Graphs $K_{1*4,5}$, $K_{1*5,4}$, $K_{1*4,4}$, $K_{2,3,4}$ have the property $M(3)$

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Abstract

Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors, $L(v)$, such that there is a unique proper coloring for G from this collection of lists, then G is called a uniquely k-list colorable graph. We say that a graph G has the property $M(k)$ if and only if it is not uniquely k-list colorable. M. Ghebleh and E. S. Mahmoodian characterized uniquely 3-list colorable complete multipartite graphs except for the graphs $K_{1*4,5}, K_{1*5,4}$, $K_{1*4,4}$, $K_{2,3,4}$, and $K_{2,2,r}$, $4 \leq r \leq 8$. In this paper we prove that the graphs $K_{1*4,5}, K_{1*5,4}$, $K_{1*4,4}$, and $K_{2,3,4}$ have the property $M(3)$.

1 Introduction

We consider simple graphs which are finite and undirected. We will use standard notations such as K_n for the complete graph on n vertices, $K_{m_1,m_2,\ldots,m_n}$ for the complete n-partite graph in which the ith part is of size m_i, and K_{s*r} for a complete r-partite graph in which each part is of size s. Notations such as $K_{s*r,t}$ etc. are used similarly. A c-coloring (proper c-coloring) of a graph G is an assignment of c different colors to the vertices of G, such that adjacent vertices have different colors. For the necessary definitions and notations we refer the reader to standard texts, such as [8]. Let G be a graph and let $L(v)$ denote a list of colors available for a vertex v of G. A *list coloring* from the given collection of lists is a proper coloring c such that the color of vertex v, $c(v)$, is in $L(v)$. We will refer to such a coloring as an *L-coloring*. The idea of list colorings of graphs was introduced in Vizing [7] and in Erdös, Rubin and Taylor [1]. We note that a list coloring of K_n is just a *system of distinct representatives* (SDR) for the collection $\mathcal{L} = \{L(v) \mid v \in V(K_n)\}$. Suppose that for each vertex v in G, there exists a list of k colors, $L(v)$, such that there is a unique proper coloring for G from this collection of lists, then G is called a *uniquely k-list colorable* graph or a $UkLC$ graph for short. We say that a graph G has *the property* $M(k)$ (M for Marshal Hall) if and only if it is not uniquely k-list colorable. So G has the property $M(k)$ if for any collection of lists assigned to its vertices, each of size at least k, (without loss of generality we can assume that the size of list is k) either there is no list coloring for G or there exist at least two list colorings. The concept of $UkLC$ graphs also arise in finding defining sets for colorings of graphs (see [6]). A minimal defining set of an n- coloring of $K_n \times K_n$ is just a *critical set* of a Latin square of order n (see [4]). Uniquely 2-list colorable graphs have been studied in [2, 5]. In particular, Mahdian and Mahmoodian [5] characterize $U2LC$ graphs as follows:

Theorem A *A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph or a complete bipartite graph.*

Ghebleh and Mahmoodian [3] extensively study the unique colorability for complete multipartite graphs. In particular, they characterize $U3LC$ complete multipartite graphs except for the graphs $K_{2,2,r}$, $4 \leq r \leq 8$, $K_{1*4,5}$, $K_{1*5,4}$, $K_{1*4,4}$, and $K_{2,3,4}$. They prove:

Theorem B *If G is a complete multipartite graph which has an induced*

UkLC subgraph, then G is UkLC.

Corollary C *If G is a complete multipartite graph which has the property $M(3)$, then each induced subgraph of G has the property $M(3)$.*

The following problem is stated in [3].

Problem *Verify the property $M(3)$ for the graphs* $K_{2,2,r}$, $4 \leq r \leq 8$, $K_{1*4,5}$, $K_{1*5,4}$, $K_{1*4,4}$, and $K_{2,3,4}$.

In this paper we will show that the graphs $K_{1*4,5}$, $K_{1*5,4}$, $K_{1*4,4}$, and $K_{2,3,4}$, have the property $M(3)$. Note that, without loss of generality, we can assume the size of each list of colors is precisely 3. In order to show that a graph G has the property $M(3)$, it is sufficient to show that G admits at least two different L-colorings. Finally, $|[m]|$ denotes the number of vertices of G with color m in an L-coloring c for G.

We make use of the following crucial results in the next sections.

Lemma 1 *Let G be a graph and L be a 3-list assignment to the vertices of G such that G admits an L-coloring c. Assume for a vertex $v \in V(G)$ there exists a color $r \in L(v)$ which is not used in c. Then there exists a new L-coloring $c' \neq c$ for G.*

Proof. Define a new L-coloring $c' \neq c$ as follows: $c'(x) = c(x)$ if $x \in V(G) \setminus \{v\}$ and $c'(v) = r$. ■

Lemma 2 *Suppose that G is a complete multipartite graph, L is a 3-list assignment to the vertices of G, and that G admits an L-coloring c. Let $s \neq t$ be two colors and let $X = c^{-1}(s)$ and $Y = c^{-1}(t)$. If $t \in \bigcap_{x \in X} L(x)$ and $s \in \bigcap_{y \in Y} L(y)$ then a new L-coloring $c' \neq c$ exists for G.*

Proof. Define a new L-coloring $c' \neq c$ as follows: $c'(v) = t$ if $v \in X$, $c'(v) = s$ if $v \in Y$ and $c'(v) = c(v)$ if $v \in V(G) \setminus (X \cup Y)$. ■

Lemma 3 *Let $\{x_1\}, \cdots, \{x_m\}$, and $\{v_1, v_2, ..., v_n\}$ be the partite sets of the graph $G = K_{1*m,n}$ and let L be a 3-list assignment to the vertices of G*

such that an L-coloring c exists for G. If for some $J \subset \{1, 2, ..., m\}$ *we have* $|L(x_i) \cap A| \leq 1$ *for all* $i \notin J$*, where* $A = \{c(v_1), c(v_2), ..., c(v_n)\} \cup \{c(x_i) | \ i \in J\}$*, then G admits an L-coloring* $c' \neq c$.

Proof. Suppose that $S = \{x_i | \ i \notin J\}$. Then $G[S]$, the subgraph induced by S, is a complete graph and has the property $M(2)$ by Theorem A. If we put $L'(x_i) = L(x_i) \setminus A$ then $|L'(x_i)| \geq 2$ for $i \notin J$. So there exists a new L'-coloring for the subgraph $G[S]$ which is extendible to $K_{1*m,n}$. ■

2 The graphs $K_{1*4,5}$ and $K_{1*4,4}$

In this section we show that the graphs $K_{1*4,5}$ and $K_{1*4,4}$ have the property $M(3)$. Throughout this section we assume $\{x_1\}, \cdots, \{x_4\}$, and $\{v_1, ..., v_5\}$ are the partite sets of the graph $G = K_{1*4,5}$. Moreover, we assume L is a 3-list assignment to the vertices of G such that an L-coloring c exists for G. The goal is to find an L-coloring $c' \neq c$ for G.

In Lemmas 4 and 5 we assume the following properties hold.

1. $c(x_i) = i$ for $i = 1, 2, 3, 4$ and if $A = \{c(v_1), ..., c(v_5)\}$ then $|A| \geq 2$.
2. If $c(v_i) \neq c(v_j)$ for some $1 \leq i, j \leq 5$, then $c(v_i) \notin L(v_j)$.
3. Each color in $\bigcup_{v \in V(G)} L(v)$ is used in the L-coloring c.

Lemma 4 *There exists a 2-subset E of* $\{1, 2, 3, 4\}$*, such that* $L(v_i) \cap E \neq \emptyset$ *for* $1 \leq i \leq 5$.

Proof. Let $c(v_i) = a$ for some i. Obviously $a \notin \{1, 2, 3, 4\}$. Let $b \in L(v_i)$ and $b \neq a$. If $b \notin \{1, 2, 3, 4\}$ then by Property 3 we must have $c(v_j) = b$ for some j. Now $c(v_i) \neq c(v_j)$ and $c(v_j) \in L(v_i)$ contradict with Property 2. So $b \in \{1, 2, 3, 4\}$. This forces to have $|L(v_i) \cap \{1, 2, 3, 4\}| = 2$ for $i = 1, 2, \cdots, 5$. So there is a color $i \in \{1, 2, 3, 4\}$ which appears in at least three of $L(v_1)$, $L(v_2)$, $L(v_3)$, $L(v_4)$, and $L(v_5)$. Without loss of generality we may assume the color 1 appears in $L(v_1)$, $L(v_2)$ and $L(v_3)$. If $1 \in L(v_4)$ or $1 \in L(v_5)$ then it is easy to find E with the required property. Otherwise

$L(v_4) \cap L(v_5) \cap \{2,3,4\} \neq \emptyset$. Let $a \in L(v_4) \cap L(v_5) \cap \{2,3,4\}$. Then $E = \{1, a\}$ is the required set. ■

The following lemma shows that $K_{1*4,5}$ has the property $M(3)$ under certain conditions.

Lemma 5 *If* $|L(x_i) \cap A| = 2$ *for some* $1 \leq i \leq 4$, *then there exists a new* L*-coloring* $c' \neq c$ *for* $G = K_{1*4,5}$.

Proof. By Lemma 4, there exists a 2-subset E of $\{1,2,3,4\}$ such that $L(v_i) \cap E \neq \emptyset$ for $1 \leq i \leq 5$. We can assume that $E = \{1,2\}$. If $1 \in L(x_2)$ and $2 \in L(x_1)$, the result follows by Lemma 2. Now assume that $1 \notin L(x_2)$ or $2 \notin L(x_1)$. Note that if $1 \in L(x_2)$ or $2 \in L(x_1)$ then $|L(x_2) \cap A| \leq 1$ or $|L(x_1) \cap A| \leq 1$, respectively. Now we consider three cases.

Case 1. $|L(x_1) \cap A| = 2$.
Let $L(x_1) \cap A = \{5,6\}$. If $L(x_2) \cap A \neq \emptyset$, then G admits a new L-coloring $c' \neq c$ defined by: $c'(x_2) \in L(x_2) \cap A$, $c'(x_1) \in \{5,6\} \setminus \{c'(x_2)\}$, $c'(x_3) = 3$, $c'(x_4) = 4$ and $c'(v_i) \in \{1,2\}$ for $1 \leq i \leq 5$.
If $L(x_2) \cap A = \emptyset$ then $L(x_2) \cap \{3,4\} \neq \emptyset$. Without loss of generality we may assume $3 \in L(x_2)$. If $a \in L(x_3) \cap A$, we introduce a new L-coloring $c' \neq c$ as follows: $c'(x_3) = a$, $c'(x_1) \in \{5,6\} \setminus \{a\}$, $c'(x_2) = 3$, $c'(x_4) = 4$ and $c'(v_i) \in \{1,2\}$ for $1 \leq i \leq 5$.
If $L(x_3) \cap A = \emptyset$, then $L(x_3) \cap \{2,4\} \neq \emptyset$. Now if $2 \in L(x_3)$, the result follows by Lemma 2. If $4 \in L(x_3)$ we proceed as follows. If $L(x_4) \cap A \neq \emptyset$, we define a new L-coloring $c' \neq c$ by: $c'(x_4) \in L(x_4) \cap A$, $c'(x_1) \in \{5,6\} \setminus \{c'(x_4)\}$, $c'(x_2) = 3$, $c'(x_3) = 4$ and $c'(v_i) \in \{1,2\}$ for $1 \leq i \leq 5$.
If $L(x_4) \cap A = \emptyset$, we apply Lemma 3 with $J = \{1\}$.

Case 2. $|L(x_2) \cap A| = 2$.
An argument similar to that described in Case 1 settles this case.

Case 3. $|L(x_i) \cap A| \leq 1$ for $i = 1,2$.
By the assumptions $|L(x_3) \cap A| = 2$ or $|L(x_4) \cap A| = 2$. So without loss of generality we may assume $L(x_4) = \{4, s, t\}$, where $\{s,t\} \subseteq A$. Now we consider two subcases.

Subcase 3.1. $A \cap (L(x_1) \cup L(x_2)) = \emptyset$.
Since $1 \notin L(x_2)$ or $2 \notin L(x_1)$ we can assume $3 \in L(x_1)$ and $4 \in L(x_2)$. If $L(x_3) \cap A \neq \emptyset$, we define a new L-coloring $c' \neq c$ by: $c'(x_1) = 3$, $c'(x_2) = 4$,

$c'(x_3) \in L(x_3) \cap A, c'(x_4) \in \{s,t\} \setminus \{c'(x_3)\}$ and $c'(v_i) \in \{1,2\}$ for $1 \le i \le 5$. If $L(x_3) \cap A = \emptyset$, we apply Lemma 3 with $J = \{4\}$.

Subcase 3.2. $A \cap (L(x_1) \cup L(x_2)) \neq \emptyset$.
This implies that one of the color lists $L(x_1)$ or $L(x_2)$ has a color from set A and other list has one of the colors 3 or 4. Without loss of generality we assume that $5 \in L(x_1)$ and $3 \in L(x_2)$, where $5 \in A$. If $(L(x_3) \cap A) \setminus \{5\} \neq \emptyset$, we define a new L-coloring $c' \neq c$ by: $c'(x_1) = 5$, $c'(x_2) = 3$, $c'(x_3) \in (L(x_3) \cap A) \setminus \{5\}, c'(x_4) = 4$ and $c'(v_i) \in \{1,2\}$ for $1 \le i \le 5$.
So let $(L(x_3) \cap A) \setminus \{5\} = \emptyset$. If $4 \notin L(x_1) \cup L(x_3)$, we apply Lemma 3 with $J = \{4\}$. Now let $4 \in L(x_1) \cup L(x_3)$. If $4 \in L(x_3)$, then we define a new L-coloring $c' \neq c$ by: $c'(x_1) = 5$, $c'(x_2) = 3$, $c'(x_3) = 4$, $c'(x_4) \in \{s,t\} \setminus \{5\}$ and $c'(v_i) \in \{1,2\}$ for $1 \le i \le 5$.
If $4 \notin L(x_3)$ then $4 \in L(x_1)$ and $L(x_3) \cap \{2,5\} \neq \emptyset$. If $2 \in L(x_3)$, the result follows by Lemma 2 since $3 \in L(x_2)$. If $2 \notin L(x_3)$ then $5 \in L(x_3)$. Now G admits a new L-coloring $c' \neq c$ defined by: $c'(x_1) = 4$, $c'(x_2) = 3$, $c'(x_3) = 5$, $c'(x_4) \in \{s,t\} \setminus \{5\}$ and $c'(v_i) \in \{1,2\}$. ∎

Now we are ready to prove the main result of this section.

Theorem 6 *The graph $K_{1*4,5}$ has the property $M(3)$.*

Proof. Let $\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$ and $\{v_1, ..., v_5\}$ be the partite sets of the graph $G = K_{1*4,5}$. Let L be a 3-list assignment to the vertices of G such that an L-coloring c exists for G. By Lemma 1 we can assume L has Property 3. Moreover, if $c(v_i) \neq c(v_j)$ and $c(v_i) \in L(v_j)$ for some $1 \le i,j \le 5$ then we can define a new L-coloring $c' \neq c$ by: $c'(u) = c(u)$ if $u \neq v_j$ and $c'(v_j) = c(v_i)$. Therefore, we can also assume L has Property 2. Now let $A = \{c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)\}$. If $|L(x_i) \cap A| \le 1$ for all $1 \le i \le 4$, then the result follows by Lemma 3 with $J = \emptyset$. Finally, if $|L(x_i) \cap A| = 2$ for some $i \in \{1,2,3,4\}$, then the result follows by Lemma 5. This completes the proof. ∎

The graph $K_{1*4,4}$ is an induced subgraph of $K_{1*4,5}$. So by Theorem 6 and Corollary C we have the following result.

Corollary 7 *The graph $K_{1*4,4}$ has the property $M(3)$.*

3 The graph $K_{1*5,4}$

In this section we show that the graph $G = K_{1*5,4}$ has the property $M(3)$. Throughout this section we assume $\{y_1\}, \{y_2\}, \{y_3\}, \{y_4\}, \{y_5\}$ and $\{w_1, w_2, w_3, w_4\}$ are the partite sets of the graph G. Moreover, we assume L is a 3-list assignment to the vertices of G such that an L-coloring c exists for G. The goal is to find an L-coloring $c' \neq c$ for G. In Lemmas 8 and 9 we assume the following properties hold.

1. $c(y_i) = i$, $1 \leq i \leq 5$, $c(w_1) = c(w_2) = 6$ and $c(w_3) = c(w_4) = 7$.
2. If $c(w_i) \neq c(w_j)$ for some $1 \leq i, j \leq 4$, then $c(w_i) \notin L(w_j)$.
3. Each color in $\bigcup_{v \in V(G)} L(v)$ is used in the L-coloring c.

Lemma 8 *If* $|L(w_1) \cap L(w_2)| = 1$ *and* $|L(w_3) \cap L(w_4)| = 1$, *then there exists a 2-subset* F *of* $\{1, 2, 3, 4, 5\}$, *such that* $L(w_i) \cap F \neq \emptyset$ *for* $i = 1, 2, 3, 4$.

Proof. Since $|L(w_1) \cap L(w_2)| = 1$, we can assume $L(w_1) = \{1, 2, 6\}$ and $L(w_2) = \{3, 4, 6\}$. This implies that $L(w_1) \cap L(w_3) \neq \emptyset$ or $L(w_2) \cap L(w_3) \neq \emptyset$. Without loss of generality, we may assume $L(w_2) \cap L(w_3) \neq \emptyset$ and $3 \in L(w_2) \cap L(w_3)$. If $\{4, 5\} \not\subseteq L(w_4)$, since $|L(w_3) \cap L(w_4)| = 1$, we must have $L(w_4) \cap \{1, 2\} \neq \emptyset$. Let $a \in L(w_4) \cap \{1, 2\}$. Then $F = \{a, 3\}$ is the required subset. If $\{4, 5\} \subseteq L(w_4)$ then $L(w_3) \cap \{1, 2\} \neq \emptyset$. Let $b \in L(w_3) \cap \{1, 2\}$. Then $F = \{b, 4\}$ is the required subset. ■

The following lemma shows that $K_{1*5,4}$ has the property $M(3)$ under certain conditions.

Lemma 9 *If* $L(y_j) = \{j, 6, 7\}$ *for some* $1 \leq j \leq 5$, $|L(w_1) \cap L(w_2)| = 1$ *and* $|L(w_3) \cap L(w_4)| = 1$, *then a new* L*-coloring* $c' \neq c$ *exists for* G.

Proof. By Lemma 8 there exists a 2-subset F of $\{1, 2, 3, 4, 5\}$, such that $L(w_i) \cap F \neq \emptyset$ for $1 \leq i \leq 4$. Without loss of generality we assume $F = \{1, 2\}$. If $2 \in L(y_1)$ and $1 \in L(y_2)$, the result follows by Lemma 2. So $2 \notin L(y_1)$ or $1 \notin L(y_2)$. We consider three cases:

Case 1. $L(y_1) = \{1, 6, 7\}$.
If $a \in L(y_2) \cap \{6, 7\}$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) \in \{6, 7\} \setminus \{a\}$, $c'(y_2) = a$, $c'(y_3) = 3$, $c'(y_4) = 4$, $c'(y_5) = 5$ and $c'(w_i) \in \{1, 2\}$ for $1 \leq i \leq 4$.
If $L(y_2) \cap \{6, 7\} = \emptyset$ then $L(y_2) \cap \{3, 4, 5\} \neq \emptyset$. Without loss of generality let $3 \in L(y_2)$. If $2 \in L(y_3)$, the result follows by Lemma 2. Now let $2 \notin L(y_3)$. If $b \in L(y_3) \cap \{6, 7\}$ we define a new L-coloring $c' \neq c$ as follows: $c'(y_1) \in \{6, 7\} \setminus \{b\}$, $c'(y_2) = 3$, $c'(y_3) = b$, $c'(y_4) = 4$, $c'(y_5) = 5$ and $c'(w_i) \in \{1, 2\}$ for $1 \leq i \leq 4$.
If $L(y_3) \cap \{6, 7\} = \emptyset$ then $a \in L(y_3) \cap \{4, 5\}$. Let $4 \in L(y_3)$ (the case $5 \in L(y_3)$ is similar). If $b \in L(y_4) \cap \{6, 7\}$ we define a new L-coloring $c' \neq c$ as follows: $c'(y_1) \in \{6, 7\} \setminus \{b\}$, $c'(y_2) = 3$, $c'(y_3) = 4$, $c'(y_4) = b$, $c'(y_5) = 5$ and $c'(w_i) \in \{1, 2\}$ for $1 \leq i \leq 4$.
Now we suppose that $L(y_4) \cap \{6, 7\} = \emptyset$. If $5 \notin L(y_4)$, we apply Lemma 3 with $J = \{1, 5\}$. So let $5 \in L(y_4)$. If $a \in L(y_5) \cap \{6, 7\}$ we define a new L-coloring $c' \neq c$ as follows: $c'(y_1) \in \{6, 7\} \setminus \{a\}$, $c'(y_2) = 3$, $c'(y_3) = 4$, $c'(y_4) = 5$, $c'(y_5) = a$ and $c'(w_i) \in \{1, 2\}$ for $1 \leq i \leq 4$.
Finally, if $L(y_5) \cap \{6, 7\} = \emptyset$, we apply Lemma 3 with $J = \{1\}$.

Case 2. $L(y_2) = \{2, 6, 7\}$.
An argument similar to that described in Case 1 takes care of this case.

Case 3. $|L(y_i) \cap \{6, 7\}| \leq 1$ for $i = 1, 2$.
We consider two subcases:

Subcase 3.1. $(L(y_1) \cup L(y_2)) \cap \{6, 7\} \neq \emptyset$
Since $1 \notin L(y_2)$ or $2 \notin L(y_1)$ it follows that one of the color lists $L(y_1)$ or $L(y_2)$ has a color from set $\{6, 7\}$ and the other list has a color from set $\{3, 4, 5\}$. We may assume that $6 \in L(y_1)$ and $3 \in L(y_2)$. If $7 \in L(y_3)$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 6$, $c'(y_2) = 3$, $c'(y_3) = 7$, $c'(y_4) = 4$, $c'(y_5) = 5$ and $c'(w_i) \in \{1, 2\}$ for $1 \leq i \leq 4$.
If $7 \notin L(y_3)$, since $L(y_i) = \{i, 6, 7\}$ for some $1 \leq i \leq 5$, we must have $L(y_4) = \{4, 6, 7\}$ or $L(y_5) = \{5, 6, 7\}$. Without loss of generality we assume $L(y_5) = \{5, 6, 7\}$. If $5 \in L(y_3)$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 6$, $c'(y_2) = 3$, $c'(y_3) = 5$, $c'(y_4) = 4$, $c'(y_5) = 7$ and $c'(w_i) \in \{1, 2\}$ for $1 \leq i \leq 4$.
Now let $5 \notin L(y_3)$. Consider the following two subcases.

3.1.1 $6 \notin L(y_3) \cup L(y_4)$.
Then $\{2, 4\} \cap L(y_3) \neq \emptyset$. If $2 \in L(y_3)$, the result follows by Lemma 2. If $4 \in L(y_3)$ and $a \in L(y_4) \cap \{5, 7\}$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 6$, $c'(y_2) = 3$, $c'(y_3) = 4$, $c'(y_4) = a$, $c'(y_5) \in \{5, 7\} \setminus \{a\}$ and

$c'(w_i) \in \{1,2\}$ for $1 \le i \le 4$.
If $4 \in L(y_3)$ and $L(y_4) \cap \{5,7\} = \emptyset$, we apply Lemma 3 with $J = \{1,5\}$.

3.1.2 $6 \in L(y_3) \cup L(y_4)$.
Let $6 \in L(y_3)$. (The case $6 \in L(y_4)$ can be settled in a similar fashion.) If $5 \in L(y_1)$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 5$, $c'(y_2) = 3$, $c'(y_3) = 6$, $c'(y_4) = 4$, $c'(y_5) = 7$ and $c'(w_i) \in \{1,2\}$ for $1 \le i \le 4$.
If $5 \notin L(y_1)$ and $4 \notin L(y_1) \cup L(y_3)$, we apply Lemma 3 with $J = \{4,5\}$).
Now let $5 \notin L(y_1)$ and $4 \in L(y_1) \cup L(y_3)$. We only consider the case $4 \in L(y_1)$. The case $4 \in L(y_3)$ can be settled in a similar fashion. If $a \in \{5,7\} \cap L(y_4)$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 4$, $c'(y_2) = 3$, $c'(y_3) = 6$, $c'(y_4) = a$, $c'(y_5) \in \{5,7\} \setminus \{a\}$ and $c'(w_i) \in \{1,2\}$ for $1 \le i \le 4$.
If $\{5,7\} \cap L(y_4) = \emptyset$, we apply Lemma 3 with $J = \{5\}$.

Subcase 3.2. $(L(y_1) \cup L(y_2)) \cap \{6,7\} = \emptyset$.
Without loss of generality we assume $3 \in L(y_1)$ and $4 \in L(y_2)$. Now consider two subcases.

3.2.1. $|L(y_5) \cap \{6,7\}| \le 1$.
So $L(y_3) = \{3,6,7\}$ or $L(y_4) = \{4,6,7\}$. Let $L(y_3) = \{3,6,7\}$. (The case $L(y_4) = \{4,6,7\}$ is similar.) If $a \in L(y_4) \cap \{6,7\}$ we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 3$, $c'(y_2) = 4$, $c'(y_3) \in \{6,7\} \setminus \{a\}$, $c'(y_4) = a$, $c'(y_5) = 5$ and $c'(w_i) \in \{1,2\}$ for $1 \le i \le 4$.
Now let $L(y_4) \cap \{6,7\} = \emptyset$. If $5 \in L(y_4)$ and $a \in L(y_5) \cap \{6,7\}$, we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 3$, $c'(y_2) = 4$, $c'(y_3) \in \{6,7\} \setminus \{a\}$, $c'(y_4) = 5$, $c'(y_5) = a$ and $c'(w_i) \in \{1,2\}$.
If $5 \in L(y_4)$ and $L(y_5) \cap \{6,7\} = \emptyset$, we apply Lemma 3 with $J = \{3\}$.
If $5 \notin L(y_4)$ then $L(y_4) \cap \{1,2\} \neq \emptyset$. Now if $2 \in L(y_4)$, the result follows by Lemma 2. If $2 \notin L(y_4)$ then $1 \in L(y_4)$ and $L(y_4) = \{1,3,4\}$. Now if $5 \notin L(y_1)$, we apply Lemma 3 with $J = \{3,5\}$. So let $5 \in L(y_1)$. If $a \in L(y_5) \cap \{6,7\}$, we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 5$, $c'(y_2) = 4$, $c'(y_3) \in \{6,7\} \setminus \{a\}$, $c'(y_4) = 3$, $c'(y_5) = a$ and $c'(w_i) \in \{1,2\}$ for $1 \le i \le 4$.
Finally, if $L(y_5) \cap \{6,7\} = \emptyset$, we apply Lemma 3 with $J = \{3\}$.

3.2.2. $L(y_5) = \{5,6,7\}$.
If $(L(y_3) \cup L(y_4)) \cap \{6,7\} = \emptyset$, we apply Lemma 3 with $J = \{5\}$. Now let $L(y_3) \cap \{6,7\} \neq \emptyset$. (The case $L(y_4) \cap \{6,7\} \neq \emptyset$ can be settled in a similar fashion.) Without loss of generality we may assume $6 \in L(y_3)$.
If $a \in L(y_4) \cap \{5,7\}$, we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 3$, $c'(y_2) = 4$, $c'(y_3) = 6$, $c'(y_4) = a$, $c'(y_5) \in \{5,7\} \setminus \{a\}$ and $c'(w_i) \in \{1,2\}$ for $1 \le i \le 4$.

Now let $L(y_4) \cap \{5,7\} = \emptyset$. If $6 \notin L(y_4)$ then $L(y_4) \cap \{2,3\} \neq \emptyset$. If $2 \in L(y_4)$, the result follows by Lemma 2. If $2 \notin L(y_4)$ then $3 \in L(y_4)$. Now if $5 \notin L(y_1)$ we apply Lemma 3 with $J = \{3,5\}$. If $5 \in L(y_1)$, we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 5$, $c'(y_2) = 4$, $c'(y_3) = 6$, $c'(y_4) = 3$, $c'(y_5) = 7$ and $c'(w_i) \in \{1,2\}$.
If $6 \in L(y_4)$ and $\{5,7\} \cap L(y_3) = \emptyset$, we apply Lemma 3 with $J = \{5\}$. If $6 \in L(y_4)$ and $a \in \{5,7\} \cap L(y_3)$, we define a new L-coloring $c' \neq c$ by: $c'(y_1) = 3$, $c'(y_2) = 4$, $c'(y_3) = a$, $c'(y_4) = 6$, $c'(y_5) \in \{5,7\} \setminus \{a\}$ and $c'(w_i) \in \{1,2\}$ for $1 \leq i \leq 4$. This completes the proof. ■

Now we are ready to prove the main result of this section.

Theorem 10 *The graph $K_{1*5,4}$ has the property $M(3)$.*

Proof. Let $\{y_i\}$, $1 \leq i \leq 5$, and $\{w_1, w_2, w_3, w_4\}$ be the partite sets of the graph $G = K_{1*5,4}$. Let L be a 3-list assignment to $V(G)$ such that an L-coloring c exists for G. By Lemma 1 we can assume L has Property 3. Moreover, if $c(w_i) \neq c(w_j)$ and $c(w_i) \in L(w_j)$ for some $1 \leq i,j \leq 4$ then we can define a new L-coloring $c' \neq c$ by $c'(u) = c(u)$ if $u \neq w_j$ and $c'(w_j) = c(w_i)$. Therefore, we can also assume L has Property 2. Let $A = \{c(w_i) | 1 \leq i \leq 4\}$. If $|A| = 1$, we apply Lemma 3 with $J = \emptyset$. If $|A| = 3$ or $|A| = 4$ we add new edges between the vertices with different colors in c. The resulting graph is $K_9 \setminus e$ or K_9, respectively. These graphs have the property $M(3)$ (see [4]). So there exists a new L-coloring $c' \neq c$ for G. Now let $|A| = 2$ and $A = \{6,7\}$. We consider two cases:

Case 1. $|[6]| = |[7]| = 2$.
Without loss of generality we can assume $c(w_1) = c(w_2) = 6$ and $c(w_3) = c(w_4) = 7$. Let $L(w_1) \cap L(w_2) = \{6\}$ and $L(w_3) \cap L(w_4) = \{7\}$. If $L(y_j) = \{j, 6, 7\}$ for some $1 \leq j \leq 5$, the result follows by Lemma 9. If $|L(y_j) \cap \{6,7\}| \leq 1$, for all $1 \leq j \leq 5$, we apply Lemma 3 with $J = \emptyset$.
Let $|L(w_1) \cap L(w_2)| > 1$. Suppose that $S = \{y_1, y_2, y_3, y_4, y_5, w_1\}$. Then $G[S]$, the subgraph induced by S, is a complete graph on six vertices and has the property $M(2)$ by Theorem A. If we put $L'(u) = L(u) \setminus \{7\}$ for $u \in S \setminus \{w_1\}$ and $L'(w_1) = L(w_1) \cap L(w_2)$, then $|L'(u)| \geq 2$ for $u \in S$. Therefore, there exists a new L'-coloring for the subgraph $G[S]$ which is extendible to G.

Case 2. $|[6]| = 1$ or $|[7]| = 1$.
Let $|[6]| = 1$ and $c(w_1) = 6$. (The case $|[7]| = 1$ is similar to this case.) Suppose that $S = \{y_1, y_2, y_3, y_4, y_5, w_1\}$. Then $G[S]$, the subgraph induced

by S, is a complete graph on six vertices and has the property $M(2)$ by Theorem A. If we put $L'(u) = L(u) \setminus \{7\}$ for $u \in S$ then $|L'(u)| \geq 2$. So there exists a new L'-coloring for the subgraph $G[S]$ which is extendible to G. ■

4 The graph $K_{2,3,4}$

In this section we show that the graph $G = K_{2,3,4}$ has the property $M(3)$. Throughout this section we assume $\{a, b\}$, $\{d, e, f\}$ and $\{x, y, z, t\}$ are the partite sets of the graph G. Moreover, we assume L is a 3-list assignment to the vertices of G such that an L-coloring c exists for G. The goal is to find an L-coloring $c' \neq c$ for G.

In Lemmas 11-18 we assume the following properties hold.

1. $|c(\{a,b\})| = |c(\{d,e,f\})| = |c(\{x,y,z,t\})| = 2$. Moreover, $c(a) = 1$, $c(b) = 2$, $c(d) = c(e) = 3$ and $c(f) = 4$.
2. If $u$and v are two vertices in the same part such that $c(u) \neq c(v)$ then $c(u) \notin L(v)$.
3. Each color in $\bigcup_{v \in V(G)} L(v)$ is used in the L-coloring c.
4. Let $5, 6 \in \{c(x), c(y), c(z), c(t)\}$. If $\|[5]\| = \|[6]\| = 2$, then we assume that $c(x) = c(y) = 5$ and $c(z) = c(t) = 6$. If $\|[5]\| = 3$ and $\|[6]\| = 1$, then we assume that $c(x) = c(y) = c(z) = 5$ and $c(t) = 6$.

We make use of the following lemma which is similar to Lemmas 4 and 8.

Lemma 11 *There exists a 2-subset F of $\{1,2,3,4\}$ such that $L(v) \cap F \neq \emptyset$ for $v \in \{x, y, z, t\}$. Moreover, if $|L(x) \cap L(y) \cap L(z) \cap L(t)| \neq 2$ then there exist 2-subsets F_i, $i = 1, 2$, of $\{1,2,3,4\}$, such that $L(v) \cap F_i \neq \emptyset$ for $v \in \{x, y, z, t\}$.*

Proof. An argument similar to that described in Lemma 4 shows that there exists an $F_1 = \{r, s\} \subseteq \{1,2,3,4\}$ such that $L(v) \cap F \neq \emptyset$ for $v \in \{x, y, z, t\}$.

In order to find another 2-subset when $|L(x) \cap L(y) \cap L(z) \cap L(t)| \neq 2$, without loss of generality, we may consider two cases. If $r \in L(x) \cap L(y) \cap L(z)$ and $s \in L(t)$, we define $F_2 = \{r, s'\}$, where $s' \in \{1,2,3,4\} \setminus \{r, s\}$. If $r \in L(x) \cap L(y)$ and $s \in L(z) \cap L(t)$, we define $F_2 = \{r', s'\}$ where $\{r', s'\} \subseteq \{1,2,3,4\} \setminus \{r, s\}$. ■

Lemmas 12-18 show that the graphs $K_{2,3,4}$ has the property $M(3)$ under certain conditions.

Lemma 12 *If* $4 \in L(a) \cap L(b)$, *then there exists a new L-coloring* $c' \neq c$ *for* $K_{2,3,4}$.

Proof. If $L(f) \cap \{1,2\} \neq \emptyset$, the result follows by Lemma 2. Otherwise we must have $L(f) = \{4,5,6\}$. Now if color 4 appears in at least three of the color lists of $\{x, y, z, t\}$ then the result follows from Lemma 2. Otherwise, we can color the vertices x, y, z, t with at most two of the three colors 1, 2 and 3, the vertices a and b with color 4 and the vertices d, e, f with colors $5, 6, x$, where $x \in \{1,2,3\}$ is an unused color in partite set $\{x, y, z, t\}$. ■

Lemma 13 *If* $3 \in L(a) \cap L(b)$, *then there exists a new L-coloring* $c' \neq c$ *for* $K_{2,3,4}$.

Proof. If $4 \in L(a) \cap L(b)$ then the result follows by Lemma 12. So let $4 \notin L(a) \cap L(b)$. This forces $L(a) \cap \{5,6\} \neq \emptyset$ or $L(b) \cap \{5,6\} \neq \emptyset$. We assume $L(a) \cap \{5,6\} \neq \emptyset$ (the case $L(b) \cap \{5,6\} \neq \emptyset$ is similar). If $L(d) \cap \{1,2\} \neq \emptyset$ and $L(e) \cap \{1,2\} \neq \emptyset$ we obtain a new L-coloring c' as follows: $c'(a) = c'(b) = 3$, $\{c'(d), c'(e)\} \subseteq \{1,2\}$ and $c'(u) = c(u)$ for $u \in \{f, x, y, z, t\}$.
Otherwise, $L(d) \cap \{1,2\} = \emptyset$ or $L(e) \cap \{1,2\} = \emptyset$. Without loss of generality we assume that $L(d) \cap \{1,2\} = \emptyset$. This forces $L(d) = \{3,5,6\}$. We consider three cases.

Case 1. $|L(f) \cap \{5,6\}| = 1$.
Assume $L(f) = \{1,4,5\}$ (the cases $L(f) = \{1,4,6\}$, $L(f) = \{2,4,5\}$ and $L(f) = \{2,4,6\}$ are similar). Consider the following two subcases.

Subcase 1.1. $L(e) \cap \{5,6\} \neq \emptyset$.
We define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 3$, $\{c'(d), c'(e), c'(f)\} \subseteq \{5,6\}$, and $c'(v) \in \{1,2,4\}$for $v \in \{x, y, z, t\}$.

Subcase 1.2. $L(e) \cap \{5,6\} = \emptyset$.
This forces $L(e) = \{1,2,3\}$. If $L(a) \cap L(b) \cap \{5,6\} \neq \emptyset$, we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) \in \{5,6\}$, $c'(d) \in \{5,6\} \setminus \{c'(a)\}$, $c'(e) = c'(f) = 1$ and $c'(v) \in \{2,3,4\}$ for $v \in \{x,y,z,t\}$.
Otherwise, $L(a) \cap L(b) \cap \{5,6\} = \emptyset$. If $||[5]|| = 3$ (the case $||[6]|| = 3$ is similar) we define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 3$, $c'(d) = 6$, $c'(x) = c'(y) = c'(z) = 5$, $c'(t) \in \{1,2,4\}$, $c'(e) \in \{1,2\} \setminus \{c'(t)\}$ and $c'(f) \in \{1,4\} \setminus \{c'(t)\}$.
Now let $||[5]|| = ||[6]|| = 2$ and consider the following three subcases.

1.2.1. $4 \in L(x) \cap L(y)$. (The case $4 \in L(z) \cap L(t)$ is similar.)
We obtain a new L-coloring c' as follows: $c'(a) = c'(b) = 3$, $c'(d) = 5$, $c'(e) = c'(f) = 1, c'(x) = c'(y) = 4$ and $c'(z) = c'(t) = 6$.

1.2.2. $4 \in (L(x) \cup L(y)) \setminus (L(x) \cap L(y))$.
We can assume $4 \in L(x)$ (the case $4 \in L(y)$ is similar). So $L(y) \cap \{1,2\} \neq \emptyset$. Define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 3$, $c'(d) = c'(f) = 5$, $c'(x) = 4$, $c'(y) \in \{1,2\}$, $c'(z) = c'(t) = 6$ and $c'(e) \in \{1,2\} \setminus \{c'(y)\}$.

1.2.3. $4 \notin L(x) \cup L(y)$.
Then $L(x) \cap L(y) \cap \{1,2,3\} \neq \emptyset$. If $L(x) \cap L(y) \cap \{1,2\} \neq \emptyset$, we obtain a new L-coloring c' as follows: $c'(a) = c'(b) = 3$, $c'(d) = c'(f) = 5$, $c'(x) = c'(y) \in \{1,2\}$, $c'(e) \in \{1,2\} \setminus \{c'(x)\}$ and $c'(z) = c'(t) = 6$.
Now let $3 \in L(x) \cap L(y)$. If $4 \in L(z) \cap L(t)$ we use an L-coloring similar to that given in Subcase 1.2.1. Let $4 \in L(z)$ and $4 \notin L(t)$ (the case $4 \notin L(z)$ and $4 \in L(t)$ is similar). This forces $L(t) \cap \{2,3\} \neq \emptyset$. If $2 \in L(t)$, we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) = 3$, $c'(d) = 6$, $c'(e) = c'(f) = 1$, $c'(x) = c'(y) = 5$, $c'(z) = 4$ and $c'(t) = 2$.
If $3 \in L(t)$, then $c'(v) \in \{3,4\}$ for each $v \in \{x,y,z,t\}$. Define a new L-coloring $c' \neq c$ by: $c'(a) \in \{5,6\}$, $c'(b) = 2$, $c'(d) \in \{5,6\} \setminus \{c'(a)\}$, $c'(e) = c'(f) = 1$. and $c'(u) \in \{3,4\}$ for $u \in \{x,y,z,t\}$.
Finally, if $4 \notin L(z) \cup L(t)$ then $L(v) \cap \{1,2\} \neq \emptyset$ for $v \in \{x,y,z,t\}$. On the other hand $L(b) \cap \{4,5,6\} \neq \emptyset$. Now if $L(b) \cap \{5,6\} \neq \emptyset$ then a new L-coloring $c' \neq c$ is defined by: $\{c'(a), c'(b)\} \subseteq \{5,6\}$, $c'(u) = c(u)$ for $u \in \{d,e,f\}$ and $c'(v) \in \{1,2\}$ for $v \in \{x,y,z,t\}$. If $4 \in L(b)$ then a new L-coloring $c' \neq c$ is defined by: $c'(a) = 1$, $c'(b) = 4$, $c'(d) = c'(f) = 5$, $c'(e) = 2$, $c'(x) = c'(y) = 3$ and $c'(z) = c'(t) = 6$.

Case 2. $\{5,6\} \subseteq L(f)$.
In this case we have $L(f) = \{4,5,6\}$. If $L(e) \cap \{5,6\} \neq \emptyset$, we take the new L-coloring $c' \neq c$ given in Subcase 1.1. If $L(e) \cap \{5,6\} = \emptyset$ then we have $L(e) = \{1,2,3\}$. Now if $||[5]|| = ||[6]|| = 2$, then new L-colorings can

be obtained similar to those described in Subcases 1.2.1, 1.2.2 and 1.2.3. Finally, let $|[5]| = 3$ (the case $|[6]| = 3$, is similar). If $4 \in L(t)$ then the result follows by Lemma 2. If $4 \notin L(t)$, then $L(t) \cap \{1,2\} \neq \emptyset$. Now we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) = 3$, $c'(d) = c'(f) = 6$, $c'(x) = c'(y) = c'(z) = 5$, $c'(t) \in \{1,2\}$ and $c'(e) \in \{1,2\} \setminus \{c'(t)\}$.

Case 3. $L(f) \cap \{5,6\} = \emptyset$.
Then $L(f) = \{1,2,4\}$. Consider two subcases.

Subcase 3.1. $L(e) \cap \{5,6\} \neq \emptyset$.
We assume $5 \in L(e)$ (the case $6 \in L(e)$ is similar). If $|[5]| = |[6]| = 2$, we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) = 3$, $c'(d) = c'(e) = 5$, $c'(z) = c'(t) = 6$, $\{c'(x), c'(y)\} \subseteq \{1,2,4\}$ and $c'(f) \in L(f) \setminus \{c'(x), c'(y)\}$.
Now let $|[5]| = 3$ (the case $|[6]| = 3$ is similar). If $3 \in L(x) \cap L(y) \cap L(z)$, we obtain a new L-coloring c' as follows: $c'(a) = 1$, $c'(b) = 2$, $c'(d) = c'(e) = 5$, $c'(f) = 4$, $c'(t) = 6$ and $c'(x) = c'(y) = c'(z) = 3$.
If $3 \notin L(x) \cap L(y) \cap L(z)$, we first color the vertices x, y and z with at most two of the three colors 1, 2 and 4. Then we color the other vertices as follows: $c'(a) = c'(b) = 3$, $c'(d) = c'(e) = 5$, $c'(t) = 6$ and $c'(f) \in L(f) \setminus \{c'(x), c'(y), c'(z)\}$.

Subcase 3.2. $L(e) \cap \{5,6\} = \emptyset$.
Then we have $L(e) = \{1,2,3\}$. If $|[5]| = |[6]| = 2$, then new L-colorings are similar to those described in Subcases 1.2.1, 1.2.2. and 1.2.3. If $|[5]| = 3$ (the case $|[6]| = 3$ is similar) we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) = 3$, $c'(d) = 6$, $c'(x) = c'(y) = c'(z) = 5$ and $c'(t) \in \{1,2,4\}$, $c'(e) = c'(f) \in \{1,2\} \setminus \{c'(t)\}$. ■

Lemma 14 *If* $L(d) \cap L(e) \cap L(f) \cap \{1,2\} \neq \emptyset$, *then there exist a new* L*-coloring* $c' \neq c$ *for* $K_{2,3,4}$.

Proof. Without loss of generality, we may assume that $1 \in L(d) \cap L(e) \cap L(f)$. If $L(a) \cap \{3,4\} \neq \emptyset$, the result follows by Lemma 2. Otherwise, $L(a) = \{1,5,6\}$. If $L(b) \cap \{5,6\} \neq \emptyset$, we define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) \in \{5,6\}$, $c'(d) = c'(e) = c'(f) = 1$, $c'(v) \in \{2,3,4\}$ for each $v \in \{x,y,z,t\}$.
Now let $L(b) \cap \{5,6\} = \emptyset$. If color 1 appears in at least three color lists of the vertices x, y, z, t then the result follows by Lemma 2. Otherwise, we can color the vertices x, y, z, t with at most two of the three colors 2, 3 and 4, and use a remaining color for vertex b. Then we color vertex a with 5 and vertices d, e, f with 1. This completes the proof. ■

Lemma 15 *If $L(a) \cap L(b) \cap \{5,6\} \neq \emptyset$, then there exists a new L-coloring $c' \neq c$ for $K_{2,3,4}$.*

Proof. Without loss of generality, we may assume that $5 \in L(a) \cap L(b)$. If $L(a) \cap L(b) \cap \{3,4\} \neq \emptyset$, the result follows by Lemmas 12 or 13. If $L(d) \cap L(e) \cap L(f) \cap \{1,2\} \neq \emptyset$, the result follows by Lemma 14. If $6 \in L(f)$ a new L-coloring $c' \neq c$ can be defined as follows: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 3$, $c'(f) = 6$ and $c'(v) \in \{1,2,4\}$ if $v \in \{x,y,z,t\}$.
Now let $6 \notin L(f)$. We consider two cases.

Case 1. $5 \notin L(f)$.
This forces $L(f) = \{1,2,4\}$. If $||[5]|| = ||[6]|| = 2$, we define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 3$, $c'(z) = c'(t) = 6$ and $\{c'(x), c'(y), c'(f)\} \subseteq \{1,2,4\}$.
Now let $||[5]|| = 3$ (the case $||[6]|| = 3$ is similar). If $4 \in L(a) \cup L(b)$, the result follows by Lemma 2. If $4 \notin L(a) \cup L(b)$, then we have $6 \in L(a) \cup L(b)$. Let $6 \in L(a)$ (the case $6 \in L(b)$ is similar.) If $1 \in L(t)$, the result follows by Lemma 2. If $4 \in L(t)$, a new L-coloring $c' \neq c$ can be defined by: $c'(a) = 6$, $c'(b) = 2$, $c'(d) = c'(e) = 3$, $c'(f) = 1$, $c'(x) = c'(y) = c'(z) = 5$ and $c'(t) = 4$.
If $L(t) \cap \{1,4\} = \emptyset$ then $L(t) = \{2,3,6\}$. If $3 \in L(x) \cap L(y) \cap L(z)$, we define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 5$, $\{c'(d), c'(e), c'(f)\} \subseteq \{1,2,4,6\}$, and $c'(v) = 3$ if $v \in \{x,y,z,t\}$.
Finally, if $3 \notin L(x) \cap L(y) \cap L(z)$, we color the vertices x, y and z by at most two of the three colors 1, 2 and 4. Therefore, we can obtain a new L-coloring c' as follows: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 3$, $c'(t) = 6, \{c'(x), c(y), c'(z)\} \subseteq \{1,2,4\}$ and $c'(f) \in L(f) \setminus \{c'(x), c'(y), c'(z)\}$.

Case 2. $5 \in L(f)$.
Let $L(f) = \{1,4,5\}$ (the case $L(f) = \{2,4,5\}$ is similar). If $4 \in L(a)$, the result follows by Lemma 2. Let $4 \notin L(a)$ and consider the following three subcases.

Subcase 2.1. $\{5,6\} \subseteq L(d) \cap L(e)$.
We define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 6$, $c'(f) = 4$ and $c'(v) \in \{1,2,3\}$ if $v \in \{x,y,z,t\}$.

Subcase 2.2. $L(d) = \{3,5,6\}$ and $\{5,6\} \not\subseteq L(e)$. (The case $L(e) = \{3,5,6\}$ and $\{5,6\} \not\subseteq L(d)$ is similar.)
So we have $L(e) \cap \{1,2\} \neq \emptyset$. If $1 \in L(e)$, we define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 5$, $c'(d) = 6$, $c'(e) = c'(f) = 1$ and $c'(v) \in \{2,3,4\}$ if

$v \in \{x, y, z, t\}$.
If $1 \notin L(e)$ then $2 \in L(e)$ and $L(e) \cap \{5,6\} \neq \emptyset$. If $6 \in L(e)$ a new L-coloring $c' \neq c$ is defined by: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 6$, $c'(f) = 4$ and $c'(v) \in \{1,2,3\}$ if $v \in \{x, y, z, t\}$.
Now let $5 \in L(e)$ and consider the following two subcases.

2.2.1. $6 \in L(a) \cup L(b)$.
Let $6 \in L(a)$ (the case $6 \in L(b)$ is similar). Define a new L-coloring $c' \neq c$ by: $c'(a) = 6$, $c'(b) = 2$, $c'(d) = c'(e) = c'(f) = 5$ and $c'(v) \in \{1,3,4\}$ if $v \in \{x, y, z, t\}$.

2.2.2. $6 \notin L(a) \cup L(b)$.
Then $L(a) \cap \{3,4\} \neq \emptyset$ and $L(b) \cap \{3,4\} \neq \emptyset$. This forces $L(a) = \{1,3,5\}$, $L(b) = \{2,4,5\}$, $L(d) = \{3,5,6\}$, $L(e) = \{2,3,5\}$ and $L(f) = \{1,4,5\}$. Now one can apply Lemma 11 to find a new L-coloring $c' \neq c$ for $K_{2,3,4}$.

Subcase 2.3. $|L(d) \cap \{5,6\}| \leq 1$ and $|L(e) \cap \{5,6\}| \leq 1$.
If $6 \notin L(a) \cup L(b)$ then $L(v) \cap \{3,4\} \neq \emptyset$ for $v \in \{a, b\}$. Define a new L-coloring $c' \neq c$ by: $\{c'(a), c'(b)\} \subseteq \{3,4\}$, $\{c'(d), c'(e), c'(f)\} \subseteq \{1,2\}$ and $c'(u) = c(u)$ for $u \in \{x, y, z, t\}$.
Now assume $6 \in L(a)$ (the case $6 \in L(b)$ is similar). We leave for the reader to find a new L-coloring $c' \neq c$ for $K_{2,3,4}$ if $L(x) \cap L(y) \cap L(z) \cap L(t) = 2$. Now let $L(x) \cap L(y) \cap L(z) \cap L(t) \neq 2$. By Lemma 11 there exist 2-subsets $F_i \subseteq \{1,2,3,4\}$, $i = 1,2$, such that $L(v) \cap F_i \neq \emptyset$ for $v \in \{x, y, z, t\}$. It is straightforward to define a new L-coloring $c' \neq c$ for $K_{2,3,4}$ when $F_i = \{1,2\}$, $F_i = \{1,4\}$, $F_i = \{2,4\}$ or $F_i = \{3,4\}$ for some $i \in \{1,2\}$. Finally, let $F_1 = \{1,3\}$ and $F_2 = \{2,3\}$. We consider two subcases.

2.3.1. $2 \in L(d) \cap L(e)$.
Define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 2$, $c'(f) = 4$ and $c'(v) \in \{1,3\}$ for $v \in \{x, y, z, t\}$.

2.3.2. $2 \notin L(d) \cap L(e)$.
We can assume $2 \in L(d)$ and $1 \in L(e)$. (The case $1 \in L(d)$ and $2 \in L(e)$ is similar.) If $6 \in L(d) \cap L(e)$ we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) = 5$, $c'(d) = c'(e) = 6$, $c'(f) = 4$ and $c'(v) \in \{1,3\}$ for $v \in \{x, y, z, t\}$.
If $6 \in L(d)$ and $5 \in L(e)$, we define a new L-coloring $c' \neq c$ by: $c'(a) = c'(b) = 5$, $c'(d) = 6$, $c'(e) = 1$, $c'(f) = 4$ and $c'(u) \in \{2,3\}$ for $u \in \{x, y, z, t\}$.
If $6 \in L(e)$ we obtain a new L-coloring $c' \neq c$ as follows: $c'(a) = c'(b) = 5$, $c'(d) = 2$, $c'(e) = 6$, $c'(f) = 4$ and $c'(v) \in \{1,3\}$ for $v \in \{x, y, z, t\}$.

If $6 \notin L(d) \cup L(e)$ then $5 \in L(d) \cap L(e)$. Define a new L-coloring $c' \neq c$ by: $c'(a) = 6$, $c'(b) = 2$, $c'(d) = c'(e) = c'(f) = 5$ and $c'(v) \in \{1,3\}$ for $v \in \{x,y,z,t\}$. ■

Lemma 16 *If $L(d) \cap L(e) \cap L(f) \cap \{5,6\} \neq \emptyset$, then there exist a new L-coloring $c' \neq c$ for $K_{2,3,4}$.*

Proof. Without loss of generality we assume $5 \in L(d) \cap L(e) \cap L(f)$. If $L(a) \cap L(b) \cap \{3,4,5,6\} \neq \emptyset$, the result follows by Lemma 12, 13 or 15. Otherwise, $6 \in L(a) \cup L(b)$. Let $6 \in L(a)$ (the case $6 \in L(b)$ is similar). Define a new L-coloring $c' \neq c$ by: $c'(a) = 6$, $c'(b) = 2$, $c'(d) = c'(e) = c'(f) = 5$ and $c'(v) \in \{1,3,4\}$ for $v \in \{x,y,z,t\}$. ■

Lemma 17 *Let $L(a) = \{1,5,6\}$ or $L(b) = \{2,5,6\}$. If $L(a) \cap L(b) = \emptyset$ and $L(d) \cap L(e) \cap L(f) = \emptyset$ then there exists a new L-coloring $c' \neq c$ for $K_{2,3,4}$.*

Proof. Let $L(a) = \{1,5,6\}$ (the case $L(b) = \{2,5,6\}$ is similar). By the assumptions we have $L(b) = \{2,3,4\}$. If $L(d) = L(e)$ or $2 \in L(f)$, the result follows by Lemma 2. Otherwise, $L(d) \neq L(e)$ and $L(f) \cap \{5,6\} \neq \emptyset$. Let $5 \in L(f)$ (the case $6 \in L(f)$ is similar). If color 1 appears in at least three color lists of the vertices x, y, z, t the result follows by Lemma 2. Otherwise, the vertices x, y, z, t can be colored with at most two of the three colors 2, 3 and 4 and a remaining color is used for vertex b. In order to color the other vertices we consider two cases:

Case 1. $L(v) \cap \{5,6\} \neq \emptyset$ for $v \in \{d,e\}$.
We use color 1 for vertex a and colors 5, 6 for vertices d, e and f.

Case 2. Only one of the relations $L(d) \cap \{5,6\} \neq \emptyset$ or $L(e) \cap \{5,6\} \neq \emptyset$ holds.
Let $L(d) \cap \{5,6\} \neq \emptyset$ (the other case is similar). This forces $L(e) = \{1,2,3\}$. If $5 \in L(d)$, we color vertex a with 6, vertices d and f with 5 and vertex e with 1. If $6 \in L(d)$ since $L(f) \cap \{1,6\} \neq \emptyset$ we can use color 5 for vertex a and colors 1 and 6 for vertices d, e and f. ■

Lemma 18 *Let $|L(v) \cap \{5,6\}| = 2$ for some $v \in \{d,e,f\}$, $L(a) \cap L(b) = \emptyset$ and $L(d) \cap L(e) \cap L(f) = \emptyset$. Then there exists a new L-coloring $c' \neq c$ for $K_{2,3,4}$.*

Proof. If $L(a) = \{1,5,6\}$ or $L(b) = \{2,5,6\}$, the result follows by Lemma 17. So let $L(u) \cap \{5,6\} \neq \emptyset$ and $L(u) \cap \{3,4\} \neq \emptyset$ for $u \in \{a,b\}$. Without loss of generality we may assume $L(a) = \{1,3,5\}$ and $L(b) = \{2,4,6\}$. If $L(d) = L(e)$ or $2 \in L(f)$, the result follows by Lemma 2. Now let $L(d) \neq L(e)$ and $2 \notin L(f)$. This forces $L(f) \cap \{5,6\} \neq \emptyset$. Let $5 \in L(f)$ (the case $6 \in L(f)$ is similar). By the assumptions we have $L(d) = \{3,5,6\}$, $L(e) = \{3,5,6\}$ or $L(f) = \{4,5,6\}$. First let $L(d) = \{3,5,6\}$ (the case $L(e) = \{3,5,6\}$ is similar). Since $L(d) \cap L(e) \cap L(f) = \emptyset$ it follows that $5 \notin L(e)$ and so $L(e) \cap \{1,2\} \neq \emptyset$. Let $2 \in L(e)$ (the case $1 \in L(e)$ is similar). Using Lemma 11 it is now straightforward to define a new L coloring $c' \neq c$ for $K_{2,3,4}$.
Finally, let $L(f) = \{4,5,6\}$. Since $L(d) \neq L(e)$, we can assume $L(d) \cap \{1,2\} \neq \emptyset$ and $L(e) \cap \{5,6\} \neq \emptyset$. Let $1 \in L(d)$ and $5 \in L(e)$ (the other cases are similar). We leave for the reader to find a new L-coloring $c' \neq c$ for $K_{2,3,4}$ if $L(x) \cap L(y) \cap L(z) \cap L(t) = 2$. Now let $L(x) \cap L(y) \cap L(z) \cap L(t) \neq 2$. By Lemma 11 there exist 2-subsets $F_i \subseteq \{1,2,3,4\}$, $i = 1,2$, such that $L(v) \cap F_i \neq \emptyset$ for $v \in \{x,y,z,t\}$. It is straightforward to define a new L-coloring $c' \neq c$ for $K_{2,3,4}$ when $F_i = \{1,2\}$, $F_i = \{1,4\}$ or $F_i = \{2,4\}$ for some $i \in \{1,2\}$. Now let $\{F_1, F_2\} \subset \{\{1,3\},\{2,3\},\{3,4\}\}$. We consider two cases:

Case 1. $L(d) \cap L(e) \cap \{1,2\} \neq \emptyset$.
If $1 \in L(d) \cap L(e)$ the result follows by Lemma 2. So let $2 \in L(d) \cap L(e)$. If $F_i = \{1,3\}$ for some $i = 1,2$, we define a new L-coloring $c' \neq c$ by: $c'(a) = 5$, $c'(b) = 6$, $c'(d) = c'(e) = 2$, $c'(f) = 4$ and $c'(u) \in \{1,3\}$ for $u \in \{x,y,z,t\}$.
If $F_i = \{3,4\}$ for some $i = 1,2$, we define a new L-coloring $c' \neq c$ by: $c'(a) = 1$, $c'(b) = 6$, $c'(d) = c'(e) = 2$, $c'(f) = 5$ and $c'(u) \in \{3,4\}$ for $u \in \{x,y,z,t\}$.

Case 2. $L(d) \cap L(e) \cap \{1,2\} = \emptyset$.
So $L(d) \cap \{5,6\} \neq \emptyset$ and $L(e) \cap \{5,6\} \neq \emptyset$. If $F_i = \{2,3\}$ for some $i = 1,2$, we define a new L-coloring $c' \neq c$ by: $c'(a) = 1$, $c'(b) = 4$, $\{c'(d), c'(e), c'(f)\} \subseteq \{5,6\}$ and $c'(v) \in \{2,3\}$ for $v \in \{x,y,z,t\}$.
If $F_i = \{3,4\}$ for some $i = 1,2$, we define a new L-coloring $c' \neq c$ by: $c'(a) = 1$, $c'(b) = 2$, $\{c'(d), c'(e), c'(f)\} \subseteq \{5,6\}$ and $c'(v) \in \{3,4\}$ for $v \in \{x,y,z,t\}$. ■

We are now ready to prove the main result of this section.

Theorem 19 *The graph $K_{2,3,4}$ has the property $M(3)$.*

Proof. Let $\{a, b\}, \{d, e, f\}$ and $\{x, y, z, t\}$ be the partite sets of the graph $G = K_{2,3,4}$. Let L be a 3-list assignment to the vertices of G such that an L-coloring c exists for G. By Lemma 1 we can assume L has Property 3. Moreover, if x and y are two vertices in the same part such that $c(x) \neq c(y)$ and $c(x) \in L(y)$ then we define a new L-coloring $c' \neq c$ by: $c'(u) = c(u)$ if $u \neq y$ and $c'(y) = c(x)$. Therefore, we can also assume L has Property 2. Now we consider two cases.

Case 1. All vertices of a part of $K_{2,3,4}$ have the same color t in c.
If we remove the vertices of this part the remaining graph H is a complete bipartite graph. Define $L'(u) = L(u) \setminus \{t\}$ for $u \in V(H)$. Since the restriction of c on $V(H)$ is an L'-coloring for H, by Theorem A there exists a new L'-coloring c' for H. Obviously, c' is extendible to $K_{2,3,4}$.

Case 2. Each part has at least two colors in c.
Let $c(i, j, k)$ denote an L-coloring which has i colors in partite set $\{a, b\}$, j colors in partite set $\{d, e, f\}$ and k colors in partite set $\{x, y, z, t\}$. We consider three subcases.

Subcase 2.1. The L-coloring is of the form $c(2, 3, 4)$, $c(2, 3, 3)$ or $c(2, 2, 4)$. Add a new edge between any two vertices in the same partite set with different colors in c. The resulting graph is K_9 or $K_9 \setminus e$. Since both K_9 and $K_9 \setminus e$ have the property $M(3)$ (see [4]) we obtain a new L-coloring $c' \neq c$ for $K_{2,3,4}$.

Suncase 2.2. The L coloring is of the form $c(2, 2, 2)$.
If $L(a) \cap L(b) \neq \emptyset$ we apply Lemmas 12, 13 or 15. If $L(d) \cap L(e) \cap L(f) \neq \emptyset$ we apply Lemmas 14 or 16. Otherwise, if $|L(u) \cap \{5, 6\}| = 2$ for some $u \in \{a, b, d, e, f\}$ we apply Lemma 17 or Lemma 18. Finally, let $|L(u) \cap \{5, 6\}| \leq 1$ for $u \in \{a, b, d, e, f\}$. Remove the vertices x, y, z, t from $K_{2,3,4}$ and remove colors $5, 6$ from $L(v)$ for $v \in \{a, b, d, e, f\}$. Then the resulting graph is the complete bipartite graph $K_{2,3}$. This graph has the property $M(2)$ by Theorem A. Obviously, this coloring is extendible to $K_{2,3,4}$.

Subcase 2.3. The L-coloring is of the form $c(2, 3, 2)$ or $c(2, 2, 3)$.
A method similar to that described in Case 2 can be applied for these cases.
■

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