A square-covering problem

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Abstract

Erdős and Soifer [3] and later Campbell and Staton [1] considered a problem which was a favorite of Erdős [2]: Let S be a unit square. Inscribe n squares with no common interior point. Denote by $e_1, e_2, \cdots, e_n$ the side lengths of these squares. Put $f(n) = max \sum_{i=1}^{n} e_i$. And they discussed the bounds for $f(n)$. In this paper, we consider its dual problem– covering a unit square with squares.

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1. Introduction

Erdős and Soifer [3] and later Campbell and Staton [1] considered a problem which was a favorite of Erdős [2]: Let S be a unit square. Inscribe n squares with no common interior point. Denote by $e_1, e_2, \cdots, e_n$ the side length of these squares. Put $f(n) = max \sum_{i=1}^{n} e_i$. They discussed the bounds for $f(n)$. Inspired by [3], we discuss a problem on square-covering and give corresponding functions $g_i(n)(i = 1, 2)$.

First, we give the definition of a minimal square-covering.

Definition 1.1. Let S be a unit square. If n squares $S_1, S_2, \cdots, S_n$ cover S, in such a way which satisfies:

(1)Each S_i has side of length $s_i (0 < s_i < 1)$ and is placed so that its sides are parallel to those of S;

(2)Each S_i can not be smaller; that is, there does not exist any $S_{i1} \subset S_i$ such that $\{S_j, j = 1, 2, \cdots, i-1, i+1, \cdots, n\} \bigcup \{S_{i1}\}$ can cover S admitting translation.

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We call this kind of covering a minimal square-covering.

With this definition of minimal square covering, define:

$$g_1(n) = min \sum_{i=1}^{n} s_i,\ g_2(n) = max \sum_{i=1}^{n} s_i.$$

When $n \le 3$, since $0 < s_i < 1$, each $S_i (i = 1, 2, \cdots, n)$ can only cover one corner of a unit square, but it has four corners, so $S_1, S_2, \cdots, S_n$ can not cover S. That is, when $n \le 3$, $g_i(n) (i = 1, 2)$ has no meaning. So in the following, let $n \ge 4$.

2. The bounds of $g_1(n)$

Proposition 2.2. $g_1(n) \ge 2$.

Proof. Obviously, if n squares cover the unit square, they must cover its two opposite sides. Since $0 < s_i < 1$, no square can cover the points of two opposite sides in the same time, so $g_1(n) \ge 2$ must hold. □

Theorem 2.3. *When n is even, $g_1(n) \le 3 - \frac{4}{n}$.*

Proof. Consider a minimal square-covering of a unit square S with $n - 2$ squares $S_2, S_3, \cdots, S_{n-1}$ each of which has side of length x, a square S_1 with side of length $1 - x$, and a square S_n with side of length $1 - (\frac{n}{2} - 1)x$. Since $s_1 + s_n \ge 1$, we have $x \le \frac{2}{n}$. When $n = 6$, see Figure 1 for the placement. It's easy to see this is a minimal-square covering. So by the definition of $g_1(n)$, $g_1(n) \le (1-x)+(n-2)x+[1-(\frac{n}{2}-1)x] = 2+(\frac{n}{2}-2)x \le 3 - \frac{4}{n}$.

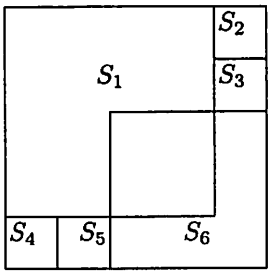

Figure 1: A unit square covered by six squares

□

Corollary 2.4. $g_1(4) = 2$.

Theorem 2.5. *When n is odd, $g_1(n) < \frac{5}{2} - \frac{1}{2(2n-7)}$.*

Proof. Consider a minimal square-covering of a unit square S with $n-3$ squares $S_2, S_3, \cdots, S_{n-2}$ each of which has side of length x, a square S_1 with side of length $1-x$, a square S_n with side of length $1-(n-3)x$ and a square with side of length $(n-3)x$. Since $s_1+s_n \geq 1$ and $s_{n-1} < s_{n-2}+s_n$, we have $x < \frac{1}{2n-7}$. When $n=7$, see Figure 2 for the placement. It's easy to see this is a minimal-square covering. So by the definition of $g_1(n)$, $g_1(n) \leq (1-x)+(n-3)x+1-(n-3)x+(n-3)x = 2+(n-4)x < 2+\frac{n-4}{2n-7} = \frac{5}{2} - \frac{1}{2(2n-7)}$.

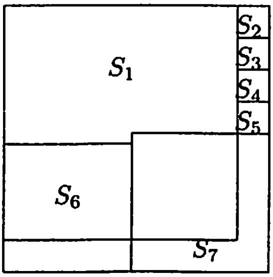

Figure 2: A unit square covered by seven squares

□

3. The bounds of $g_2(n)$

Proposition 4.6. $g_2(k^2) \geq k$.

Proof. It is obvious that all the $S_i (i = 1, 2, \cdots, n)$ must be equal, whence $ne_i = 1$. That is, for each i, $e_i = \frac{1}{\sqrt{n}}$. So n is a perfect square, say, $n = k^2$, and the optimal covering is the standard n-covering, that is, a $k \times k$ grid of squares each of which has side of length $\frac{1}{k}$. Obviously, k^2-1 such squares can not cover the unit square. By the definition of $g_2(n)$, $g_2(k^2) \geq k$. □

Proposition 4.7. $g_2(k^2+1) > k$.

Proof. From a standard k^2-covering, remove a 2×2 grid and replace it with five squares $S_{i1}, S_{i2}, \cdots, S_{i5}$ covering the same area, and which are placed as in Figure 3 such that S_{i1} is the largest square of $\{S_{ij} \mid j = 1, 2, \cdots, 5\}$ and $s_{i2} = s_{i3} = \frac{2}{k} - s_{i1}$, $s_{i4} = \frac{2}{k} - 2(\frac{2}{k} - s_{i1}) = 2s_{i1} - \frac{2}{k}$, $s_{i5} = 2(\frac{2}{k} - s_{i1}) = \frac{4}{k} - 2s_{i1}$, $0 < s_{ij} < \frac{2}{k} (j = 1, 2, \cdots, 5)$, $s_{i4} + s_{i1} \geq \frac{2}{k}$, $s_{i5} < s_{i4} + s_{i3}$. The result is a covering with k^2+1 squares, the sum of whose lengths is $s = k - \frac{4}{k} + s_{i1} + 2(\frac{2}{k} - s_{i1}) + (2s_{i1} - \frac{2}{k}) + (\frac{4}{k} - 2s_{i1}) = k + \frac{2}{k} - s_{i1}$. The inequalities above imply that $\frac{3}{2k} < s_{i1} < \frac{2}{k}$, so $s > k + \frac{2}{k} - \frac{2}{k} = k$.

Obviously, this covering is a minimal covering, so we have $g_2(k^2+1) > k$.

□

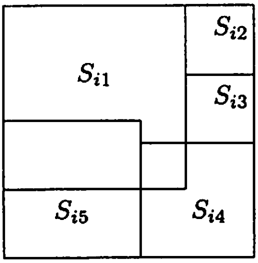

Figure 3: A 2×2 grid covered by five squares

Proposition 4.8. $g_2(k^2-1) \geq k - \frac{3}{k}$.

Proof. From a standard k^2-covering, remove a 3×3 grid and replace it with eight squares $S_{i1}, S_{i2}, \cdots, S_{i8}$ covering the same area, and which are placed as in Figure 4 such that S_{i1} is the largest square of $\{S_{ij} \mid j = 1, 2, \cdots, 8\}$ and $s_{i2} = s_{i3} = s_{i4} = s_{i5} = s_{i6} = s_{i7} = \frac{3}{k} - s_{i1}$, $s_{i8} = \frac{3}{k} - 3(\frac{3}{k} - s_{i1}) = 3s_{i1} - \frac{6}{k}$, $0 < s_{ij} < \frac{3}{k}(j = 1, 2, \cdots, 8)$, $s_{i8} + s_{i1} \geq \frac{3}{k}$. The result is a covering with $k^2 - 9 + 8 = k^2 - 1$ squares, the sum of whose lengths is $s = k - \frac{9}{k} + s_{i1} + 6(\frac{3}{k} - s_{i1}) + (3s_{i1} - \frac{6}{k}) = k + \frac{3}{k} - 2s_{i1}$. The inequalities above imply that $\frac{9}{4k} \leq s_{i1} < \frac{2}{k}$, so $s \geq k + \frac{3}{k} - 2\frac{2}{k} = k - \frac{3}{k}$.

Obviously, this covering is a minimal covering, so we have $g_2(k^2 - 1) \geq k - \frac{3}{k}$.

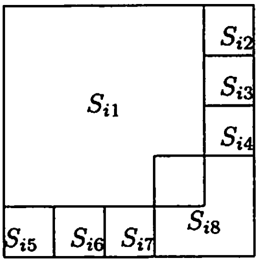

Figure 4: A 3×3 grid covered by eight squares

□

When neither $n-1$ nor $n+1$ is a perfect square, we have the following result:

Theorem 4.9. *If neither $n-1$ nor $n+1$ is a perfect square, then $g_2(n) > \sqrt{n-1}$.*

In the proof of Theorem 4.9, we borrow the main idea of [1].

Proof. When $n = k^2$, by Theorem 4.6, $g_2(n) \geq \sqrt{n} > \sqrt{n-1}$.

When $n \neq k^2$, k must lie between two perfect squares of different parity. That is, there is an integer k such that $k^2 < n < (k+1)^2$, $n - k^2$ and $(k+1)^2 - n$ have different parity. Consider the values of n where $k^2 + 1 < n < (k+1)^2 - 1$. There are two cases which provide upper bounds for all n on the interval $[k^2 + 2, (k+1)^2 - 2]$:

Case 1. $(k+1)^2 - n$ is odd. Say, $(k+1)^2 - n = 2a + 1 (a \geq 1)$, $k^2 < n \leq (k+1)^2 - 3$. From a standard $(k+1)^2$-covering of S, remove an $(a+1) \times (a+1)$ grid and replace it with an $a \times a$ grid covering the same area. The result is a covering with $(k+1)^2 - (a+1)^2 + a^2 = n$ squares, the sum of whose lengths is

$$[(k+1)^2 - (a+1)^2]\frac{1}{k+1} + a^2(\frac{a+1}{a})(\frac{1}{k+1}) = k + 1 - \frac{a+1}{k+1}.$$

Obviously, no one of these n squares can be smaller. So

$$g_2(n) \geq k + 1 - \frac{a+1}{k+1}, g_2^2(n) \geq (k + 1 - \frac{a+1}{k+1})^2 > n - 1.$$

That is, $g_2(n) > \sqrt{n-1}$.

Case 2. $n - k^2$ is odd. Say, $n - k^2 = 2a - 1 (a \geq 2)$, $k^2 + 3 \leq n < (k+1)^2$. From a standard k^2-covering of S, remove an $(a-1) \times (a-1)$ grid and replace it with an $a \times a$ grid covering the same area. The result is a covering with $k^2 - (a-1)^2 + a^2 = k^2 + 2a - 1 = n$ squares of the unit square S. The sum of the lengths of sides is

$$[k^2 - (a-1)^2]\frac{1}{k} + a^2(\frac{a-1}{a})(\frac{1}{k}) = k + \frac{a-1}{k}.$$

Obviously, no $n - 1$ squares of these n squares can cover S. So

$$g_2(n) \geq k + \frac{a-1}{k}, g_2^2(n) \geq (k + \frac{a-1}{k})^2 = k^2 + 2a - 1 + (\frac{a-1}{k})^2 - 1 > n - 1.$$

That is, $g_2(n) > \sqrt{n-1}$.

□

The following lemma is a well-known result [4]:

Lemma 4.10. *Finitely many squares whose total area is equal to* 3 *can cover a unit square.*

Theorem 4.11. $g_2(n) \leq 3\sqrt{n}$.

Proof. Let $\{S_i\}_{i=1}^n$ be a minimal covering of the unit square S, and s_i denote the length of the side of $S_i (i = 1, 2, \cdots, n)$. We first prove that $\sum_{i=1}^{n} s_i^2 \leq 3$. Otherwise, if $\sum_{i=1}^{n} s_i^2 > 3$, there exists a $S_{i1} \subset S_i$, such that $s_{i1} < s_i$ and $s_{i1}^2 + \sum_{j=1}^{i-1} s_j^2 + \sum_{j=i+1}^{n} s_j^2 \geq 3$. By Lemma 4.10, it is obvious that $S_1, S_2, \cdots, S_{i-1}, S_{i1}, S_{i+1}, \cdots, S_n$ can cover the unit square S, which contradicts the definition of a minimal covering. So $\sum_{i=1}^{n} s_i^2 \leq 3$.

Let s be the vector $(s_1, s_2, \cdots, s_n)$, and let v be the vector $(1, 1, \cdots, 1)$. Now $\sum_{i=1}^{n} s_i \leq \|s_i\|\|\mathrm{v}\| \leq \sum_{i=1}^{n} s_i^2 n^{\frac{1}{2}} = n^{\frac{1}{2}} \sum_{i=1}^{n} s_i^2 \leq 3\sqrt{n}$, so $g_2(n) \leq 3\sqrt{n}$.

□

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