On Closed and Upper Closed Geodetic Numbers of Graphs

Ferdinand P. Jamil
Department of Mathematics
MSU - Marawi
Marawi City

Imelda S. Aniversario *
Department of Mathematics
MSU - IIT
9200 Iligan City
e-mail: imeldaaniversario@yahoo.com

Sergio R. Canoy, Jr.
Department of Mathematics
MSU - IIT
9200 Iligan City

ABSTRACT

Let G be a connected graph. For $S \subseteq V(G)$, the geodetic closure $I_G[S]$ of S is the set of all vertices on geodesics (shortest paths) between two vertices of S. We select vertices of G sequentially as follows: Select a vertex v_1 and let $S_1 = \{v_1\}$. Select a vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$. Then successively select vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, \ldots, v_i\}$. We define the *closed geodetic number* (resp. *upper closed geodetic number*) of G, denoted $cgn(G)$ (resp. $ucgn(G)$), to be the smallest (resp. largest) k whose selection of v_k in the given manner yields $I_G[S_k] = V(G)$. In this paper, we show that for every pair a, b of positive integers with $2 \leq a \leq b$, there always exists a connected graph G such that $cgn(G) = a$ and $ucgn(G) = b$, and if $a < b$, the minimum order of such graph G is b. We characterize those connected graphs G with the property: If $cgn(G) < k < ucgn(G) = b$, then there is a selection of vertices v_1, v_2, ..., v_k as in the above manner

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such that $I_G[S_k] = V(G)$. We also determine the closed and upper closed geodetic numbers of some special graphs and the joins of connected graphs.

1 Introduction

Let G be a connected graph. A $u-v$ *geodesic*, for vertices u and v in G, is any shortest path in G joining u and v. The length of a $u-v$ geodesic is called the *distance* $d_G(u,v)$ between u and v. For a vertex v of G, the *eccentricity* $e_G(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* $rad(G)$ and the maximum eccentricity is its *diameter* $diam(G)$. We denote by $I_G[u,v]$ the set of all vertices lying on any of the $u-v$ geodesics. The set of vertices in the interior of a $u-v$ geodesic is a *geodetic interior.* If $S \subseteq V(G)$, we define the *closure* of S to be the set $I_G[S]$ given by $I_G[S] = \cup\{I_G[u,v] : u, v \in S\}$. By a *geodetic cover* of G we mean a subset S of $V(G)$ such that the geodetic closure $I_G[S]$ is $V(G)$, i.e., $I_G[S] = V(G)$. The number $gn(G)$ given by $gn(G) = min\{|S| : I_G[S] = V(G)\}$ is called the *geodetic number* of G. Such geodetic cover of G that determines the number $gn(G)$ is called *minimum geodetic cover or basis* of G. We refer to [2] - [7] for concepts and results on geodetic numbers.

Let G be a connected graph. A subset S of $V(G)$ is a *closed geodetic* subset if $S = \emptyset$ or there is a positive integer k and a sequence of sets $S_1 = \{v_1\}$, $S_2 = \{v_1, v_2\}$, ..., $S_k = \{v_1, v_2, \ldots, v_k\}$ such that $S_k = S$ and $v_i \notin I_G[S_{i-1}]$ for all $i = 2, 3, 4, \ldots, k$. A closed geodetic subset S of $V(G)$ is a *closed geodetic cover* of G if $V(G)$ is a geodetic closure of S. If S is a closed geodetic cover of G and S_1, S_2, ..., S_k are the sets described above, then we refer to the set $S_k = S$ as a *canonical representation of* S. We denote by $C^*(G)$ the set of all closed geodetic covers of G. The *closed geodetic number of* G, denoted $cgn(G)$, is defined to be

$$cgn(G) = min\{|S| : S \in C^*(G)\}.$$

The *upper closed geodetic number of* G, denoted $ucgn(G)$, is defined as

$$ucgn(G) = max\{|S| : S \in C^*(G)\}.$$

A set $S \in C^*(G)$ with $|S| = cgn(G)$ is called a *closed geodetic basis* of G. A set $S \in C^*(G)$ with $|S| = ucgn(G)$ is called a *maximum closed geodetic cover* of G.

The idea of closed geodetic number comes from two classes of graphical games called *achievement and avoidance games* presented by Harary in

[10]. These games were examined for the geodetic number by Buckley and Harary, and by Necaskova in [11]. A study on closed geodetic numbers is done in [1]. Among the results obtained in [1] is the characterization of connected graphs G for which $cgn(G) = p, p-1$, 2 or 3. It is also shown that for any positive integers k and n for which $4 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there always exists a connected graph G where $|V(G)| = n$, $gn(G) = 4$ and $cgn(G) = k$. And for integers n, m and k with $5 \leq m \leq k$ and $2k-m+4 \leq n$, there exists a (connected) graph G such that $|V(G)| = n$, $gn(G) = m$ and $cgn(G) = k$.

In this paper, the authors consider the upper closed geodetic number of a graph. The reader can easily check that $ucgn(K_p) = p$, $ucgn(P_n) = n$, $ucgn(C_n) = \lceil \frac{n}{2} \rceil + 1$, and if $G = K_1 + \bigcup m_j K_j$, where $2 \leq \sum m_j$, then $ucgn(G) = |V(G)|$. It is worth noting that $cgn(G) \leq ucgn(G) \leq |V(G)|$ for any connected graph G.

2 Connected Graphs

Theorem 2.1 *Let G be a connected graph. Then $cgn(G) = ucgn(G)$ if and only if $G = K_p$.*

Proof. It is known (see [1]) that if $G = K_p$, then $cgn(G) = p$. Thus $cgn(G) = ucgn(G)$. Conversely, suppose that $G \neq K_p$. Then there exist vertices u and v in $V(G)$ such that $d_G(u,v) = 2$. Let w be an interior vertex in a $u-v$-geodesic. Let $v_1 = u$, and let $v_2 = v$. Put $S_1 = \{v_1\}$, and $S_2 = \{v_1, v_2\}$. For $i \geq 3$, we choose $v_i \in V(G) \setminus I_G[S_{i-1}]$ such that

$$d_G(w, v_i) = min\{d_G(w,x) : x \in V(G) \setminus I_G[S_{i-1}]\}.$$

Let k be that positive integer with $v_k \notin I_G[S_{k-1}]$ and $I_G[S_k] = V(G)$. Then $cgn(G) \leq k$.

Define a new sequence $S'_1 = \{u_1\}$, $S'_2 = \{u_1, u_2\}$, ..., $S'_{k+1} = \{u_1, u_2, \ldots, u_{k+1}\}$, where $u_1 = v_1$, $u_2 = w$, and $u_i = v_{i-1}$ for $i = 3, \ldots, k+1$. Suppose that there exist i, j, and l with $1 \leq i < j < l \leq k+1$ such that $u_l \in I_G[u_i, u_j]$. Then $u_i = w$, $u_j = v_{j-1}$ and $u_l = v_{l-1}$, and $d_G(w, v_{j-1}) = d_G(w, v_{l-1}) + d_G(v_{l-1}, v_{j-1})$. However, by the above construction, $d_G(w, v_{j-1}) \leq d_G(w, v_{l-1})$, a contradiction. Thus, $u_i \notin I_G[S'_{i-1}]$ for all $i = 1, 2, \ldots, k+1$, and $S'_{k+1} \in C^*(G)$. Thus, $cgn(G) \leq k < k+1 \leq ucgn(G)$. ∎

It is known (see [1]) for a connected graph G of order p that, $cgn(G) = p$ if and only if $G = K_p$ [1]. If $|V(G)| = p \geq 3$, then $ucgn(G) = p$ if and only if $G = K_p$ or $G = P_p$. Our first attempt in this paper is to characterize connected graphs G of order $p \geq 4$ with $ucgn(G) = p$.

Theorem 2.2 *Let G be a connected graph of order $n \geq 4$. If there exists $A \subsetneq V(G)$ such that the induced subgraph $\langle A \rangle$ is a cycle of order $k \geq 4$, then $ucgn(G) < n$.*

Proof. Suppose that there exists $A \subsetneq V(G)$ with $\langle A \rangle = C_k$ where $k \geq 4$. Let $S \in C^*(G)$ with a canonical representation $S = \{v_1, v_2, \ldots, v_m\}$. Suppose that $A \subseteq S$. Let $j = max\{i : v_i \in A\}$. Since $\langle A \rangle$ is a cycle, there exist vertices $u, w \in A$ such that $d_G(u,w) = 2$ and $v_j \in I_G[u,w]$. However, for some integers r and s, $1 \leq r, s \leq m$, $u = v_r$ and $w = v_s$. By definition of j, we have $r, s < j$. This means that $v_j \in I_G[S_{j-1}]$, a contradiction. Therefore, $|A \cap S| < k$, and by the arbitrary nature of S, $ucgn(G) < n$. ■

A vertex v in a connected graph G is an *extreme vertex* if the neighborhood $N(v) = \{u \in V(G) : d_G(u,v) = 1\}$ of v induces a complete subgraph of G. The set of all extreme vertices in G is denoted by $Ext(G)$.

Accordingly, every minimum geodetic cover of a connected graph contains its extreme vertices [6]. This is, in fact, true for non-minimum geodetic covers, and follows directly from the fact that an extreme vertex v is either an initial or terminal vertex of any geodesic containing v.

Theorem 2.3 [7] *Every geodetic cover of a connected graph G contains all its extreme vertices.*

Let G be a connected graph. The symbol G' denotes the resulting subgraph of G after removing all extreme vertices of G. For $k \geq 2$, the symbol $G^{(k)}$ denotes the resulting subgraph of $G^{(k-1)}$ after removing all extreme vertices of $G^{(k-1)}$. For convenience, we also write $G^0 = G$. We remark that there exists a nonnegative integer k such that either $G^{(k)} = K_p$ or $Ext(G^{(k)}) = \emptyset$. Note that if $Ext(G^{(k)}) = \emptyset$, then $G^{(k)} = G^{(n)}$ for all $n \geq k$. Let $\rho(G) = min\{k : G^{(k)} = K_p \;\; or \;\; Ext(G^{(k)}) = \emptyset\}$. We call $\rho(G)$ the *extremity number* of G.

Lemma 2.4 *Let G be a connected graph, and let $u \in Ext(G^{(i)})$ and $v \in Ext(G^{(j)})$ for some nonnegative integers i and j. If w is an interior vertex in $I_G[u,v]$, then $w \in V(G^{(k)})$, for some $k > i, j$.*

Theorem 2.5 *Let G be a connected graph of order $n \geq 4$. Then $ucgn(G) = n$ if and only if $G^{(\rho(G))}$ is complete.*

Proof. Suppose that $G^{(\rho(G))}$ is complete. If $\rho(G) = 0$, then G is complete; hence, $ucgn(G) = n$. Suppose that $\rho(G) > 0$. Then the sets $Ext(G^0)$, $Ext(G')$, $Ext(G'')$, …, $Ext(G^{(\rho(G))})$ are pairwise disjoint. In fact, $\bigcup_{i=0}^{\rho(G)} Ext(G^{(i)}) = V(G)$. Put $S = \{v_1, v_2, \ldots, v_n\}$, where the first

$|Ext(G^{(\rho(G))})|$ are exactly the vertices in $Ext(G^{(\rho(G))})$; the next $|Ext(G^{(\rho(G)-1)})|$ vertices in S are exactly the vertices in $Ext(G^{(\rho(G)-1)})$; and so on, and finally, the last $|Ext(G^0)|$ vertices in S are exactly the vertices in $Ext(G^0)$. By Lemma 2.4, $S \in C^*(G)$. Thus $ucgn(G) = n$.

Conversely, suppose that $G^{(\rho(G))}$ is not a complete graph. Then $Ext(G^{(\rho(G))}) = \emptyset$. Thus, $G^{(\rho(G))}$, and consequently the graph G, contains an induced cycle C_m, $m \geq 4$. By Theorem 2.2, $ucgn(G) < n$. ■

A graph G is an *extreme geodesic graph* if every vertex of G lies on some $u-v$ geodesic in G where u and v are extreme vertices of G. More precisely, G is an extreme geodesic graph if and only if $Ext(G)$ is a geodetic basis of G.

In view of the last statement in Lemma 2.4, if G is extreme geodesic, then $Ext(G)$ is a closed geodetic cover of G.

Theorem 2.6 *Let G be a noncomplete extreme geodesic graph and $S \subseteq V(G)$. Then $S \in C^*(G)$ if and only if $S = Ext(G) \cup A \cup B$, where $A \subseteq \bigcup_{i=1}^{\rho(G)-1} Ext(G^{(i)})$ and B is a closed geodetic subset of $V(G^{(\rho(G))})$.*

Proof. Suppose that $S \in C^*(G)$. By Lemma 2.3, $Ext(G) \subseteq S$. Since G is noncomplete, $V(G) \setminus Ext(G) \neq \emptyset$. We write $S \setminus Ext(G) = A \cup B$, where $A \subseteq \bigcup_{i=1}^{k-1} Ext(G^{(i)})$ and B a subset of $V(G^{(\rho(G))})$. Since $S \in C^*(G)$, B is a closed geodetic subset of $V(G^{(k)})$.

Conversely, suppose that $S = Ext(G) \cup A \cup B$, where $A \subseteq \bigcup_{i=1}^{\rho(G)-1} Ext(G^{(i)})$ and B is a closed geodetic subset of $V(G^{(\rho(G))})$. By an extreme geodesic graph, we have $V(G) = I_G[Ext(G)] \subseteq I_G[S]$, and hence $V(G) = I_G[S]$. To proceed on, we may assume that both A and B are nonempty. We note that a geodesic in $G^{(\rho(G))}$ is also a geodesic in G, and conversely. Thus, B is a closed geodetic subset of $V(G)$. Write $B = \{w_1, w_2, \ldots, w_j\}$ so that $w_l \notin I_G[w_i, w_{i'}]$ for all $i, i' < l$ for all $l = 3, 4, \ldots, j$. Let $k_1 < k_2 < \cdots < k_n \leq k - 1$ be the sequence of all positive integers such that $A \cap Ext(G^{(k_i)}) \neq \emptyset$, $i = 1, 2, \ldots, n$. Put $S = \{v_1, v_2, \ldots, v_k\}$, where the first j vertices are exactly the vertices $w_1, w_2, \ldots, w_j$ in B; the next $|A \cap Ext(G^{(k_{n-1})})|$ vertices being the vertices in $A \cap Ext(G^{(k_{n-1})})$; and so on. And finally, after considering all vertices in $\bigcup_{j=1}^{n}(A \cap Ext(G^{(k_j)}))$, we complete S with the vertices in $Ext(G)$. By Lemma 2.4, $S \in C^*(G)$. ■

Corollary 2.7 *Let G be a noncomplete extreme geodesic graph. Then $cgn(G) = |Ext(G)|$; $ucgn(G) = \sum_{i=0}^{\rho(G)-1} |Ext(G^{(i)})| + ucgn(G^{(\rho(G))})$. In particular, if $G^{(\rho(G))}$ is a complete graph, then $ucgn(G) = |V(G)|$.*

Corollary 2.8 *If T is a tree and $S \subseteq V(T)$, then $S \in C^*(T)$ if and only if $S = Ext(T) \cup A$ for some $A \subseteq V(T) \setminus Ext(T)$. Consequently, $cgn(T) = |Ext(T)|$ and $ucgn(T) = |V(T)|$.*

Proof. Let T be a tree, and $S \subseteq V(T)$. The statement is trivial if $T = P_1$ or P_2. Otherwise there exists a positive number k such that $T^{(k)} = K_1$ or K_2, and the subgraphs T^0, T', T'', ..., $T^{(k)}$ are distinct. By Theorem 2.6, $S \in C^*(T)$ if and only if $S = Ext(T) \cup A$, $A \subseteq V(T) \setminus Ext(T)$. Since $ucgn(T^{(k)}) = \left|V(T^{(k)})\right|$, Corollary 2.7 implies that $ucgn(T) = |V(T)|$. ■

Let $K_{p_1}, K_{p_2}, \ldots, K_{p_n}$ be complete graphs, each containing a complete subgraph K_r $(r \geq 1)$. The graph G obtained from the union of these n complete graphs by identifying the K_r's (one from each complete graph) in an arbitrary way is called the *K_r-gluing* of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n}. If G is a K_r-gluing of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n}, then $V(G)$ is the disjoint union of $Ext(G)$ and $V(K_r)$.

Lemma 2.9 [4] *If G is a K_r-gluing of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n}, then $Ext(G)$ is a geodetic cover of G.*

In view of Lemma 2.9, a K_r-gluing of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n} is an extreme geodesic graph.

Corollary 2.10 *Let r, p_1, p_2, ..., p_n be positive integers with $r < p_1 \leq p_2, \ldots \leq p_n$. Let G be a K_r-gluing of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n}, and let $S \subseteq V(G)$. Then $S \in C^*(G)$ if and only if $S = Ext(G) \cup A$ for some $A \subseteq V(K_r)$. Consequently, $cgn(G) = |Ext(G)|$ and $ucgn(G) = |V(G)|$.*

Proof. Let G be a K_r-gluing of $K_{p_1}, K_{p_2}, \ldots,$ and K_{p_n}, and $S \subseteq V(G)$. Then $G' = K_r$. By Theorem 2.6, $S \in C^*(G)$ if and only if $S = Ext(G) \cup A$ for some $A \subseteq V(K_r)$. By Corollary 2.7, $cgn(G) = |Ext(G)|$ and $ucgn(G) = |V(G)|$. ■

Lemma 2.11 *Let G be a connected graph with $|V(G)| \geq 4$, and let $S \subseteq V(G)$. If $S \in C^*(G)$, then no distinct vertices u, v, w, z in S satisfy $u, v \in I_G[w, z]$ and $w, z \in I_G[u, v]$.*

Let K_p be a complete graph of order $p \geq 3$ and Ω a family of complete proper subgraphs of K_p. We say that Ω is an *independent family* if no two distinct subgraphs in Ω have a common vertex. If Ω is an independent family of complete proper subgraphs of K_p, each of order at least 2, the graph G obtained from K_p by deleting the edges in Ω is denoted by

$$K_p \setminus E(\Omega),$$

and is the graph of order p with the property: $xy \in E(G)$ if and only if xy is not an edge in any subgraph in Ω.

Lemma 2.12 *Let K_p be a complete graph of order $p \geq 3$ and Ω an independent family of complete proper subgraphs of K_p, each of order at least 2. Let $G = K_p \setminus E(\Omega)$. If $K_r \in \Omega$ and $u, v \in V(K_r)$, then $w \in I_G[u, v]$ for all $w \in V(G) \setminus V(K_r)$.*

Theorem 2.13 *Let K_p be a complete graph of order $p \geq 3$ and Ω an independent family of complete proper subgraphs of K_p, each of order at least 2. Let $G = K_p \setminus E(\Omega)$ and let $S \subseteq V(G)$. Then $S \in C^*(G)$ if and only if either $S = V(K_r)$ for some $K_r \in \Omega$ or $S = V(K_r) \cup V(K_m)$ for some $K_r \in \Omega$ and some subgraph K_m of G with $V(K_r) \cap V(K_m) = \emptyset$.*

Proof. Let $S \in C^*(G)$. Since G is not complete, there exists a vertex $u, v \in S$ such that $d_G(u, v) = 2$. This means that, in particular, v is a vertex of some subgraph K_r in Ω. Let $A_v = \{u \in S : d_G(u, v) = 2\}$. Then $A_v \neq \emptyset$. We claim that $A_v \cup \{v\} = V(K_r)$. Since $d_G(v, x) = 1$ for all $x \in V(G) \setminus V(K_r)$, it follows that $A_v \cup \{v\} \subseteq V(K_r)$. On the other hand, suppose that $u \in V(K_r) \setminus \{v\}$. Then $d_G(u, v) = 2$. We will show that $u \in S$. Suppose that $u \notin S$. Then there exist $x, y \in S$ such that $d_G(x, y) = 2$ and $u \in I_G[x, y]$. Since xu and uy are edges of G, we have $x, y \notin V(K_r)$. Consequently, x and y are vertices of some complete subgraph of G in Ω other than K_r. By Lemma 2.12, in fact, $A_v \cup \{v\} \subseteq V(K_r) \subseteq I_G[x, y]$. However, Lemma 2.12 also implies that $x, y \in I_G[A_v \cup \{v\}]$. By Lemma 2.11, this is a contradiction. Thus $u \in S$. This implies that $u \in A_v$. Therefore, $V(K_r) = A_v \cup \{v\}$.

If $S = V(K_r)$ or $S = V(K_r) \cup V(K_1)$, then the desired conclusion already holds. Suppose that $S_0 = S \setminus V(K_r)$ is at least a doubleton. Let x, y be distinct vertices in S_0. By Lemma 2.12, for each $u \in A_v$, $x, y \in I_G[u, v]$. Now, if $d_G(x, y) = 2$, then $v, u \in I_G[x, y]$ for all $u \in A_v$. By Lemma 2.11, this is impossible since $S \in C^*(G)$. Thus, $d_G(x, y) = 1$. The arbitrary nature of x and y implies that $S_0 = V(K_m)$ for some subgraph K_m of G.

Conversely, suppose that $S = V(K_r) \cup V(K_m)$ for some $K_r \in \Omega$ and some subgraph K_m of G with $V(K_r) \cap V(K_m) = \emptyset$. We first claim that $V(K_r) \in C^*(G)$. Let $w \in V(G) \setminus V(K_r)$. Since no two distinct subgraphs in Ω have a common vertex, wv is not a edge of any subgraph other than K_r, for all $v \in V(K_r)$. Thus wv is an edge of G, for all $v \in V(K_r)$. Consequently, $w \in I_G[V(K_r)]$. This means that $V(G) = I_G[V(K_r)]$. Moreover, since the distance in G between any two distinct vertices in $V(K_p)$ is 2, it is impossible to have $w \in I_G[u, v]$ for distinct vertices $w, u, v \in V(K_r)$. Therefore, any way of enumerating the vertices in $V(K_r)$ yields $V(K_r) \in C^*(G)$.

Finally, write $V(K_r) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_m) = \{w_1, w_2, \ldots, w_l\}$. Put $S = \{v_1, v_2, \ldots, v_{m+l}\}$, where $v_i = w_i$ for $i = 1, 2, \ldots, l$ and

$v_{l+i} = u_i$ for $i = 1, 2, \ldots, n$. Then $S \in C^*(G)$. ■

Corollary 2.14 *Let K_p be a complete graph of order $p \geq 3$ and Ω an independent family of complete proper subgraphs of K_p, each of order at least 2. Let $G = K_p \setminus E(\Omega)$. Then $cgn(G) = min\{r : K_r \in \Omega\}$ and $ucgn(G) = max\{r + m(r) : K_r \in \Omega\}$, where $m(r) = max\{n : K_n$ is subgraph of G with $V(K_n) \cap V(K_r) = \emptyset\}$.*

Corollary 2.15 *$cgn(K_{m,n}) = min\{m, n\}$ and $ucgn(K_{m,n}) = 1+max\{m, n\}$ for $m, n \geq 2$.*

Proof. Suppose that $m, n \geq 2$. Let $G = K_{m+n} \setminus E(\Omega)$, where $\Omega = \{K_m, K_n\}$. Then $G = K_{m,n}$. By Corollary 2.14, $cgn(K_{m,n}) = min\{m, n\}$ and $ucgn(K_{m,n}) = 1 + max\{m, n\}$. ■

Theorem 2.16 *For any pair of positive integers m and n with $2 \leq m \leq n$, there exists a connected graph G such that $cgn(G) = m$ and $ucgn(G) = n$. Such graph G can be chosen such that $|V(G)| = n$ or $|V(G)| > n$.*

Proof. If $m = n$, then, by Theorem 2.1, K_m is the desired graph G. Suppose that $m < n$. We consider the graph G which is the K_{n-m}-gluing of the m copies of the complete graph K_{n-m+1}, or we take G being a tree T with n vertices and with m endvertices. In any case, $|V(G)| = n$ and, by Corollary 2.10 or Corollary 2.8, $cgn(G) = m$ and $ucgn(G) = n$.

We may also consider $G = K_{m,n-1}$. In this case, $|V(G)| = m + n - 1$, and by Corollary 2.15, $cgn(G) = m$ and $ucgn(G) = n$. ■

Corollary 2.17 *For any pair of positive integers m and n with $2 \leq m \leq n$, the smallest order of a graph G with $cgn(G) = m$ and $ucgn(G) = n$ is n.*

In Corollary 2.8 and Corollary 2.10, we find that if G is a tree or is a K_r- gluing of some complete graphs and if k is a positive integer such that $cgn(G) < k < ucgn(G)$, then there is an $S \in C^*(G)$ such that $|S| = k$. Any such property is being referred to as *Intermediate Value Property*. However, not all connected graphs possess such property. In particular, we consider $G = K_7 \setminus E(\Omega)$, where $\Omega = \{K_2, K_5\}$. From Theorem 2.13, we know that $cgn(G) = 2$ and $ucgn(G) = 6$, and no closed geodetic closure S of G with $|S| = 4$.

Theorem 2.18 *(Intermediate Value Theorem) Let G be a connected noncomplete graph. Then G possesses the Intermediate Value Property if and only if for each nonmaximum closed geodetic cover S of G there exists $S' \in C^*(G)$ such that $|S'| = 1 + |S|$.*

Proof. Suppose that G has the Intermediate Value Property. Let S be a nonmaximum closed geodetic cover of G with $|S| = k$. Then $cgn(G) \leq k \leq ucgn(G) - 1$. If $k < ucgn(G) - 1$, then by the Intermediate Value Property, there exists $S' \in C^*(G)$ such that $|S'| = k + 1$. If $k = ucgn(G) - 1$, then we take a maximum closed geodetic cover S' of G.

Conversely, suppose that $cgn(G) < k < ucgn(G)$. Let $m = max\{|S| \leq k : S \in C^*(G)\}$. By the hypothesis, there exists $S' \in C^*(G)$ such that $|S'| = m + 1$. By the definition of m, $m + 1 > k$. Consequently, $m = k$, and the conclusion follows. ∎

We note that, in general, $K_{m,n}$ $(m, n \geq 2)$ does not satisfy the condition in Theorem 2.18.

Corollary 2.19 *Let* $2 \leq m \leq n$. *Then* $K_{m,n}$ *possesses the Intermediate Value Property if and only if* $n - m \leq 2$.

3 Join of Graphs

In [1], the closed geodetic number of the join of two graphs were determined. In this present note, we characterize all closed geodetic covers of a join.

Let G be a connected graph. Let $S \subseteq V(G)$. The **2-path closure** $P_2[S]_G$ of S is that set $P_2[S]_G = S \cup \{w \in V(G) : w \in I_G[u, v]$ for some $u, v \in S \cap N(w)\}$. A set S is called **2-path closure absorbing** if $P_2[S]_G = V(G)$.

It is worth noting that a 2-path closure absorbing subset of the vertex set of a connected graph is a geodetic cover of the graph. In [1], the closed geodetic numbers of the join of graphs were determined.

Lemma 3.1 *If* G *is a connected graph and* $diam(G) = 2$, *then every geodetic cover of* G *is a 2-path closure absorbing set in* G.

Theorem 3.2 *Let* H *be a connected noncomplete graph, and* $G = H + K_p$. *Let* $S \subseteq V(G)$. *If* $S \in C^*(G)$, *then* $S \cap V(H) \in C^*(G)$ *and is a 2-path closure absorbing set in* H.

Proof. Let $S \in C^*(G)$. If $\langle S \rangle$ is complete, then $I_G[S] = S \neq V(G)$, a contradiction. Thus, there exist at least two distinct vertices u and v in S such that $d_G(u, v) = 2$. Clearly, $u, v \in V(H)$. Let $A = S \cap V(H)$. Then $V(K_p) \subset I_G[A]$. We claim that A is a 2-path closure absorbing in H. In view of Lemma 3.1, S is a 2-path closure absorbing set in G. Let $w \in V(H) \setminus A$. Then $w \in V(G) \setminus S$. Thus, there exist $u, v \in S$ such that

$d_G(u,v) = 2$ and $w \in I_G[u,v]$. Incidentally, $u, v \in A$. If $[u,w,v]$ is a $u-v$ geodesic in G, then it is a $u-v$ geodesic also in H. That is, $w \in I_H[u,v]$ and $d_H(u,v) = 2$. Hence, $V(H) = P_2[A]_H$.

Finally, suppose that, in canonical form, $S = \{v_1, v_2, \ldots, v_n\}$. Suppose further that $m = |A|$. Accordingly, $m \geq 2$. Now, let $i_1 = min\{k : v_k \in V(H)\}$, and for $j = 2, 3, \ldots, m$, let $i_j = min\{k : v_k \in V(H) \setminus \{v_{i_1}, v_{i_2}, \ldots, v_{i_{j-1}}\}\}$. Put $u_j = v_{i_j}$ for $j = 1, 2, \ldots, m$. Then $A = \{u_1, u_2, \ldots, u_m\} \in C^*(G)$. ■

Corollary 3.3 *Let H be a connected noncomplete graph, and $G = H + K_p$. If S is a closed geodetic basis of G, then $S \subseteq V(H)$ and S is a 2-path closure absorbing set in H.*

Corollary 3.4 *Let H is a connected noncomplete graph, and $G = H + K_p$. Then*

$$cgn(H + K_p) = min\{|S| : S \subseteq V(H), S \in C^*(G) \text{ and } P_2[S]_H = V(H)\}.$$

Theorem 3.5 [1] *Let H be a connected noncomplete graph, and let $G = H + K_p$. Let $S \subseteq V(H)$. If S is a 2-path closure absorbing set in H and $S \in C^*(H)$, then $S \in C^*(G)$.*

Theorem 3.6 *Let H be a connected noncomplete graph, and let $G = H + K_p$. Let $S \subseteq V(H)$. If $S \in C^*(H)$ and is 2-path closure absorbing set in H, then $S \cup B \in C^*(G)$ for every $B \subseteq V(K_p)$.*

Proof. Let $S \subseteq V(H)$, and suppose $S \in C^*(H)$ and is a 2-path closure absorbing in H. By Theorem 3.5, $S \in C^*(G)$. Suppose that S, in canonical form, is given by $S = \{u_1, u_2, \ldots, u_n\}$. We may write $B = \{w_1, w_2, \ldots, w_m\}$. Put $v_i = w_i$ for $i = 1, 2, \ldots, m$, and $v_{m+i} = u_i$ for $i = 1, 2, \ldots, n$. Then $S = \{v_1, v_2, \ldots, v_{m+n}\} \in C^*(G)$. ■

Corollary 3.7 *Let H is a connected noncomplete graph, and $G = H + K_p$. Then*

$$ucgn(H + K_p) = p + \ max\{|S| : S \subseteq V(H), S \in C^*(G) \text{ and } P_2[S]_H = V(H)\}.$$

Corollary 3.8 *If H is a connected noncomplete graph and $diam(H) = 2$, then $ucgn(H + K_p) = p + \ ucgn(H)$.*

Example 3.9 $ucgn(P_n + K_p) = p + n$.

Example 3.10 $ucgn(C_n + K_p) = p + n - 1$, $n > 3$.

Example 3.11 *$ucgn(K_{m,n} + K_p) = p + 1 + max\{m, n\}$, for all $m, n \geq 2$.*

Theorem 3.12 *Let $G = H + K$, where H and K are connected noncomplete graphs. Let $S \subseteq V(G)$. If $S \in C^*(G)$, then S is one of the following:*

(1) $S \subseteq V(H)$ and is a 2-path closure absorbing set in H;

(2) $S = A \cup V(K_p)$, where $A \subseteq V(H)$ is a 2-path closure absorbing set in H and K_p is a subgraph of K;

(3) $S \subseteq V(K)$ and is a 2-path closure absorbing in K;

(4) $S = V(K_p) \cup B$, where $B \subseteq V(K)$ is a 2-path closure absorbing set in K and K_p is a subgraph of H.

Proof. Let $G = H+K$, where H and K are connected noncomplete graphs, and let $S \subseteq V(G)$. Suppose that $S \in C^*(G)$. If $\langle S \rangle$ is a complete subgraph of G, then $\langle S \rangle = G$, a contradiction. Thus, there exist vertices $u, v \in S$ such that $d_G(u, v) = 2$. Either both $u, v \in V(H)$ or both $u, v \in V(K)$. Suppose $u, v \in V(H)$. Let $A = S \cap V(H)$. Then $V(K) \subset I_G[A]$. If $A = S$, then $S \subseteq V(H)$. Suppose $A \neq S$. We claim that $S = A \cup V(K_p)$, where K_p is a subgraph of K. To this end, we write $S = A \cup (S \setminus A)$. If $S \setminus A$ is a singleton, then we are done. Suppose that $S \setminus A$ is at least a doubleton, and let $x, y \in S \setminus A$. If $d_G(x, y) = 2$, then $V(H) \subset I_G[x, y]$. This is impossible since $S \in C^*(G)$. Thus, $d_G(x, y) = 1$. Since x and y are arbitrary, $\langle S \setminus A \rangle$ is a complete subgraph of K. Write $K_p = \langle S \setminus A \rangle$, and the claim is established.

Let $w \in V(H) \setminus A$. Since $w \notin V(K)$, $w \in V(G) \setminus S$. By Lemma 3.1, there exist vertices u and v in S such that $w \in I_G[u, v]$ and $d_G(u, v) = 2$. In this, we note that whether $S = A$ or $S = A \cap V(K_p)$ we have $u, v \in A$. Then $[u, w, v]$ is a $u - v$ geodesic in H, and so $d_H(u, v) = 2$. This means that A is a 2-path closure absorbing set in H.

Similarly, if $u, v \in V(K)$, then either $S \subseteq V(K)$, which is a 2-path closure absorbing set in K, or $S = V(K_p) \cup B$, where $B \subseteq V(K)$ which is closure absorbing in K and K_p is a subgraph of H. ■

Theorem 3.13 [1] *Let $G = H + K$, where H and K are connected noncomplete graphs. If either*

(1) $S \subseteq V(H)$, S is a 2-path closure absorbing set in H and $S \in C^*(H)$ or

(2) $S \subseteq V(K)$, S is a 2-path closure absorbing set in K and $S \in C^*(K)$, then $S \in C^*(G)$.

Theorem 3.14 *Let $G = H + K$, where H and K are connected noncomplete graphs. If either*

(1) $S = A \cap V(K_p)$ for some 2-path closure absorbing set $A \subseteq V(H)$ in H with $A \in C^*(H)$ and some subgraph K_p of K; or

(2) $S = V(K_p) \cap B$ *for some 2-path closure absorbing set* $B \subseteq V(K)$ *in* K *with* $B \in C^*(K)$ *and some subgraph* K_p *of* H,
then $S \in C^*(G)$.

The proof of Theorem 3.14 is parallel to the proof of Theorem 3.6. In this case, we use Theorem 3.13.

Corollary 3.15 *Let* $G = H + K$, *where* H *and* K *are connected and non-complete graphs. Then*

$$ucgn(G) = max\{p_1 + \eta, p_2 + \kappa\},$$

where

$$\eta = max\{|S| : S \subseteq V(H), S \in C^*(G) \text{ and } P_2[S]_H = V(H)\},$$

$$\kappa = max\{|S| : S \subseteq V(K), S \in C^*(G) \text{ and } P_2[S]_K = V(K)\},$$

$$p_1 = max\{p : K_p \text{ is a subgraph of } K\},$$

and

$$p_2 = max\{p : K_p \text{ is a subgraph of } H\}.$$

Example 3.16 $ucgn(P_m + P_n) = 2 + max\{m, n\}$.

Example 3.17 $ucgn(P_m + C_n) = 2 + max\{m, n - 1\}$.

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