Subdivision of Edges and Matching Size

D. Bauer [1]

E. Schmeichel [2]

T. Surowiec [1]

[1] *Department of Mathematical Sciences*
Stevens Institute of Technology
Hoboken, NJ 07030, U.S.A.

[2] *Department of Mathematics*
San Jose State University
San Jose, CA 95192, U.S.A.

February 28, 2007

Abstract

The well-known formula of Tutte and Berge expresses the size of a maximum matching in a graph G in terms of the deficiency $\max_{X\subseteq V(G)}\{\omega_0(G-X)-|X|\}$ of G, where $\omega_0(H)$ denotes the number of odd components of H. Let G' be the graph formed from G by subdividing (possibly repeatedly) a number of its edges. In this note we study the effect such subdivisions have on the difference between the size of a maximum matching in G and the size of a maximum matching in G'.

Keywords: maximum matchings, Tutte sets

AMS Subject Classifications (2004): 05C70

1 Introduction and Preliminary Results

In this note we consider only simple graphs. Our terminology will be standard. A good reference for any undefined terms is [7].

Given a graph G, let $\omega(G)$ (resp., $\omega_0(G)$) denote the number of components (resp., odd components) of G. An important result in matching theory is due to Tutte [6].

Theorem 1.1 *(Tutte's Theorem) A graph G has a perfect matching if and only if $\omega_0(G - X) \leq |X|$ for all $X \subseteq V(G)$.*

In 1958, Berge [3] extended Tutte's Theorem to give the exact size of a maximum matching in a graph G. Define the **deficiency** of G, denoted $\text{def}(G)$, by $\max_{X \subseteq V(G)}\{\omega_0(G - X) - |X|\}$.

It can be shown that $\text{def}(G)$ is the number of vertices unmatched by a maximum matching in G, and thus we have the following.

Theorem 1.2 *(Berge-Tutte formula) The maximum size of a matching in a graph G is $\frac{|V(G)| - def(G)}{2}$.*

Since 1958, matching theory has received considerable attention. Much of this work is described in [4].

Motivated by the formula in Theorem 1.2, we define a **Tutte set** in G as a set $X \subseteq V(G)$ such that $\omega_0(G-X) - |X| = \text{def}(G)$. These sets are referred to as **barriers** in [4]. We denote by T_G the set of all Tutte sets in a graph G. A theoretical and a computational study of maximal Tutte sets appears in [1, 2].

We seek to gain some insight into the following question. Suppose G is a graph with a maximum matching M. Let G' be the graph formed from G by subdividing (possibly repeatedly) a number of its edges. How does the size of a maximum matching M' in G' compare to the size of M?

Let Q be the set of degree two vertices that are added to $V(G)$ by subdividing the edges of G. We say that these vertices are **inserted** into the edges of G to form G'. Let $q = |Q|$, $m = |M|$, and $m' = |M'|$. Of course, $m' \geq m$. However, what can be said about $m' - m$? Indeed, how large must q be to insure that $m' > m$? This leads to the following.

Let G be a graph with maximum matching of size m. We define $f(G)$ to be the maximum number of degree two vertices that can be inserted into the edges of G to form a graph G' with $m = m'$.

In Section 2 we investigate these questions for regular graphs. If G is k-regular with edge-connectivity $\lambda = k - 1$, we determine $m^{'} - m$, and show that our result is best possible. For regular graphs with smaller edge-connectivity, we find $f(G)$. In Section 3 we consider general graphs, i.e., graphs that are not necessarily regular, and show that $f(G)$ is closely related to the maximum number of edges in a Tutte set of G. We conclude in Section 4 with some open questions.

2 Regular Graphs

We begin by studying regular graphs. The following result of Petersen is well-known [5].

Theorem 2.1 *Let G be a 3-regular, 2-edge-connected graph on n vertices. Then G has a perfect matching.*

This can easily be generalized to k-regular graphs, for $k \geq 3$.

Theorem 2.2 *Let G be a k-regular, $(k-1)$-edge-connected graph on n vertices, where n is even. Then G has a perfect matching.*

Note that if G is a 3-regular, 2-edge-connected graph on n vertices, it is possible to form $G^{'}$ by inserting $q \geq 6$ vertices into the edges of G without $G^{'}$ having a perfect matching. Simply let $G = K_4$ and insert a vertex into each edge. However if four degree 2 vertices are inserted into any 3-regular, 2-edge-connected graph, the resulting graph will have a perfect matching. This is a special case of the following result.

We define

$$g(n,q,k) = \begin{cases} \frac{n}{2}, & n \text{ even, } q = 1 \\ \lfloor \frac{n}{2} \rfloor + \frac{q}{k}, & \text{otherwise.} \end{cases}$$

Theorem 2.3 *Let $k \geq 3$, and G be a k-regular, $(k-1)$-edge-connected graph on $n \geq 4$ vertices. Form G' by inserting q vertices of degree two into the edges of G. Then G' has a matching M' with $m^{'} \geq g(n,q,k)$.*

PROOF: Let S' be a smallest Tutte set for G' with deficiency $d^{'} = \mathrm{def}(G^{'}) = \omega_0(G' - S') - |S'|$. We let $n' = n + q$ and $t^{'} = \omega_0(G^{'} - S^{'})$.

Case 1: $S' = \emptyset$.

If $S' = \emptyset$, then

$$d' = \begin{cases} 0, & \text{if } n' \text{ is even} \\ 1, & \text{if } n' \text{ is odd} \end{cases}$$

and by Berge's Theorem,

$$m' = \frac{n' - d'}{2} = \lfloor \frac{n+q}{2} \rfloor \geq g(n,q,k).$$

Case 2: $S' \neq \emptyset$.

We first show that $m' \geq \frac{n}{2} + \frac{q}{k}$.

Suppose otherwise. Then

$m' = \frac{n' - d'}{2} = \frac{n+q-d'}{2} < \frac{n}{2} + \frac{q}{k}$, and thus $d' = t' - |S'| > q - \frac{2q}{k}$.

Hence

$$|S'| < t' - q + \frac{2q}{k}. \tag{1}$$

Now let $H_i, 1 \leq i \leq t'$, be the odd components of $G' - S'$, and let Q be the set of q inserted vertices of degree two. The t' odd components H_i of $G' - S'$ can be partitioned into three categories:

1. $|H_i| > |H_i \cap Q|$ and $|H_i \cap Q|$ odd;
2. $|H_i| > |H_i \cap Q|$ and $|H_i \cap Q|$ even;
3. $|H_i| = |H_i \cap Q|$.

Let $H_1, \ldots, H_p$ be the odd components of the first type, $H_{p+1}, \ldots, H_{p+y}$ be the odd components of the second type, and $H_{p+y+1}, \ldots, H_{t'}$ be the odd components of the third type. Of course, p and/or y may be zero.

We will henceforth assume that n is *even* and k is *odd*; The other cases are proved similarly.

We begin by noting that $S' \cap Q = \emptyset$. Otherwise, if $u \in S' \cap Q$, then $S' - u$ is a smaller Tutte set for G'.

We next establish the following. Let $E(A, B)$ denote the set of edges that join a vertex of A to a vertex of B, and $e(A, B) = |E(A, B)|$; $E(\langle A \rangle)$ and $e(\langle A \rangle)$ are analogously defined.

Claim:

$$e(H_i, S') \geq \begin{cases} \text{k-1}, & \text{if } 1 \leq i \leq p+y \\ 2, & \text{if } p+y+1 \leq i \leq t'. \end{cases}$$

PROOF OF CLAIM: Suppose $1 \leq i \leq p+y$ and consider H_i. Since $S' \cap Q = \emptyset$, let $S = S'$ and note that $H_i - Q$ induces a component of $G - S$. Since G is $(k-1)$-edge-connected, $e(H_i, S) \geq k-1$ in G. If $l \in E_G(H_i, S)$ was not subdivided in G to form G', then $l \in E_{G'}(H_i, S)$. Otherwise, if $l = r_i s_i$, where $r_i \in H_i, s_i \in S$, was subdivided by a vertex $q_i \in Q$, then $q_i s_i \in E_{G'}(H_i, S)$. Hence $e_{G'}(H_i, S') = e_G(H_i, S) \geq k-1$.

Finally, since G' is 2-edge-connected, $e(H_i, S') \geq 2$ if $p+y+1 \leq i \leq t'$.

This proves the claim.□

Hence $e(H_i, S') \geq k-1$ for $1 \leq i \leq p+y$. However if $e(H_i, S') = k-1$ for $p+1 \leq i \leq p+y$, then $\sum_{v_i \in H_i} deg_{H_i}(v_i)$ is odd, a contradiction. Hence $e(H_i, S') \geq k$ for $p+1 \leq i \leq p+y$. Thus, since G is k-regular, we conclude

$$k|S'| \geq e(G' - S', S') \geq (k-1)p + ky + 2(t' - y - p), \tag{2}$$

and from (1) and (2) we get

$$(k-2)(p+y+q) < (k-2)t' + p. \tag{3}$$

Now let $\tilde{Q} = \cup_{i=p+y+1}^{t'}(Q \cap H_i) = \cup_{i=p+y+1}^{t'} H_i$ and $|\tilde{Q}| = \tilde{q} \leq q$.

Also, let $\tilde{t} = t' - p - y$. Then by (3),

$$(k-2)(q - \tilde{t}) < p. \tag{4}$$

However $k \geq 3$, and since each $H_i, 1 \leq i \leq p$, contains at least one vertex in Q, $q - \tilde{t} \geq q - \tilde{q} \geq p$, contradicting (4).

Hence

$$m' \geq \frac{n}{2} + \frac{q}{k}.$$

It is now easy to show that

$$\frac{n}{2} + \frac{q}{k} \geq g(n, q, k),$$

completing the proof. □

Theorem 2.3 is easily seen to be best possible. For $n \geq 4$, let $G = K_n$ and form G' by inserting a single vertex of degree two into each edge of G. Since $m' = n$, and since $k = n - 1$ and $q = \frac{n(n-1)}{2}$, we have $m' = \lceil g(n, q, k) \rceil$.

The Corollary below follows easily.

Corollary 2.4 *Let $k \geq 3$, and G be a k-regular, $(k-1)$-edge-connected graph on $n \geq 4$ vertices. Then*

$$f(G) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

We next consider regular graphs with smaller edge-connectivity. As noted in Theorem 2.2, if G is a k-regular, $(k-1)$-edge-connected graph on n vertices, where n is even, then G has a perfect matching. If G is a k-regular graph with edge connectivity $\lambda \leq k - 2$, however, there is no simple formula to determine the size of a maximum matching in G.

Suppose we have a k-regular graph G with $\lambda \leq k-2$, and we again form G' from G by subdividing (possibly repeatedly) a number of its edges by inserting q vertices of degree two. How does the size of a maximum matching m' in G' compare to the size of m as a function of q? Indeed, what is $f(G)$? We have an answer to the latter question.

Let

$$x = \begin{cases} \dfrac{n}{2\left(\dfrac{k+3}{(k-\lambda-1)}\right)}, & k \text{ odd}, \lambda \text{ even} \\[2ex] \dfrac{n}{2\left(\dfrac{k+2}{(k-\lambda-1)}\right)}, & k \text{ even}, \lambda \text{ odd} \\[2ex] \dfrac{n}{2\left(\dfrac{k+3}{k-\lambda}\right)}, & k \text{ odd}, \lambda \text{ odd} \\[2ex] \dfrac{n}{2\left(\dfrac{k+2}{k-\lambda}\right)}, & k \text{ even}, \lambda \text{ even.} \end{cases}$$

Theorem 2.5 *Let $k \geq 3$, and G be a k-regular, λ-edge-connected graph on $n \geq 4$ vertices, where $\lambda \leq k-2$. Then $f(G) = x$.*

PROOF: We first show that $f(G) \leq x$.

We will assume that k and λ are both odd; analogous arguments can be used in the other three cases. Let Q be a set of degree two vertices inserted into the edges of G to form G', where $|Q| = q$. We show that if $q > x$, then $m' > m$.

Suppose otherwise, i.e., that $q > x$ and $m' = m$.

Let S' be a smallest Tutte set for G' with deficiency $d' = t' - |S'|$, where $t' = \omega_0(G' - S')$.

We partition the t' odd components of $G' - S'$ into three categories. Let $t' = p + y + \tilde{t}$, and

1. P_j be those odd components with $|P_j| > |P_j \cap Q|$ and $|P_j \cap Q|$ odd, $1 \leq j \leq p$.
2. Y_j be those odd components with $|Y_j| > |Y_j \cap Q|$ and $|Y_j \cap Q|$ even, $1 \leq j \leq y$.
3. $\tilde{T}_j$ be those odd components with $|\tilde{T}_j| = |\tilde{T}_j \cap Q|$, $1 \leq j \leq \tilde{t}$.

Note that $S' \cap Q = \emptyset$. Otherwise, if $u \in S' \cap Q$, then $S' - u$ is a smaller Tutte set for G'.

Now let $S = S'$ and consider $G - S$. Clearly $d \geq y - |S|$, where d denotes the deficiency of G. By Theorem 1.2,

$$m' = \frac{n + q - d'}{2}, \quad m = \frac{n - d}{2}.$$

Let $d = e + y - |S|$. Then $m' > m \Leftrightarrow q > d' - d = p + \tilde{t} - e$. However each P_i component, $1 \leq i \leq p$, and each $\tilde{T}_i$ component, $1 \leq i \leq \tilde{t}$, contain at least one vertex in Q. Hence $q \geq p + \tilde{t}$. Since $m' = m$, it follows that $e = 0$ and $q = p + \tilde{t}$. Thus each component P_i, $1 \leq i \leq p$, and $\tilde{T}_i, 1 \leq i \leq \tilde{t}$, contains exactly one vertex in Q. In addition, $e = 0$ and $d \geq 0$ imply $y \geq |S|$.

We now let

- p_{2i} be the number of components of $G - S$ containing $2i$ vertices, $1 \leq i \leq (k-1)/2$;
- p_{k+1} be the number of components of $G - S$ containing at least $k + 1$ vertices;
- y_{2i+1} be the number of components of $G - S$ containing $2i+1$ vertices, $0 \leq i \leq (k-1)/2$;

- y_{k+2} be the number of components of $G-S$ containing at least $k+2$ vertices.

Note that $p = \sum_{i=1}^{(k+1)/2} p_{2i}$ and $y = \sum_{i=0}^{(k+1)/2} y_{2i+1}$.

We now consider $E(G^{'} - S, S)$. Since each vertex in S has degree k,

$$\begin{aligned} k|S| &\geq e(G^{'} - S, S) \geq \sum_{j=1}^{(k-1)/2} 2j(k-(2j-1))p_{2j} \\ &+ \sum_{j=0}^{(k-1)/2} (2j+1)(k-2j)y_{2j+1} + (\lambda+1)p_{k+1} + \lambda y_{k+2} + 2\tilde{t}. \quad (5) \end{aligned}$$

Since G is λ-edge-connected, the coefficient of y_{k+2} is λ. The coefficient of p_{k+1} is $\lambda+1$ due to a simple parity argument.
Since $y \geq |S|$, $ky \geq k|S|$, and thus

$$\begin{aligned} (k-\lambda)y_{k+2} &\geq \sum_{j=1}^{(k-1)/2} 2j(k-(2j-1))p_{2j} + \sum_{j=0}^{(k-1)/2} 2j(k-(2j+1))y_{2j+1} \\ &+ (\lambda+1)p_{k+1} + 2\tilde{t}. \quad (6) \end{aligned}$$

Clearly $n \geq \left(\sum_{j=1}^{(k+1)/2} 2jp_{2j} \right) + (k+2)y_{k+2} + |S|$. Using (6) and (5) we obtain lower bounds for y_{k+2} and $|S|$, respectively. Hence

$$n \geq \sum_{j=1}^{(k-1)/2} \eta_j p_{2j} + \eta_{k+1} p_{k+1} + \eta_{\tilde{t}} \tilde{t},$$

where

- $\eta_j = 2j + (\frac{k+2}{k+\lambda})2j(k-(2j-1)) + \frac{2j}{k}(k-(2j-1)) + \frac{\lambda}{k(k-\lambda)}2j(k-(2j-1))$
- $\eta_{k+1} = k+1+(\frac{k+2}{k-\lambda})(\lambda+1) + \frac{\lambda+1}{k} + \frac{\lambda(\lambda+1)}{k(k-\lambda)}$
- $\eta_{\tilde{t}} = \frac{2(k+2)}{k-\lambda} + \frac{2}{k} + \frac{2\lambda}{k(k-\lambda)}$.

Note that $2j(k-(2j-1)) \geq 4 > 2$ for $3 \leq k,\ 1 \leq j \leq \frac{k-1}{2}$. Hence

$$\eta_j \geq 2j + \frac{2(k+2)}{k-\lambda} + \frac{2}{k} + \frac{2\lambda}{k(k-\lambda)} \geq \eta_{\bar{t}}.$$

Also, $\lambda \geq 1$ implies $\eta_{k+1} \geq \eta_{\bar{t}}$. Since $\eta_{\bar{t}} = \frac{2(k+3)}{k-\lambda}$, it follows that

$$n \geq \frac{2(k+3)}{k-\lambda}\left(\bar{t} + \sum_{j=1}^{(k+1)/2} p_{2j}\right) \geq \frac{2(k+3)}{k-\lambda} q = \frac{nq}{x}.$$

Thus $q \leq x$, a contradiction. Hence $f(G) \leq x$.

We now construct examples to show that $f(G) \geq x$. For each k and λ, $1 \leq \lambda \leq k-2$, we describe a k-regular, λ-edge-connected graph $G_{k,\lambda}$ in the table below. In each case, $V(G_{k,\lambda}) = V(S) \cup V(Y)$, where S is a complete graph on $|S|$ vertices and Y is the disjoint union of $|S|$ identical graphs, each having the prescribed degree sequence in the table.

k , λ	$\lvert S \rvert$	Degree seq. of each Y component
odd , even	$k-\lambda$	$k^{k-\lambda+1}(k-1)^{\lambda+1}$
odd , odd	$k-\lambda+1$	$k^{k-\lambda+2}(k-1)^{\lambda}$
even , even	$k-\lambda+1$	$k^{k-\lambda+1}(k-1)^{\lambda}$
even , odd	$k-\lambda$	$k^{k-\lambda}(k-1)^{\lambda+1}$

It is clear from the degree sequence of each component in Y that a graph $G_{k,\lambda}$ exists with a perfect matching. If G' is formed from $G_{k,\lambda}$ by inserting a single degree two vertex into each edge of $\langle S \rangle$, it is easy to see that $m' = n/2$, and thus $f(G) \geq x$. □

3 Characterizing $f(G)$ via Tutte sets

We next consider the problem of characterizing $f(G)$ in general graphs, i.e., graphs that are not necessarily regular. Recall that a Tutte set in a graph G is a set $X \subseteq V(G)$ such that $\omega_0(G-X) - |X| = \text{def}(G)$, and that T_G denotes the set of all Tutte sets in G. Let $\mathbf{Z(G)}$ denote the maximum size of any Tutte set in a graph G, i.e., $Z(G) = max_{S \in T_G} e(\langle S \rangle)$.

Theorem 3.1 *Let G be a graph on n vertices. Then $f(G) = Z(G)$ unless all of the following hold;*

1. n is even

2. $\emptyset \in T_G$

3. $Z(G) = 0$,

in which case $f(G) = 1 = Z(G) + 1$.

PROOF: Let $S \in T_G$. If G' is obtained from G by inserting a degree two vertex into each edge of $\langle S \rangle$, $m' = m$. Hence $f(G) \geq Z(G)$. Thus, it suffices to show that $f(G) \leq Z(G)$.

Let $k = Z(G) \geq 0$, and suppose that $f(G) \geq k+1$. Form G' by inserting a set $Q = \{q_1, q_2, \cdots, q_{k+1}\}$ of degree two vertices into $k+1$ different edges of G in such a way that $m' = m$. Hence,

$$\frac{n + (k+1) - d'}{2} = \frac{n-d}{2}$$

and

$$d' = d + (k+1). \tag{7}$$

Now let $S' \in T_{G'}$. We may assume S' has been chosen such that

1. $S' \cap Q = \emptyset$.
   Otherwise, iteratively replace S' by $S' - u$, for any $u \in S' \cap Q$. Each time we again obtain a Tutte set for G'.
2. $S' \neq \emptyset$.
   By (7), $d' = d + (k+1)$, and so $d' \geq 2$ and hence $\phi \notin T_{G'}$ unless $d = 0$ and $k = 0$, i.e.,
   (a) n is even,
   (b) $\emptyset \in T_G$, and
   (c) $Z(G) = 0$.

Claim: $S' \in T_G$

PROOF OF CLAIM: Note that $G - S'$ contains at least $\omega_0(G' - S') - (k+1)$ odd components. Thus

$$\begin{aligned} d &= d' - (k+1) \\ &= \omega_0(G' - S') - |S'| - (k+1) \\ &\leq \omega_0(G - S') - |S'| \\ &\leq d, \end{aligned}$$

and so $d = \omega_0(G - S') - |S'|$. This proves the claim. □

Now consider the components of $G - S'$. Suppose $G - S'$ has p even components and t odd components. Since $d = t - |S'|$, by (7) we have $d' = t + (k+1) - |S'|$. Also, since S' is a Tutte set for G', $d' = \omega_0(G' - S') - |S'|$. Now let y denote the number singelton components of $G' - S'$, i.e., the number of vertices of Q inserted into the edges of $\langle S' \rangle$. Clearly

$$e(\langle S' \rangle) \geq y. \tag{8}$$

Also,

$$\begin{aligned} t + (k+1) - |S'| &= d' \\ &= \omega_0(G' - S') - |S'| \\ &\leq (t + p + y) - |S'|, \end{aligned}$$

and thus

$$p + y \geq k + 1. \tag{9}$$

Finally, since $S' \neq \emptyset$, we can form a new Tutte set S'' by removing one vertex from each of the p even components of $G - S'$ and placing them into S'. Now by (8) and (9),

$$e(\langle S'' \rangle) \geq e(\langle S' \rangle) + p \geq y + p \geq k + 1,$$

contradicting $Z(G) = k$. Thus $f(G) = Z(G)$.

Finally, suppose each of the following hold:

1. n is even
2. $\emptyset \in T_G$
3. $Z(G) = 0$.

Since n is even and G has a perfect matching, inserting a single degree two vertex into an edge of G will not result in a graph with a larger matching. Hence $f(G) \geq 1$.

Suppose $f(G) \geq 2$. Then we can add two degree two vertices to distinct edges of G to form G' with $m = m'$. Thus by (5), $d' = d + 2 = 2$. We again have $S' \neq \emptyset$, and similar arguments lead to a contradiction. Thus $f(G) = 1$, completing the proof. □

4 Further results and open questions

We begin with two simple results concerning the structure of Tutte sets in k-regular, $(k-1)$-edge-connected graphs. Let G be such a graph on $n \geq 3$ vertices. If n is even, Theorem 3.1 and Corollary 2.4 imply that a Tutte set can have at most one edge. We can strengthen this as follows.

Theorem 4.1 *Let G be a k-regular, $(k-1)$-edge-connected graph on n vertices, with n even. Then every Tutte set of G is independent, i.e., $Z(G) = 0$.*

PROOF: Suppose otherwise, i.e., let S be a Tutte set for G with $e(\langle S \rangle) > 0$. Then $e(S, G-S) < k|S|$. Since $G-S$ has exactly $|S| + \text{def}(G) \geq |S|$ odd components, at least one of them, say H, must satisfy $e(H,S) \leq k-1$. Since G is $(k-1)$-edge-connected, $e(H,S) = k-1$. Thus $\sum_{v \in H} deg_H(v) = \sum_{v \in H} deg_G(v) - e(H,S) = k|H| - (k-1)$. Since $|H|$ is odd, $\sum_{v \in H} deg_H\ v$ is odd regardless of the parity of k, a contradiction. □

If n is odd, Theorem 3.1 and Corollary 2.4 already yield that $f(G) = Z(G) = 0$, and thus all Tutte sets in G are independent. But once again, this can be strengthened. We call a graph G **factor-critical** if $G - v$ has a perfect matching for every $v \in G$. It is known (cf. [7, Ex. 3.1.10]) that G is factor-critical if and only if the only Tutte set in G is $\emptyset$.

Theorem 4.2 *Let G be a k-regular, $(k-1)$-edge-connected graph on n vertices, with n odd. Then G is factor-critical.*

PROOF: It suffices to show that the only Tutte set of G is $\emptyset$. Suppose otherwise, and let S be a nonempty Tutte set of G. Since n is odd, $\text{def}(G) \geq 1$, and thus $\omega_0(G-S) > |S| \geq 1$. We conclude, as above, that $G-S$ has an odd component H satisfying $e(H,S) = k-1$, a contradiction. □

The converse of Theorem 4.1 is easily seen to be false by just examining diametrically opposite vertices in a 6-cycle. This suggests an interesting open problem. If G is a k-regular, $(k-1)$-edge-connected graph on n vertices, with n even, which independent sets in G are Tutte sets?

Another interesting question concerns k-regular graphs with edge-connectivity $\lambda \leq k-2$. In Theorem 2.5 we determined $f(G)$ for such graphs. Thus we know that if we insert $q > f(G)$ degree two vertices into the edges of G, $m' - m > 0$. However, can we precisely determine $m' - m$ as a function of q?

References

[1] D. Bauer, H. J. Broersma, A. Morgana, and E. Schmeichel. Tutte sets in graphs I: maximal Tutte sets and D-graphs. *J. Graph Theory*, to appear.

[2] D. Bauer, H. J. Broersma, A. Morgana, E. Schmeichel, and T. Surowiec. Tutte sets in graphs II: the complexity of finding maximum Tutte sets in graphs. *Discrete Appl. Math.*, to appear.

[3] C. Berge. Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. (Paris)*, 247:258 - 259, 1958.

[4] L. Lovász and M. D. Plummer. Matching theory. *Ann. Discrete Math. 29.* North-Holland, Amsterdam, 1986.

[5] J. Petersen. Die theorie der regularen graphs. *Acta Math.*, 15:193 - 220, 1891.

[6] W. T. Tutte. The factorization of linear graphs. *J. London Math. Soc.*, 22:107 - 111, 1947.

[7] D.B. West. *Introduction to Graph Theory.* Prentice Hall, Upper Saddle River, NJ, 2^{nd} edition, 2001.