Colouring walls may help to make good schedules

Andrea Vietri*

Abstract

We present some applications of wall colouring to scheduling issues. In particular, we show that the chromatic number of walls has a very clear meaning when related to certain real-life situations.

1 Introduction

Walls and their *brick colourings* were initially employed to rephrase, in terms of adjacency matrices, a notion of arc colouring for directed hypergraphs [1, 4, 7]. Some further research on walls was carried out in [5] and, more recently, in [6]. In the former paper, among other things, a connection with latin squares was outlined for possible future investigation. In the latter, the chromatic classification problem for a certain subclass of walls was shown to be seemingly as tough and compelling as the similar problem of the edge colouring for graphs [3]. In the following pages we investigate a more applicative aspect of wall theory, namely by interpreting certain *scheduling problems* as brick colouring problems. In particular, in each example we show that a precise, practical meaning can be given to the least number of colours needed. Some basic theoretical results are applied, along the way, to the various contexts. The formal definition of wall, and all related notions, will be provided right after managing the first scheduling problem.

2 Scheduling problems and walls

Scheduling Problem 1. An educational project involving some research institutes provides that every such structure hosts a number of students, each one in a separate period of time. For any fixed institute periods

*Dipartimento Me.Mo.Mat., Università Roma1, via A. Scarpa 16, 00161 Roma, Italia; e-mail: vietri@dmmm.uniroma1.it http://www.dmmm.uniroma1.it/~vietri

are continuative, although they may have different durations and may be separated by spells of inactivity, in keeping with the institute regulations. During a fixed period the involved student will carry out a monothematic project, with the help of the local staff. Different institutes may award different numbers of periods. Although the research areas of any two institutes are assumed to be essentially distinct, they are all part of a unique, more general research area (e.g. extremal graphs, generating functions, permutation groups, and some other branches of Combinatorics). Figure 1 represents a 12-month educational project involving 7 institutes. The labelling provides a feasible solution to the assignment of students (numbers) to periods (rectangles and squares).

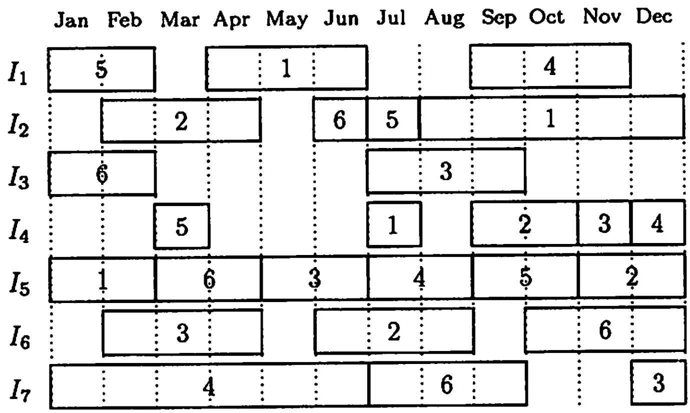


Figure 1: Scheduling an educational project

QUESTION **1.a.** *What is the smallest number of students that can be involved in the project schematised in Figure 1?*

[A] In Figure 1 we have actually succeeded in involving the least possible number of students, namely 6 (this might be regarded as a good achievement if, for example, we were interested in maximising the overall activity of any single student). We could not do better; indeed, a quick look to the diagram shows that in February, as well as in July and September, there are precisely 6 positions to hold. Furthermore, even without those constraints, the fifth institute should receive 6 different students.

QUESTION **1.b.** *Could we have answered "6" to the first question after simply looking at the February column, or at the fifth row, or at both, thus without embarking on the entire labelling?*

[A] Certainly not. For it is enough to replace the second institute schedule with a unique 12-month period, which would increase by 1 the number of students needed.

QUESTION **1.c.** *Using the only hypothesis that no more than 6 distinct periods appear in any row or column, is it possible to evaluate the least number of required students?*

[A] In the worst case 11 students will be needed (the reader is not supposed to check this answer in real time. At the moment, he/she should in any case trust wall theory and carry on until Proposition 2.2).

As mentioned at the beginning, scheduling problems like the one above can be effectively formalised in terms of *walls*. This is exactly what we are going to do in the following lines. Reasoning in the wall environment will enable us – among other things – to give a fully justified answer to the third question, as well as to successfully manage other scheduling problems.

Definition 2.1. A *wall* is a partial chessboard whose squares have been labelled under the condition that no label occurs in more than one row. Maximal set of equally labelled squares are termed *bricks*. A wall whose bricks are all connected is termed *coherent*. The *degree* of a wall W, in symbols $\delta(W)$, is the largest number of different labels in any row or column of W. A *brick colouring* of W is a map which assigns numbers (colours) to bricks in such a way that no colour occurs in two or more bricks of some row, nor it occurs in two or more squares in some column. The *chromatic number* of W, in symbols $\rho(W)$, is the least number of colours needed for a brick colouring of W.

It is then clear that Figure 1 shows a 6-colour brick colouring of a wall. Let us now look at the above scheduling problem from the wall-theoretic viewpoint. The three questions and answers can be rephrased as follows.

QUESTION **1′.a.** *Let W denote the coherent wall representing the overall offer of the institutes. Evaluate $\rho(W)$.*

[A] $\rho(W) = 6$.

QUESTION **1′.b.** *In the present example, $\rho(W) = \delta(W)$. Does this equality hold for every coherent wall?*

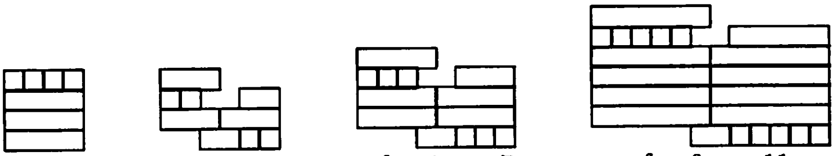
[A] By replacing the bricks in row 6 with a unique brick of maximal length, we increase ρ by 1 while keeping δ equal to 6. Thus, the answer is negative. Notice, however, that in any case $\rho(W) \geq \delta(W)$. Such inequality holds for every wall W (not necessarily coherent) and is an elementary consequence of the above definitions.

QUESTION **1′.c.** *Is there any upper bound for $\rho(W)$ when $\delta(W) = 6$?*

[A] We appeal to the basic

Proposition 2.2. [4] $\rho(W) \leq 2\delta(W) - 1$ *for every coherent wall W, the upper bound being tight for every fixed degree.*

Such result, which was obtained with few difficulties using induction on the wall length, explains why 11 colours are enough for any coherent wall of degree 6. In Figure 2 we exhibit one trivial and three nontrivial coherent walls whose chromatic numbers are the largest possible, subject to constant degree. In particular, the wall on the right side is one of the "worst cases" mentioned in the answer to Question 1.c. The three examples are part of an infinite family $\{W_i : i \geq 2\}$ which can be easily described, and whose generic element satisfies $\rho(W_i) = 2\delta(W_i) - 1$. Proving the correctness of such equalities might be a fairly pleasant exercise.



$\delta = 4, \rho = 7$ (*trivial*) $\delta = 3, \rho = 5$ $\delta = 4, \rho = 7$ $\delta = 6, \rho = 11$

Figure 2: Walls with largest chromatic number

We invite the reader to do his best for producing new examples of coherent walls having $\rho = 2\delta - 1$ and, more generally, for constructing coherent walls of prescribed degree and chromatic number. Still we observe that this kind of problem reminds us of the well-known, and unsettled, *classification problem* for the chromatic index of graphs, namely, the problem of characterising all graphs of degree d that are edge-colourable in d colours. Such a problem traces back to Vizing's theorem (see e.g. [3]). It is then conceivable that a full understanding of the interplay between δ and ρ is currently beyond reach – similarly to what happens on the graph-theoretical side. Finally, we remark that the coherence hypothesis is essential for the existence of an upper bound when the degree is fixed. In fact, it is not hard to construct a generic wall of degree d and chromatic number r for any positive integers $r \geq d \geq 2$. In particular, concerning the above schedule, there would be no hope of effectively upper bounding the number of students if each period was allowed to split into disconnected sub-periods, thus with possible "intertwinings" between sub-periods of distinct students in the same institute. On the other hand, using König's theorem on bipartite graphs it can be shown [4] that $\rho(W) = \delta(W)$ whenever W has all *square* bricks. Therefore, 6 students are enough if *none of the periods intersects two or more other periods*, regardless of whether or not some periods are continuative. Indeed, under that condition, by suitably permuting columns and possibly shortening some bricks one can easily obtain square bricks only, without affecting the incidence between bricks.

The educational project scheduling has motivated the introduction of walls in the present context. The definitions and results so far pointed out pave the way for managing two other scheduling problems, which can be likewise interpreted as colouring problems for certain coherent walls. We just state them, in the following lines.

Scheduling Problem 2. A food factory produces some species of vegetables. Each species requires a characteristic treatment and, therefore, a prescribed sequence of working periods during all its growth process (for example, the first three periods might be ploughing, sowing and first thinning). The number of workers on a fixed species during a certain period is assumed to be the same (say n) for all species and periods. The factory employees are heterogeneous in their origin, education, and experience. For this reason, the management policy is that of involving as many employees as possible in treating the same species. By doing so, the overall product of the factory is expected to be homogeneous enough, while, on the other hand, continuing the work done by others will increase the workers' expertise as times goes on. It is then clear that the coherent wall model is a natural candidate for scheduling the factory activity over a fixed, possibly long, period. In this case rows correspond to species, while the bricks in each row determine the prescribed sequence of working periods. Once the staff has been partitioned into stable teams of n workers, any correct brick colouring shows how many working teams (colours) will be needed and which species will be cured by a given team at a certain time. The chromatic number ρ is strongly related to the overall productivity, for if ρ teams suffice and there are w workers, then the same schedule can be carried out parallely in $\lfloor w/(n\rho) \rfloor$ different agricultural grounds (the symbol $\lfloor q \rfloor$ denotes the largest integer $z \leq q$).

Scheduling Problem 3. The art directors of some important theaters meet together for scheduling the activity in each theater for the coming year. Every theater has to be assigned some performances which will take place during one of the available periods. Such periods had been established before the meeting, by each director, independently. They may have different lengths, and their number may vary from theater to theater. In fact, during the meeting every director shows his own schedule, which looks like a row of a coherent wall, thus with connected bricks (to fill with performances) and possibly empty spaces. In this case, every colouring of the wall provides a consistent assignment of performances, while the chromatic number is the tight lower bound for the number of art companies required (assuming that distinct performances will be held by distinct companies).

3 Schedules manageable with regular walls

In all the above examples periods were conceivably allowed to vary. As a consequence, the resulting walls turned out to be as much general as possible. In particular, they had no regularity property nor symmetry. Admittedly, the rather commonplace structure the word *wall* reminds us was never referred to in the previous lines. It would be discouraging not to succeed in finding a schedule which corresponds to a *real* wall, namely a wall with no empty spaces and all bricks of the same length – except the extreme ones, possibly – suitably shifted in each row. The formal definition of this kind of coherent wall can be given as follows [5].

Definition 3.1. Let m, n, k be positive integers. The (m, n, k)-*regular wall*, in symbols $W^k_{m,n}$, is a complete chessboard with m rows and n columns, which we regard as a matrix $(m_{ij}, 1 \leq i \leq m, 1 \leq j \leq n)$, and whose (i, j)-square is labelled by $\left(i, \lfloor \frac{j-i}{k} \rfloor\right)$.

In the next two figures we have depicted $W^3_{3,36}$ and $W^6_{10,60}$, the latter wall having been optimally coloured. Actually, the four lower extremal bricks of the former wall should be properly shortened and both figures should be rescaled, so as to yield the required chessboards. However, what really counts is that the incidence relationship among bricks is not compromised. The additional symbology in the figures will be clarified in due course. Incidentally, we observe that every $W^1_{m,n}$ optimally coloured is just an $m \times n$ latin rectangle.

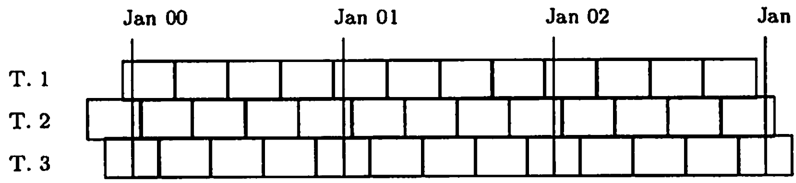


Figure 3: $W^3_{3,36}$ "untrimmed"

Leaving aside the – probably questionable – aesthetic viewpoint, schedules yielding regular walls would be heartily welcome because of the easiness of computing the related chromatic numbers. Indeed, the following holds.

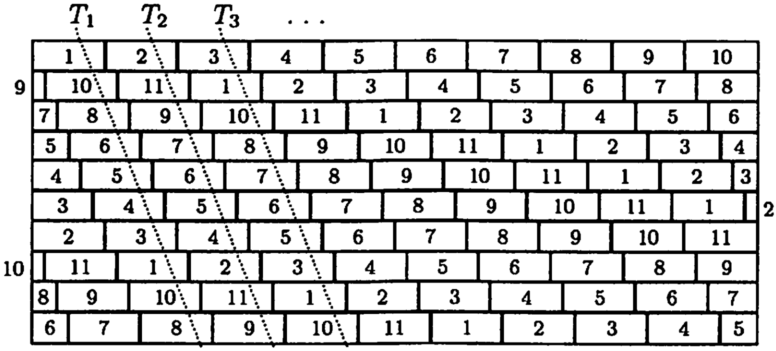


Figure 4: An optimal colouring of $W^6_{10,60}$

Theorem 3.2. [5] *If $m > k > 1$ and $k \not| m$, then*

$$\rho(W^k_{m,n}) = \begin{cases} m & \text{if} \quad n \leq m-k \\ m+1 & \text{if} \quad m-k+1 \leq n \leq km+1 \\ \lceil \frac{n-1}{k} \rceil + 1 & \text{if} \quad n \geq km+2 \end{cases},$$

where $\lceil q \rceil$ denotes the smallest integer $z \geq q$. If $k|m$, the upper bound in the first condition is changed to $m-k+1$ and the lower bound in the second condition becomes $m-k+2$; the rest is unchanged. If $1 < m < k$, the first condition is changed to $n = 1$ and the lower bound in the second condition becomes 2; the rest is unchanged. Finally,

$$\rho(W^k_{1,n}) = \left\lceil \frac{n}{k} \right\rceil .$$

Notice that if the first condition holds, then – roughly speaking – the related wall is comparatively high, whence the number of colours is conditioned by the number of rows only; similarly, if the third condition holds, then every row contains a large number of bricks which make the wall rather long and account for the chromatic number by themselves. Therefore, in both cases the chromatic number is not difficult to anticipate. Instead, the second condition is more delicate in that it describes what happens when passing from one trivial case to the other. During such "crossing", the chromatic number stands still while the length of the wall increases (see, for instance, the optimal colouring in Figure 4; every optimal colouring of $W^6_{10,n}$ with $5 \leq n \leq 59$ does require 11 colours, and can be obtained by simply removing the last $60-n$ columns of $W^6_{10,60}$).

Luckily enough, there exists a great amount of scheduling problems which lead to regular walls. We are going to describe the first of two examples.

Scheduling Problem 4. In order to collect some feedback data (comments, suggestions, etc.) about the 3-month initial training courses held in a factory from November 1999 to February 2003, all employees have been partitioned according to the month and year of their engagement (engagements are assumed to take place every month). Due to obvious overlappings, courses had been held parallely by 3 trainers, as shown in Figure 3. Rows correspond to trainers, while each brick covers the 3 months of course for some fixed group. Each group will now be administered a questionnaire, two questionnaires being different for any two groups which had the same trainer or whose training periods overlapped (while the former condition will result in a great variety of data for each trainer, the latter condition prevents questionnaires to be affected by events not strictly related to some fixed trainer, e.g. an influenza epidemy, a factory closing period, a fast sequence of strikes, etc.). In the present context, Theorem 3.2 quickly provides the least number of questionnaires needed to carry out such a test. More precisely, we have $m = 3, n = 36, k = 3$, whence $\rho = 13$. As remarked earlier, such a number could have been easily guessed without using the above theorem. Indeed, while the lowest row clearly requires 13 distinct colours, it is rather elementary to obtain a total colouring with no further colour, once that row has been coloured.

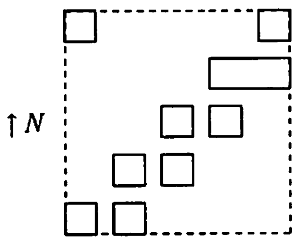
Before proceeding to the second example, we observe that in the so far employed walls distinct rows have never been associated to distinct periods of time. In fact, in all cases the time variable was precisely the column index. However, due to the typically recursive placement of bricks in regular walls, such structures lend themselves to represent partitions of objects which evolve over time, this variable being represented by the row index. A typical situation occurs, for example, when a new regulation provides that the staff in some office be divided into groups, according to the date of engagement (such a division might be related to pension schemes, task force formations, etc.). If the groups so formed are supposed to stay unchanged for the coming years, then the future engagements will result in new groups to form, whereas some old groups will reduce or disappear because of retirements. The evolution of all these groups over the years is easily representable by means of a regular wall, as shown in the next example.

Scheduling Problem 5. In accordance with a far reaching project, the marketing staff of a big firm has been partitioned into 10 task forces $\{T_i\}_{1\leq i\leq 10}$ with respect to the duration of the working period of each employee. In details, $w \in T_i$ if and only if w has been working in that firm for α years with $\alpha \in \mathbf{R}$ and $\lceil \alpha/3 \rceil = i$ (e.g. workers engaged from $18+\varepsilon$ to 21 years ago – with ε arbitrarily small – are all grouped together in T_7). Because most retirements in that firm occur before the 30th working year, such a partition seems sensible enough. The members of any fixed task force will meet once a semester, for 5 years running, and discuss each time a different topic (equipment maintenance, productivity, wages, and so forth). At the end of each meeting they will draft a document, which is expected to contain important suggestions for future developments. In order to optimise the overall outcome, some further task forces will be formed from the newly engaged workers as times passes. Therefore, at the end of the 5-year period two new groups will be present, while T_{10} will be disappeared (leaving aside some exceptions) and T_9 will be reasonably smaller than at the beginning. The regular wall in Figure 4 represents the evolution of groups during the 5 years. In order to make clear the meaning of the brick colouring it still remains to say that, according to the above project, no two groups are allowed to discuss the same topic in the same semester, and a given topic can be discussed in distinct meetings as long as the working periods of the related groups are disjoint. Whereas the former constraint may sound quite natural, the latter has perhaps a more concealed meaning, as it ensures that each topic is discussed at most once by people with the same working experience. Notice that this constraint does not prevent any group to discuss the same argument twice (e.g. in the 1st and 8th meeting). In conclusion, every correct brick colouring provides a number of topics (colours) to assign to all groups over the whole period, while the chromatic number (11, by Theorem 3.2) returns the least number of topics to assign in compliance with the above constraints. In other words, choosing 11 topics is the best one can do in order to concentrate upon the least number of topics without causing the mentioned redundancies.

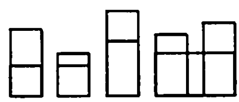
4 An urbanistic problem

At this point we hope that wall colourings have gained the reader's sympathy as valuable tools for managing scheduling problems. In the last example we try to extend the scope of wall colourings by pointing to architects as to possible further users of wall theory (although, as we optimistically expect it, some architects may have already enjoyed the previous examples). For our purposes, both the wall dimensions are now regarded as space measures, thus with no use of time variables.

Scheduling Problem 6. The left side of Figure 5 represents a very essential top view of a housing estate. Clearly, a few changes could turn it into a wall whose bricks correspond to the blocks of the residential complex. For advertising purposes, it would be now desirable to create a poster in which the estate view is first taken from the West and, subsequently, from the South – in both cases from ground level. However, using a simple axonometry is not recommended for either views, because some blocks would turn out to be partly or totally hidden; alternatively, in order to avoid any data loss, some drawings should overlap (see the right side of Figure 5). It is then decided to sort blocks into disjoint groups, in such a way that any group admits an axonometry with no overlaps, both from the South and from the West. All the resulting pairs of axonometries (one pair for each group) will then show up together in a single poster. As it can be quickly realised, every correct brick colouring of the top view wall provides a suitable partition of blocks. In particular, according to the chromatic number, at least 3 pairs of axonometries will be needed.



$\delta = 2, \rho = 3$



View from the South, with overlaps

Figure 5: Top view, and ground level view, of a housing estate

Actually, the above schedule was chosen with more general purposes than the ones mentioned at the beginning. In fact, the top view wall is part of an infinite class of coherent walls which, up to suitable permutations of rows and columns, and to proper removals of inessential bricks and shortenings of other bricks, represent the entire class of coherent walls of degree 2 and chromatic number 3. Such representative walls are the analogue of the odd-length circuits in graph theory. The relevant result has been established in [6]. In Figure 6 we limit ourselves to present four further walls of the above mentioned class. The sequence of ones and zeroes, appearing as a subscript, uniquely determines every representative wall, up to circular permutations of the digits. The number of ones must be odd and refers to the number of longer bricks making up the "stair". In plain words, every classifying wall of height n corresponds to a necklace with n beads of two colours, where at least one monochromatic set consists of an odd number of beads.

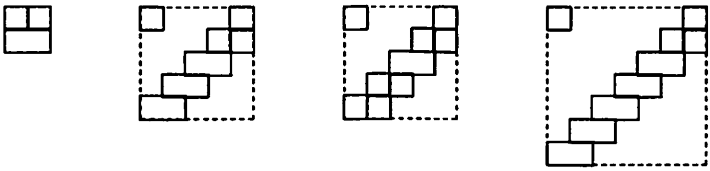


Figure 6: The walls $W_{01}, W_{00111}, W_{00100}$, and $W_{0011111}$

References

[1] G. Ausiello, P.G. Franciosa and D. Frigioni, *Directed Hypergraphs: Problems, Algorithmic Results, and a Novel Decremental Approach*, Lecture Notes in Theor. Comp. Sci. **2202** (2001), pp. 312–327.

[2] V.W. Bryant, *Aspects of Combinatorics, a Wide-Ranging Introduction*, Cambridge University Press, Cambridge, 1993.

[3] S. Fiorini and R.J. Wilson, *Edge-colourings of graphs*, Research Notes in Math., vol. 16, Pitman, London, 1977.

[4] A. Vietri, *Arc colouring of directed hypergraphs and chromatic number of walls*, preprint of the MeMoMat department, Rome, 2005, http://www.dmmm.uniroma1.it/preprints.it.php?tipopubb=p.

[5] A. Vietri, *Chromatic number of regular walls*, Ars Comb. **76** (2005), pp. 97–111.

[6] A. Vietri, *Coherent walls of maximal class*, submitted.

[7] A. Vietri, *The complexity of arc-colourings for directed hypergraphs*, Discr. Appl. Math. **143** (2004), pp. 266–271.