i \le n\}$ where
$E(K_n \odot \overline{K_2}) = \{u_i u_j : 1 \le i < j \le n\} \cup \{u_i v_{1i} : 1 \le i \le n\} \cup \{u_i v_{2i} : 1 \le i \le n\}$.

Simply, we may write the vertices of $K_n \odot \overline{K_2}$ in triples (u_i, v_{1i}, v_{2i}), $1 \le i \le n$ and their labelings written also in triples. For $1 \le n \le 11$, let us define a labeling function as follows:

$$(1,2,3),(5,4,6),(7,8,9),(11,10,12),(13,14,15),$$
$$(17,16,18),(19,20,21),(23,22,24),(27,26,25),(29,30,28),$$
$$(31.32.33)$$

It is clear that this labeling gives the prime labeling for all $1 \le n \le 11$ by removing each time the vertices of greater labeling. For $n = 12$. The following is a prime labeling:

$$(1,4,6),(2,9,15),(3,8,10),(5,12,14),(7,16,18),(11,20,21),$$
$$(13,22,24),(17,25,26),(19,27,28),(23,30,32)$$
$$(29,33,34),(31,35,36)$$

For $13 \le n \le 16$, we add for $K_{12} \odot \overline{K_2}$ the following triples $(37,38,39),(41,40,42),(43,44,45),(47,46,48)$. This completes the proof. □

Theorem 2.5

If G is a prime graph of order n, then $\overline{G}$ is not prime if

$n \geq 21$ or n even ≥ 16 .

Proof.

Let G be a prime graph of order n, then $\beta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$. If $n \geq 21$, then $\omega(\overline{G}) \geq \left\lfloor \frac{n}{2} \right\rfloor > \left\lceil \frac{n}{2} \right\rceil - 2 \geq \pi(n)+1$ that is, $\omega(\overline{G}) > \pi(n)+1$. If n even ≥ 16, then $\omega(\overline{G}) \geq \left\lfloor \frac{n}{2} \right\rfloor > \left\lceil \frac{n}{2} \right\rceil - 1 \geq \pi(n)+1$ that is, $\omega(\overline{G}) > \pi(n)+1$. Hence by Corollary 2.2, $\overline{G}$ is not prime. □

Note that we may find graphs such that neither these graphs nor its complement is prime. For example, $K_{n,n}$, $n \geq 3$. Also we may find graphs in which these graphs are not prime but its complement are. For example, K_n , $n \geq 4$.

Corollary 2.6

If G is a prime graph of order n, $n \geq 4$, such that $\beta(G) > \left\lceil \frac{n}{2} \right\rceil + 1$, then $\overline{G}$ is not prime.

Proof.

Since $\beta(G) > \left\lceil \frac{n}{2} \right\rceil + 1$, then $\omega(\overline{G}) > \left\lceil \frac{n}{2} \right\rceil + 1 \geq \pi(n)+1$ for every $n \geq 4$. Hence $\overline{G}$ is not prime by Corollary 2.2 . □

3. Families of prime disconnected graphs.

The following results concern with the prime labelings of some disconnected graphs.

Theorem 3.1

(a) $C_m \cup S_n$ is prime for all $m \geq 3$ and $n \geq 1$

(b) $S_m \cup S_n$ is prime for all $m, n \geq 1$.

Proof.

(a) If m is even, then label the vertices of the cycle C_m consecutively with the labels 2, 3, . . ., m+1, and then label the vertices of the star S_n with the labels 1, m+2, m+3, . . ., m+n+1, where 1 is the label of the center of the star. If n is odd, then label the vertices of the cycle C_m consecutively with the labels 2, 3, . . ., m, m+2, and then label the vertices of the star S_n with the labels 1, m+1, m+3, m+4, . . ., m+n+1, where 1 is the label of the center of the star.

(b) Let $V(S_m) = \{u_0, u_1, \ldots, u_m\}$ and $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ where u_0 and v_0 are the centers of the stars S_m and S_n respectively. Suppose that $1 \leq m \leq n$ and define a labeling function ,

$$f : V(S_m \cup S_n) \longrightarrow \{1, 2, \ldots, m+n+2\}$$

as follows

$$f(u_0) = 2 , \qquad f(v_0) = 1$$

$$f(u_i) = 2i+1 \quad , \quad 1 \leq i \leq m$$

$$f(v_j) = \begin{cases} 2(j+1) & , \quad 1 \leq j \leq m \\ m+2+j & , \quad m < j \leq n \end{cases}$$

then f is bijection and the graph is prime. □

Theorem 3.2

$K_m \cup P_n$ is prime if and only if ($1 \leq m \leq 3$ and $n \geq 1$) or($m = 4$ and n is odd ≥ 1).

Proof.

If $K_m \cup P_n$ is prime, then, $\beta(K_m \cup P_n) \geq \left\lfloor \frac{m+n}{2} \right\rfloor$, that is $\left\lceil \frac{n}{2} \right\rceil + 1 \geq \left\lfloor \frac{m+n}{2} \right\rfloor$ and this implies that $1 \leq m \leq 3$ and $n \geq 1$, or $m = 4$ and n is odd ≥ 1.

Conversely, if $1 \leq m \leq 3$ and $n \geq 1$, then label K_m with 1, 2, . . ., m and label the vertices of the path P_n consecutively with the labels m+1, m+2, . . ., m+n, and if m = 4 and n is odd ≥ 1, then label the vertices of K_4 with the labels 1, 2, 3, 5 and then label the vertices of P_n consecutively with the labels 6, 7, . . ., n+4, 4. □

Theorem 3.3

For m, n ≥ 1, $K_m \cup S_n$ is prime if and only if $\pi(m+n+1) \geq m$

Proof.

$\Rightarrow$ Let $G = K_m \cup S_n$ is prime, then G is a spanning subgraph of R_{m+n+1} and by Corollary 2.2, we have $m \leq \omega(G) \leq \omega(R_{m+n+1}) = \pi(m+n+1) + 1$, so that $\pi(m+n+1) \geq m-1$. We show that $\pi(m+n+1) \neq m-1$. Suppose that $\pi(m+n+1) = m-1$, then $\omega(G) = \omega(R_{m+n+1}) = m$ and we may assume that the vertices of K_m receive the labels 1, $p_1, p_2, \ldots, p_{m-1}$, where $p_1, p_2, \ldots, p_{m-1}$ are the first (m−1) prime numbers less than or equal to m+n+1, hence the center vertex of S_n receive a composite vertex label and we may assume that p_i be the smallest prime divisor of the vertex label of the center vertex of S_n, for some $1 \leq i \leq m-1$, then we have $2p_i$ or $3p_i$ is a vertex label of a pendent vertex of S_n, which is a contradiction.

$\Leftarrow$ Let $\pi(m+n+1) \geq m$ and $p_1 < p_2 \ldots < p_m$ be the first m prime numbers less than or equal to m+n+1, then label the vertices of K_m with the labels $p_1, p_2, \ldots, p_m$ and label the vertices of S_n with the labels $\{1, 2, \ldots, m+n+1\} \setminus \{p_1, p_2, \ldots, p_m\}$ where 1 is the label of the center vertex of S_n, hence $K_m \cup S_n$ is prime. □

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