On List (2,1)-Labeling of Some Planar Graphs*

Sumei Zhang , Qiaoling Ma†

School of Science, University of Jinan, Jinan, Shandong 250022, P.R.China

Abstract

A list (2,1)-labeling L of graph G is an assignment list $L(v)$ to each vertex v of G such that G has a (2,1)-labeling f satisfying $f(v) \in L(v)$ for all v of graph G. If $|L(v)| = k+1$ for all v of G, we say that G has a k-list (2,1)-labeling. The minimum k taken over all k-list (2,1)-labelings of G, denoted $\lambda_l(G)$, is called the list label-number of G. In this paper, we study the upper bound of $\lambda_l(G)$ of some planar graphs. It is proved that $\lambda_l(G) \leq \Delta(G) + 6$ if G is an outerplanar graph or h_1-graph; And $\lambda_l(G) \leq \Delta(G) + 9$ if G is an h_2-graph or Halin graph.

1 Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. Graphs in this paper are simple, unless otherwise stated, i.e., they have no loops or multiple edges. A graph is called *planar* if it can be embedded in the plane. For a graph G, let $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ denote, respectively, its vertex set, edge set, maximum degree and minimum degree. Let $d(u,v)$ denote the distance of u and v. We use $N(v)$ to denote the neighborhood of v and $N^2(v) = \{u \in V(G) | d(u,v) = 2\}$. Let

*This work is supported by the Nature Science Foundation of Shandong Province of China(Y2003A01)and the Doctoral Foundation of University of Jinan(B0615,Y0625).

†The corresponding author: Qiaoling Ma, E-mail: ss_maql@ujn.edu.cn

$d(v) = |N_G(v)|$ be the degree of v in G. A vertex v is called a k-vertex if $d(v) = k$, let $V_k(G) = \{v \in V(G) | d(v) = k\}$. A $(2,1)$-labeling of graph G is an assignment f of nonnegative integers to the vertices of G such that

(1) $|f(u) - f(v)| \geq 2$ if u and v are adjacent,and

(2) $|f(u) - f(v)| \geq 1$ if u and v are distance two apart.

Elements of the image of f are called *labels*, and the *span* of f, denoted $s(f)$, is the difference between the largest and smallest labels. The minimum span taken over all $(2,1)$-labelings of G, denoted $\lambda(G)$, is called the label-number of G. Unless otherwise stated, we shall assume with no loss of generality that the minimum label of $(2,1)$-labelings of G is 0.

A list (2,1)-labeling L of graph G is an assignment label set $L(v)$ to each vertex v of G such that G has a (2,1)-labeling f satisfy $f(v) \in L(v)$ for all v of graph G. If $|L(v)| = k+1$ for all v of G, we say that G has a k-list (2,1)-labeling. The minimum k taken over all k-list (2,1)-labelings of G, denoted $\lambda_l(G)$, is called the list label-number of G. Clearly, if $L(v) = \{0, 1, 2, \cdots, k\}$ for all $v \in V(G)$ yields a k-list (2,1)-labeling of G, then this labeling must be a (2,1)-labeling of G, and we have $\lambda(G) \leq \lambda_l(G)$.

The problem of labeling a graph with a condition at distance two, was first investigated by Griggs and Yeh. They showed that $\lambda(G) \leq \Delta^2(G) + 2\Delta(G)$ and conjectured that $\lambda(G) \leq \Delta^2(G)$ for $\Delta(G) \geq 2$. In [3], Chang and Kuo improved the the bound of Griggs and Yeh to $\Delta^2(G) + \Delta(G)$. Jan van den Heuvel and McGuinness [5] studied the label-number of planar graph and obtained that $\lambda(G) \leq 2\Delta + 34$ for any planar graph. Zhou and Wang [8] studied list $(2,1)$-labeling of trees and cycles and obtained the following results.

Theorem 1.1.[8] *If G is a graph with maximum degree Δ. Then*

$$\lambda_l(G) \leq \Delta^2 + 2\Delta - 2.$$

Especially, for a tree T and a cycle C_n, $\lambda_l(T) \leq \Delta(T) + 3$ and $\lambda_l(C_n) \leq 7$ hold.

In this paper, we study the upper bound of the λ_l-number of some planar graphs and obtain some interesting results. Our main method of some proofs is the following: we induct on the number of vertices. We find a vertex u that can be deleted leaving the induction hypothesis valid. Then we calculate $|S_G(u)|$, the number of forbidden labels for u. In a list $(2,1)$-labeling we always have $|S_G(u)| \leq 3|N(u)| + |N^2(u)|$.

2 The List $(2,1)$-labeling of outerplanar graphs

A planar graph is called *outerplanar* graph if there exists a face f_0 such that $V(G) \subseteq b(f_0)$, where $b(f)$ denote the boundary of f. The following useful lemma can be found in [2].

Lemma 2.1.[2] *Let G be a 2-connected outerplanar graph of order at least 5. Then one of the following conditions holds:*

(1) *G has two adjacent 2-vertices u and v;*

(2) *G has a 2-vertex u adjacent to a 3-vertex v such that $N(u) \subset N(v)$;*

(3) *G has two nonadjacent 2-vertices u and v adjacent to a common 4-vertex w such that $(N(u) \bigcup N(v)) \setminus \{w\} = N(w) \setminus \{u, v\}$.*

The following theorem is one of our main results, which estimates an upper bound of the λ_l-number of outerplanar graphs.

Theorem 2.1. *Let G be a 2-connected outerplanar graph. Then*

$$\lambda_l(G) \leq \Delta(G) + 6.$$

Proof. Suppose that L is a list of G with $|L(v)| = \Delta(G) + 7$ for all $v \in V(G)$. We shall prove the theorem by induction on the number of vertices of graph G.

If $|V(G)| \leq 4$, since G is the subgraph of complete graph K_4. It is easy to prove that

$$\lambda_l(G) \leq \lambda_l(K_4) \leq 9.$$

Let G ba an outerplanar graph such that for all outerplanar graphs H with $|V(H)| < |V(G)|$ the theorem is true. We note first that we can assume that $|V(G)| \geq 5$. It suffices to produce a $(\Delta(G)+6)$-list $(2,1)$-labeling of G. By Lemma 2.1, we consider the following three cases.

Case 1. G has two adjacent 2-vertices u and v.

Let $H = G - u + wv$, where w is the other adjacent vertex of u. Obviously, H is an outerplanar graph with $|V(H)| < |V(G)|$. By the induction hypothesis, H has a $(\Delta(H)+6)$-list $(2,1)$-labeling. Now we label vertex u in G. For every labeled vertex $x \in (N(u) = \{v, w\})$, there are 3 consecutive labels $f(x)-1$, $f(x)$, $f(x)+1$ that are forbidden for use on u. Similarly, for every labelled vertex $y \in (N(w) \bigcup N(v) \setminus \{u\})$, there are one label $f(y)$ that is forbidden for use on u. Then

$$|S_G(u)| \leq 3|N(u)|+|N^2(u)| = 2\times 3+|N(w)\bigcup N(v)|-1 = 6+\Delta(G) < |L(u)|.$$

We can choose a label in list $L(u)$ for vertex u and obtain a $(\Delta(G)+6)$-list $(2,1)$-labeling of G.

Case 2. G has a 2-vertex u adjacent to a 3-vertex v such that $N(u) = \{v, w\}$ and $N(u) \subset N(v)$.

Case 3. G has two nonadjacent 2-vertices u and v adjacent to a common 4-vertex w such that $(N(u) \bigcup N(v)) \setminus \{w\} = N(w) \setminus \{u, v\}$.

For case 2 and 3 let $H = G - u$. Then H is an outerplanar graph with $|V(H)| < |V(G)|$. Similarly to Case 1 we have

$$|S_G(u)| \leq 6 + \Delta(G) < |L(u)|.$$

We can choose a label in list $L(u)$ for vertex u and obtain a $(\Delta(G)+6)$-list $(2,1)$-labeling of G.

This completes the proof of the Theorem.

3 The List $(2,1)$-labeling of planar graphs with high degree

Let G be a planar graph, we denote its face set by $F(G)$. For $f \in F(G)$, we use $d(f)$ to denote the number of vertices on the boundary of f. A face f is called a k-face if $d(f) = k$. For $k = 1, 2, \cdots$, we call G an h_k-graph, if

$\Delta(G) = |V(G)| - k$. Next, we will study the list $(2,1)$-labeling of h_1-graphs and h_2-graphs.

3.1 The List $(2,1)$-labeling of h_1-graphs

Wang[6] studied the structure properties of h_1-graphs and given the following results.

Lemma 3.1.[6] *Let G be an h_1-graph. Then $|V_\Delta(G)| \le \delta(G)$.*

Lemma 3.2.[6] *Let G be an h_1-graph and $d(u) = \delta(G)$. Then $G - u$ is also an h_1-graph.*

Lemma 3.3.[7] *Let G be an h_1-graph with $|V(G)| \ge 2$ and let w be a Δ-vertex of G. Then at least one of the following cases is true for G:*

(1) $\delta(G) = 1$;

(2) *there is a 2-vertex u on a 3-face uwv;*

(3) *there is a 3-vertex u with $N(u) = \{w, v_1, v_2\}$ such that $uwv_1, uwv_2 \in F(G)$.*

The following theorem is one of our main results, which estimates an upper bound of the λ_l-number of h_1-graphs.

Theorem 3.1. *Let G be an h_1 graph with $|V(G)| \ge 2$. Then $\lambda_l(G) \le \Delta(G) + 6$.*

Proof. Suppose that L is a list of G with $|L(v)| = \Delta(G) + 7$ for all $v \in V(G)$. We shall prove the theorem by induction on the number of vertices of graph G.

By enumeration, we can prove the theorem holds for $|V(G)| \le 4$. Assume that it is true for all h_1-graphs with fewer than $|V(G)| = k$ vertices, and let G be an h_1-graph of order k. By Lemma 3.3, we consider the following three cases.

Case 1. There is a 1-vertex u adjacent to a Δ-vertex w.

Let $H = G - u$, by Lemma 3.2, H is also an h_1-graph and $|V(H)| = k-1$. By the induction hypothesis, H has a $(\Delta(H)+6)$-list $(2,1)$-labeling. Now we label vertex u in G. It is easy to calculate that

$$|S_G(u)| \le 3 + \Delta(G) - 1 < |L(u)|.$$

Then we can choose a label in list $L(u)$ for vertex u and obtain a $(\Delta(G)+6)$-list $(2,1)$-labeling of G.

Case 2. There is a 2-vertex u on a 3-face uwv.

By case 1, we may assume that $\delta(G) = 2$. Let $H = G - u$, by Lemma 3.2, H is also an h_1-graph and $|V(H)| = k - 1$. By the induction hypothesis, H has a $(\Delta(H) + 6)$-list $(2,1)$-labeling. Since $|N(u)| = |\{w, v\}| = 2$, $|N^2(u)| \le \Delta(G) - 2$, we have

$$|S_G(u)| \le 6 + \Delta(G) - 2 < |L(u)|.$$

We can choose a label in list $L(u)$ for vertex u and obtain a $(\Delta(G)+6)$-list $(2,1)$-labeling of G.

Case 3. There is a 3-vertex u with $N(u) = \{w, v_1, v_2\}$ such that $uwv_1, uwv_2 \in F(G)$.

Since G is an h_1-graph, by Case 1 and 2, we may assume that $\delta(G) = 3$. Let $H = G - u$, by Lemma 3.2, H is also an h_1-graph and $|V(H)| = k - 1$. Similarly to Case 2, we have

$$|S_G(u)| \le \Delta(G) + 6 < |L(u)|.$$

We can also obtain a $(\Delta(G) + 6)$-list $(2,1)$-labeling of G.

This completes the proof of the Theorem.

3.2 The List $(2,1)$-labeling of h_2-graphs

Lemma 3.4.[7] *Let G be an h_2-graph. Then $|V_\Delta(G)| \le 2$.*

Lemma 3.5.[7] *Let G be an h_2-graph with $|V(G)| \ge 8$ and a unique Δ-vertex w and let $N^c(w) = V(G) \setminus \{N(w), w\} = \{x\}$ with $d(x) \ge 2$. Then at least one of the following cases is true for G:*

(1) *there is a vertex u such that $d(u) = 1$;*

(2) *there is a 2-vertex u on a 3-face uwv;*

(3) *there is a 3-vertex u with $N(u) = \{w, v_1, v_2\}$ such that $uwv_1, uwv_2 \in F(G)$.*

Lemma 3.6.[7] *Let G be an h_2-graph with $|V(G)| \geq 9$ and two adjacent Δ-vertices w_1, w_2. Then at least one of the following cases holds for G:*

(1) *there is a 2-vertex $u \in N(w_1) \bigcap N(w_2)$ such that $uw_1w_2 \in F(G)$;*

(2) *there is a 3-cycle vw_1w_2v such that its interior contains only a vertex u and three edges uv, uw_1, uw_2 and $d(v) \leq 6$;*

(3) *there are three vertices $u, v_1, v_2 \in N(w_1) \bigcap N(w_2)$ such that $d(u) \leq 4$, $d(v_1) \leq 5$, $d(v_2) \leq 5$ and the interior of the 4-cycle $v_1w_1v_2w_2v_1$ contains only u and the edges incident to u.*

Lemma 3.7. *Let G be an h_2-graph with $|V(G)| \geq 9$ and two adjacent Δ-vertices w_1, w_2. Then $\lambda_l(G) \leq \Delta(G) + 9$.*

Proof. Suppose that L is a list of G with $|L(v)| = \Delta(G) + 10$ for all $v \in V(G)$. We shall prove the theorem by induction on the number of vertices of graph G.

By enumeration, we can prove the theorem holds for $|V(G)| \leq 9$. Assume that it is true for all h_2-graphs with fewer than $|V(G)| = k$ vertices and with two adjacent Δ-vertices. Let G be an h_2-graph of order k with two adjacent Δ-vertex w_1, w_2. By Lemma 3.6, we consider the following three cases.

Case 1. There is a 2-vertex $u \in N(w_1) \bigcap N(w_2)$, such that $uw_1w_2 \in F(G)$.

Let $H = G - u$, then $\Delta(H) = |V(H)| - 2$, and $|V(H)| = |V(G)| - 1 \geq 9$. We know that H is also an h_2-graph with two adjacent $\Delta(H)$-vertex. By the induction hypothesis, H has a $(\Delta(H) + 9)$-list $(2,1)$-labeling. Since $|N(u)| = |\{w_1, w_2\}| = 2$, $|N^2(u)| \leq \Delta(G) - 1$, then

$$|S_G(u)| \leq \Delta(G) + 5 < |L(u)|.$$

We can obtain a $(\Delta(G)+9)$-list $(2,1)$-labeling of G.

Case 2. There is a 3-cycle vw_1w_2v such that its interior contains only a 3-vertex u and three edges uv, uw_1, uw_2.

Let $H = G - u$. Then H is also an h_2-graph with two adjacent $\Delta(H)$-vertices. By the induction hypothesis, H has a $(\Delta(H)+9)$-list $(2,1)$-labeling. For $|N(u)| = |\{v, w_1, w_2\}| = 3$, $|N^2(u)| \leq \Delta(G) - 2$, we have

$$|S_G(u)| \leq \Delta(G) + 7 < |L(u)|.$$

We obtain a $(\Delta(G)+9)$-list $(2,1)$-labeling of G.

Case 3. There are three vertices $u, v_1, v_2 \in N(w_1) \bigcap N(w_2)$ such that $d(u) \leq 4$, $d(v_1) \leq 5$, $d(v_2) \leq 5$ and the interior of the 4-cycle $v_1w_1v_2w_2v_1$ contains only u and the edges incident to u.

Let $H = G - u$. By the induction hypothesis, H has a $(\Delta(H)+9)$-list $(2,1)$-labeling. We consider three subcases and label the vertex u in G.

Case 3.3.1. $uv_1, uv_2 \notin E(G)$, same as Case 1.

Case 3.3.2. u is adjacent to only one of two vertices v_1, v_2. Similar to case 2.

Case 3.3.3. $uv_1, uv_2 \in E(G)$, since $|N(u)| = 4$, $|N^2(u)| \leq \Delta(G) - 3$, then $|S_G(u)| \leq \Delta(G) + 9 < |L(u)|$.

This completes the proof of the Theorem.

Let $C = v_1v_2 \cdots v_{k-2}v_1$ be a cycle of length $k - 2 (\geq 5)$. Add a new vertex w_1 to the interior of C and another w_2 to the exterior respectively, and then join both w_1 and w_2 to each v_i for $i = 1, 2, \cdots, k-2$. Denote the resulting graph by $\widetilde{W}_k$.

It is easy to see that $\widetilde{W}_k$ is an h_2-graph with two nonadjacent Δ-vertices. Moreover, every h_2-graph G containing two nonadjacent Δ-vertices can be induced from $\widetilde{W}_k$ by removing some edges in $E(C)$, where $k = |V(G)|$. Clearly, $\widetilde{W}_k$ has a $(k+7)$-list $(2,1)$-labeling.

Lemma 3.8. *Let G be an h_2-graph with $|V(G)| \geq 5$ and two nonadjacent Δ-vertices w_1, w_2. Then $\lambda_l(G) \leq \Delta(G) + 9$.*

Proof. Since G can be induced from $\widetilde{W}_k$ by removing some edges in $E(C)$, where $k = |V(G)|$, and $\Delta(G) = \Delta(\widetilde{W}_k) = k - 2$. Then $\lambda_l(G) \leq \lambda_l(\widetilde{W}_k) \leq \Delta(G) + 9$.

The Lemma holds.

Theorem 3.2. *Let G be an h_2-graph with $|V(G)| \geq 9$. Then $\lambda_l(G) \leq \Delta(G) + 9$.*

Proof. Suppose that L is a list of G with $|L(v)| = \Delta(G) + 10$ for all $v \in V(G)$. We shall prove the theorem by induction on the number of vertices of graph G.

By enumeration, we can prove the theorem holds for $|V(G)| \leq 9$. Assume that it is true for all h_2-graphs with fewer than $|V(G)| = k$ vertices. Let G be an h_2-graph of order k, we consider the following two cases.

Case 1. If $|V_\Delta(G)| = 2$, by Lemma 3.7 and 3.8, the theorem holds.

Case 2. If $|V_\Delta(G)| = 1$, let w be the vertex with maximum degree and let $x \in N^c(w)$.

If $d(x) \geq 2$, by Lemma 3.5, we obtain a vertex u and let $H = G - u$. Clearly, H is an h_2-graph. By Lemma 3.4, we know that $|V_\Delta(G)| \leq 2$. By the induction hypothesis and Lemma 3.7 and 3.8, H has a $(\Delta(H) + 9)$-list $(2,1)$-labeling. Similarly to the proof of Theorem 3.1, we can obtain a $(\Delta(G) + 9)$-list $(2,1)$-labeling of G.

If $d_G(x) = 1$, then $H = G - x$ is an h_1-graph. By Theorem 3.1, $\lambda_l(H) \leq \Delta(H) + 6 < \Delta(G) + 9$. Since $|N(x)| = 1$, $|N^2(x)| \leq \Delta(G) - 1$, then

$$|S_G(x)| \leq \Delta(G) + 2 < |L(x)|,$$

we can choose a label in list $L(x)$ for vertex x and obtain a $(\Delta(G) + 9)$-list $(2,1)$-labeling of G.

This completes the proof of the Theorem.

4 The List $(2,1)$-labeling of Halin graphs

For any 3-connected planar graph G with $\Delta(G) \geq 3$, if the boundary edges of face f_0 which is adjacent to the others are removed, it becomes a tree, and the degree of each vertex of $V(f_0)$ is 3, then G is called a *Halin* graph. f_0 is called the outer face of G, and the others called interior faces. The vertices on face f_0 are called outer vertices, the others are called the interior vertices[4]. Clearly, a wheel $G = [v_0; v_1, \cdots v_\Delta]$ is a Halin graph with only one interior vertex v_0.

By the definition of Halin graph, the following lemma is true.

Lemma 4.1. *Let G be a Halin graph. Then*

(1) *The degree of all outer vertices is* 3;

(2) *If G is not a wheel, there are at least two interior vertices of G, and there always exists an interior vertex w which is only adjacent to one interior, and $N(w) = \{u, u_1, \cdots u_k\}$, $u_1v_1, u_kv_2 \in E(G)$, $v_1 \neq u_2$, $v_2 \neq u_{k-1}$, $2 \leq k \leq \Delta(G) - 1$, where u is the interior vertex adjacent to w, and $u_1, \cdots u_k$ are outer vertices adjacent to w.*

The following theorem estimates an upper bound of the λ_l-number of Halin graphs.

Theorem 4.1. *Let G be a Halin graph with maximum degree Δ. Then*

$$\lambda_l(G) \leq \Delta + 9.$$

Proof. Suppose that L is a list of G with $|L(v)| = \Delta(G) + 10$ for all $v \in V(G)$. We shall prove the theorem by induction on $|V_I(G)|$, the number of interior vertices of G.

If $|V_I(G)| = 1$, then G is a wheel. It is also an h_1-graph, by Theorem 3.1, the theorem holds.

Let G ba a Halin graph and for all Halin graphs G' with $|V_I(G')| < |V_I(G)|$, the theorem is true and $|V_I(G)| \geq 2$.

Let $G' = G - \{u_1, \cdots u_k\} + \{v_1w, v_2w\}$, by Lemma 4.1 and the definition of Halin graph, G' is a Halin graph and $|V_I(G')| = |V_I(G)| - 1$. By the induction hypothesis, G' has a $(\Delta(G') + 9)$-list $(2, 1)$-labeling.

Now we label vertices $u_1, \cdots u_k$ in order of ascending subscripts. First label the vertex u_1. Since $N(u_1) = \{w, v_1, u_2\}$ and u_2 has no label, then $|S_G(u_1)| \leq 8$. For $2 \leq i \leq k$, we note that $u_i, \cdots u_k$ have not been labelled and $u_1, \cdots u_{i-1}$ have been labelled. It suffices to prove that $|S_G(u_k)| \leq \Delta + 9$. In fact, since $N(u_k) = \{w, v_2, u_{k-1}\}$, $|N(v_2)| = 3$ and $d(w) \leq \Delta(G)$, then

$$|S_G(u_k)| \leq 9 + (\Delta(G) - 2) + 2 = \Delta(G) + 9.$$

We can choose a label in list $L(u_k)$ for vertex u_k, then we obtain a $(\Delta(G)+9)$-list $(2,1)$-labeling. This prove the Theorem.

Acknowledgments The authors are indebted to the anonymous referees for their constructive suggestions.

References

[1] J.A.Bondy and U.S.R.Murty, *Graph theory with applications*,[M] MacMillan,London (1976).

[2] O.V.Borodin and D.R.Woodall, *Thirteen colouring numbers for outerplane graphs*,[J] Bul. Inst. Combin. and Appl. **14** (1995),87-100.

[3] G.J.Chang and D.Kuo, *The L(2,1)-labelingproblem on graphs*,[J] SIAM J. Discrete Math. **9** (1996),309-316.

[4] R.Halin, *Studies an minimally n-connected graph*, In Comb. Math. and Its Applications (Proc. Conf. Oxford,1969),Academic Press, London,(1971),129-136.

[5] J.V.D.Heuvel and S.McGuinness, *Coloring the square of a planar graph*,[J] J.Graph Theory. **42**(2003), 110-124.

[6] W.F.Wang, *A characterization of the entire chromatic number of plane graphs with high maximum degree.*,[J] Acta Math. Scientia(in chinese). **20** (2000),644-649.

[7] W.F.Wang and K.M.Zhang *Edge-face chromatic number of plane graphs with high maximum degree.*,[J] Australa.J.Combin. **18** (1998),235-244.

[8] Z.F.Zhou and W.F.Wang, *The L(2,1)-labeling of mycielski graphs.*,[J] J. Zhejiang Normal university(in chinese). **5** (1992),586-595.