A NECESSARY AND SUFFICIENT CONDITION FOR A 3-REGULAR GRAPH TO BE C0RDIAL

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abstract In this paper, we give a necessary and sufficient condition for a 3-regular graph to be cordial.

Keyword Regular graph, Cordial graph

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We define a $0-1$ label f on $V(G)$ by giving each $v \in V(G)$ a label $f(v) = 0$ *or* 1, and denote $V_0 = \{v|\ v \in V(G), f(v) = 0\}$, $V_1 = \{v|\ v \in V(G), f(v) = 1\}$. From a $0-1$ label on $V(G)$, we derive a $0-1$ label on $E(G)$ by giving each $uv \in E(G)$ a label $f(u, v) = |f(u) - f(v)|$, and denote
$E_{00} = \{uv|\ uv \in E(G), f(u) = f(v) = 0\}$,
$E_{11} = \{uv|\ uv \in E(G), f(u) = f(v) = 1\}$,
$E_1 = \{uv|\ uv \in E(G), |f(u) - f(v)| = 1\}$ and $E_0 = E_{00} \cup E_{11}$. Let S be a set, we denote by $|S|$ the number of elements of S. If there exists $0-1$ label f on $V(G)$ such that $||V_0| - |V_1|| \leq 1$ and $||E_0| - |E_1|| \leq 1$, then G is said to be Cordial [1] and f is said to be a Cordial label of G. The cordiality of some graphs have been discussed in some papers [2, 3, 4]. In this paper we give a necessary and sufficient condition for a 3-regular graph to be cordial.

2 Preliminaries

Lemma 1. Let R^k be a k-regular graph and f be a $0-1$ label of R^k. If $|V_0| = |V_1|$, then $|E_{00}| = |E_{11}|$.

Proof. Since the edges incident with every vertex in V_0 belong to either E_{00} or E_1, We have

$$k|V_0| == 2|E_{00}| + |E_1| \tag{1}$$

Similarly

$$k|V_1| == 2|E_{11}| + |E_1| \tag{2}$$

Combining (1), (2) and $|V_0| = |V_1|$, we have $|E_{00}| = |E_{11}|$. □

Let $v \in V(G)$, denote by $d_G(v)$, or simply $d(v)$, the degree of v. Denote by $\Delta(G) = \max_{v \in V(G)} d(v)$ and $\overline{d}(G) = \frac{\sum v \in V(G) d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$ the maximum degree and the mean degree of G, respectively.

Lemma 2. Let G be any graph. If
(1) $\Delta(G) \leq 3$
(2) $1.5 \leq \overline{d}(G) \leq 3$ and
(3) G contains no 3-regular components, then there exist two vertices u and v such that $|E(G - \{u, v\})| = |E(G)| - 3$ where $G - V'$ is the subgraph obtained from G by deleting the vertices in the subset V' together with their incident edges.

Proof. We distinguish two cases.

Case 1. G contains no vertices of degree 3. Then contains two vertices u and v, of degree 2, adjacent to each other. Otherwise each component of G should be K_1, K_2, or a path of length 2, contradicting $\overline{d}(G) \geq \frac{3}{2}$. So u and v are two desired vertices.

Case 2. G contains vertices of degree 3.

Subcase 1. G contains isolated vertices. Let u be a vertex of degree 3, and v an isolated vertex. Then u and v are two desired vertices.

Subcase 2. G contains no isolated vertices and there is a vertex, u, of degree 1, that is adjacent to a vertex, v, of degree3. Then u and v are two desired vertices.

Subcase 3. G contains no isolates vertices and any neighbor of each vertex of degree 3 has a degree greater that 1. Since G contains no 3-regular components, there exists a vertex of degree 3 that has at least one neighbor of degree 2. Let z be such a vertex of degree 3, and u, v, w, the neighbors of z. Without loss of generality, we suppose that the degree of u is 2. If z

has u as the only neighbor of degree 2, then $d(v) = d(w) = d(z) = 3$. Since $\overline{d}(G) \leq 2$, there are at least two vertex of degree 1, at least one of which, say x, is not a neighbor of u. So u and v are two desired vertices. If z has at least two neighbors of degree 2, say u and v, then since $\overline{d}(G) \leq 2$ and $d(z) = 3$, there is at least one vertex, say x, of degree 1. At most one of u and v is adjacent to x. We may suppose that u is not a neighbor of x. Then u and x are two desired vertices. □

A subset S of V is called an independent set of G if no two vertices in S are adjacent in G.

Lemma 3. Let G be a graph of order n with $\Delta(G) \leq 3$ and S a maximal independent set of G. Then $|S| \geq \lceil \frac{n}{4} \rceil$.

Proof. Let S be an independent set of G and $|S| = k$. It suffices to show that if $k < \lceil \frac{n}{4} \rceil$, then S is not a maximal independent set of G. Since $\Delta(G) \leq 3$, it is clear that whenever $k < \lceil \frac{n}{4} \rceil$, $V(G - S - N(S))$ is not empty, where $N(S)$ is the set of neighbor of vertices of S, and every vertex in $G - S - N(S)$ is not adjacent to vertices in S. □

Lemma 4. Let R^3 be a 3-regular graph of order n. If at least one component of R^3 is not K^4, then R^3 contains an independent set S such that
(1) $|S| \geq \lceil \frac{n}{4} \rceil$
(2) $R^3 - S$ contains either two vertices of degree 1 not adjacent to each other or a vertex of degree 1 adjacent to a vertex of degree 2.

Proof. Let G be a component of R^3 which is not K^4. We distinguish four cases.

Case 1. G contains two 3-cycles, C_{xuv} and C_{yuv}, with an edge uv in common. Let z be the neighbor of x other than u and v, and z_1 and z_2 the neighbors of z other than x. Denote $G_1 = R^3 - \{x, y, u, v, z\} - \{z_1, z_2\}$. It is clear that $|G_1| \geq n - 7$. By Lemma 3, G_1 contains an independent set S_1 with $|S_1| = \lceil \frac{n}{4} \rceil - 2$. Then $S = S_1 \bigcup \{u, z\}$ is an independent set of R^3 with $|S| \geq \lceil \frac{n}{4} \rceil$, and x and v are two desired vertices.

Case 2. G contains 3-cycles and any two 3-cycles have no edges in common . Let C_{uvw} be a 3-cycle and x the neighbor of u other than v and w, y the neighbor of v other than u and w, z the neighbor of w other than u and v, respectively. Denote by x_1 and x_2 the neighbors of x other than u, and $G_1 = R^3 - \{u, v, w, x, y, z\} - \{x_1, x_2\}$. Since $|G_1| \geq n - 8$, by Lemma 3, G_1 contains an independent set S_1 with $|S_1| = \lceil \frac{n}{4} \rceil - 2$. Then $S = S_1 \bigcup \{x, v\}$ is an independent set of R^3 with $|S| \geq \lceil \frac{n}{4} \rceil$, and u and w are two desired vertices.

Case 3. G contains no 3-cycles but at least one 4-cycle. Let C_{xyuv}

be a 4-cycle in G. Let x_1 be the neighbor of x other than y and v, y_1 the neighbor of y other than u and x, u_1 the neighbor of u other than v and y, and v_1 the neighbor of v other than x and u. Denote $G_1 = R^3 - \{x, y, u, v, u_1, v_1\} - \{x_1, y_1\}$. Since $|G_1| \geq n - 8$, by Lemma 3, G_1 contains an independent set S_1 with $|S_1| = \lceil \frac{n}{4} \rceil - 2$. Then $S = S_1 \bigcup \{x, u\}$ is an independent set of R^3 with $|S| \geq \lceil \frac{n}{4} \rceil$, and y and v are two desired vertices.

Case 4. G contains neither 3-cycle nor 4-cycles. It follows $|G| \geq 10$. Let u be a vertex in G and x, y and v the neighbors of u. Denote by x_1, x_2 the neighbors of x other than u, and by y_1, y_2 the neighbors of y other than u, and by v_1, w the neighbors of v other than u, by w_1, w_2 the neighbors of w other than v, respectively. Since G contains neither 3-cycles nor 4-cycles, x, y and w are distinct and not adjacent to one another. Denote $G_1 = R^3 - \{u, v, w, x_1, x_2, y_1, y_2, v_1\} - \{w_1, w_2\}$.Since $|G_1| \geq max\{n-12, 0\}$, by Lemma 3, G_1 contains an independent set S_1 with $|S_1| = \lceil n/4 \rceil - 3$. Then $S = S_1 \cup \{x, y, w\}$ is an independent set of R^3 with $|S| \geq \lceil n/4 \rceil$, and u and v are two desired vertices. □

3 Main Results

Theorem 1. Every 3-regular of order $8n$ is cordial.

Proof. Let R^3_{8n} be a 3-regular graph of order $8n$.

Case 1. $R^3_{8n} = \cup K_4$ Since $2K_4$ is cordial, then $2nK_4$ is also cordial.

Case 2. R^3_{8n} is not the union of K_4. By Lemma 4 there is an independent set S of R^3_{8n} with $|S| = 2n$. Since each component of R^3_{8n} has at least four vertices and there are at most $2n$ components, we can choose an S such that S and each component have at least one vertex in common. Denote $G = R^3_{8n} - S$, the graph obtained by deleting S from R^3_{8n}. It is clear that G has exactly $6n$ vertices, $6n$ edges, and contains no 3-regular components. To verify the condition (2) of Lemma 2, we note that the deleting of pairs of vertices incident to $3k$ edges $(0 \leq k \leq n-1)$ results in a graph which has exactly $6n - 2k$ vertices and $6n - 3k$ edges, and whose mean degree is

$\overline{d}(G) = \frac{2(6n-3k)}{(6n-2k)} = 2 - k/(3n-k) \geq 2 - (n-1)/[3n-(n-1)] \geq \frac{3}{2}$

Thus by Lemma 2 the deleting of altogether n pairs of vertices incident to $3n$ edges results in a graph G^*, which has exactly $4n$ vertices and $3n$ edges. Label 0 to each vertex of $V(G^*)$, and 1 to the other vertices of R^3_{8n}. Then $V_0 = V(G^*), V_1 = V(R^3_{8n}) - V(G^*), E_{00} = E(G^*)$, and $E_{11} = E(R^3_{8n} - V_0)$. Clearly we have $|V_0| = |E_{00}| = 3n$. By Lemma 1, $|E_{11}| = |E_{00}| = 3n$. Thus we obtain a cordial labeling of R^3_{8n}. □

Theorem 2. Every 3-regular graph of order $8n + 2(n \geq 1)$ is cordial.

Proof. Let R^3_{8n+2} be a 3-regular graph of order $8n + 2$. Clearly R^3_{8n+2} has at least one component which is not K_4. By Lemma 4 there is an independent set S with $|S| = 2n + 1$, containing a vertex in common with each component, whose deleting from R^3_{8n+2} results in a graph with $6n + 1$ vertices and $6n$ edges that has either two vertices, x and y, of degree1, not adjacent to each other or a vertex, x, of degree 1, adjacent to a vertex, z, of degree 2. Deleting x and y, or x and z we obtain a graph, G, with $6n - 1$ vertices and $6n - 2$ edges. Note that for $k \leq n - 2, 2 \geq 2(6n - 2 - 3k)/[6n - 1 - 2k] \geq 3/2$. By Lemma 4 we can delete altogether $n - 1$ pairs of vertices incident to $3n - 3$ edges and obtain a graph G^* with $4n + 1$ vertices and $3n + 1$ edges. By labeling 0 to each vertex of $V(G^*)$ and 1 to the other vertices of R^3_{8n}, we obtain a cordial label, for we have $|V_0| = |V_1| = 4n + 1, |E_0| = 2(3n + 1) = 6n + 2$, and $|E_1| = 12n + 3 - (6n + 2) = 6n + 1$. □

Theorem 3. Every 3-regular graph of order $8n + 6$ is cordial.

Proof. Let R^3_{8n+6} be a 3-regular graph of order $8n + 6, (n \geq 0)$. By Lemma 4 there is an independent set S with $|S| = 2n+2$, containing at least one vertex in common with each component, whose deleting from R^3_{8n+6} results in a graph with $6n + 4$ vertices and $6n + 3$ edges that has a vertex, x, of degree 1. Deleting x we obtain a graph G, with $6n + 3$ vertices and $6n + 2$ edges. Note that for $0 \leq k \leq n - 1$, $2(6n + 2 - 3k)/(6n + 3 - 2k) \geq 2 - 2n/(4n + 5) \geq 3/2$.
By Lemma 2 we can delete altogether n pairs of vertices incident to $3n$ edges and obtain a graph, G^*, with $4n + 3$ vertices and $3n + 2$ edges. By labeling 0 to each vertex of $V(G^*)$ and 1 to the other vertices of R^3_{8n+6}, we obtain a cordial label, for we have $|V_0| = |V_1| = 4n + 3, |E_0| = 2|E_{00}| = 6n + 4$ and $|E_1| = (8n + 6) \times (3/2) - (6n + 4) = 6n + 5$. □

Theorem 4. Every 3-regular graph of order $8n + 4, (n \geq 0)$ is not cordial.

Proof. Let R^3_{8n+4} be a 3-regular graph of order $8n + 4$. Since $E(R^3_{8n+4}) = 12n + 6$, for any cordial label of R^3_{8n+4} we have $|E_0| = |E_1| = 6n + 3$. On the other hand, however, by Lemma 1 E_0 should be an even number, a contradiction . □

Combining Theorem 1, Theorem 2, Theorem 3 and Theorem 4 we obtain the following theorem.

Theorem 5. A 3-regular graph of order k is cordial if and only if $k \neq 8n+4$.

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