Regular Weighted Graphs Without Positive Cuts

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Abstract. For a simple and finite graph $G = (V, E)$ let $w_{\max}(G)$ be the maximum total weight $w(E) = \sum_{e \in E} w(e)$ of G over all weight functions $w : E \to \{-1, 1\}$ such that G that has no positive cut, i.e. all cuts C satisfy $w(C) \leq 0$.

For $r \geq 1$ we prove that $w_{\max}(G) \leq -\frac{|V|}{2}$ if G is $(2r-1)$-regular and $w_{\max}(G) \leq -\frac{r|V|}{2r+1}$ if G is $2r$-regular. We conjecture the existence of a constant c such that $w_{\max}(G) \leq -\frac{5|V|}{6} + c$ if G is a connected cubic graph and prove a special case of this conjecture. Furthermore, as a weakened version of this conjecture we prove that $w_{\max}(G) \leq -\frac{2|V|}{3} + \frac{2}{3}$ if G is a connected cubic graph.

Keywords. cut; negative weights

1 Introduction

We use standard graph-theoretical terminology and consider simple and finite graphs $G = (V, E)$ together with a weight function $w : E \to \{-1, 1\}$ that assigns weights of -1 or $+1$ to the edges. The pair (G, w) is called a *weighted graph.*

For a weighted graph (G, w) let $E^+ = \{e \in E \mid w(e) = 1\}$, $E^- = \{e \in E \mid w(e) = -1\}$, $G^+ = (V, E^+)$ and $G^- = (V, E^-)$. We denote the *neighbourhood* (*vertex degree*) of $v \in V$ in the graphs G, G^+ and G^- as $N(v)$, $N^+(v)$ and $N^-(v)$ ($d(v)$, $d^+(v)$ and $d^-(v)$), respectively. A graph $G = (V, E)$ is *r-regular*, if $d(v) = r$ for all $v \in V$. Let K_n and K_{n_1,n_2} denote the complete graph of order n and the complete bipartite graph with partite sets of cardinality n_1 and n_2, respectively. A *cut* of a graph $G = (V, E)$ defined by some set $U \subseteq V$ of vertices consists of the set of edges of G with exactly one endpoint in U and is denoted by $\delta_G(U)$. A cut $\delta_G(U)$ of a weighted graph (G, w) is *positive*, if

$$w(\delta_G(U)) = \sum_{e \in \delta_G(U)} w(e) = |\delta_G(U) \cap E^+| - |\delta_G(U) \cap E^-| > 0.$$

Cuts in graphs whose edges have only non-negative weights are among the most relevant and well-studied objects in graph theory. Whereas minimum cuts are fundamental - and easy to determine - because of their natural relation to flows in graphs [1, 2] the approximation of the maximum cut problem received tremendous attention (cf. [3, 4] and the references therein). Most of the arguments developed for cuts in graphs with non-negative weights fail if negative weights are allowed. The typical problems considered for the so-called *signed graphs* whose edges are labeled with a positive or negative sign do not include cuts and their weight (cf. the references in [6]).

Nevertheless, positive cuts in graphs with ± 1 weights have algorithmical relevance. To give just one simple example consider the number of **not**-gates in a boolean circuit [5] using just **nand**-, **nor**- and **not**-gates. Exchanging a single **nand**-gate with a **nor**-gate a **not**-gate has to be inserted or removed on all incident arcs (see Figure 1). This observation easily implies that the number of **not**-gates can not be reduced by exchanging some **nand**- with **nor**-gates if and only if the corresponding weighted graph has no positive cut.

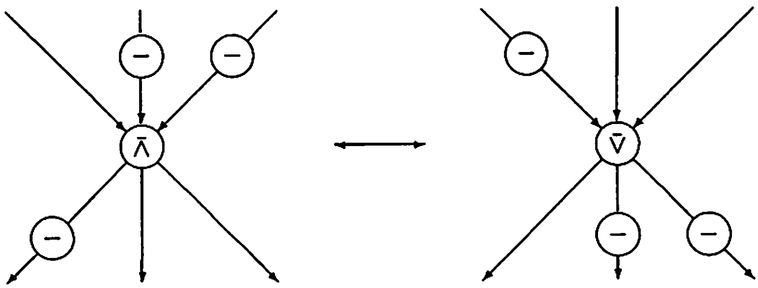

Figure 1

Now let (G, w) be a weighted graph that has no positive cut. A standard exchange argument easily implies that (G, w) has a cut of weight at least $\frac{1}{2}w(E)$. Hence (G, w) must satisfy $w(E) \leq 0$. The motivation for the results presented here is the observation that this last conclusion is not best-possible. In fact, one easily sees that $w(E)$ has to be strictly negative in order to avoid positive cuts.

In the present paper we will investigate this effect for regular graphs. In Section 2 we prove best-possible results for regular graph. In Section 3 we consider cubic (= 3-regular) graphs in more detail. We pose a conjecture for cubic graphs and prove some related results.

2 Regular graphs

We first consider regular graphs of odd degree.

Proposition 1 *For some $r \geq 1$ let $G = (V, E)$ be a connected $(2r-1)$-regular graph of order n. If $w : E \to \{\pm 1\}$ is such that (G, w) has no positive cut, then $w(E) \leq -\frac{n}{2}$ with equality if and only if $G^+ \cong K_r \cup K_r$ and $G^- \cong K_{r,r}$.*

Proof: Let (G, w) be as in the statement of the proposition. If $d^+(u) \geq r$ for some $u \in V$, then $\delta_G(\{u\})$ is a positive cut which is a contradiction. Hence $d^+(u) \leq r-1$ for all $u \in V$ and therefore $w(E) = \frac{1}{2}\sum_{u \in V}(d^+(u) - d^-(u)) \leq -\frac{n}{2}$.

Now let $w(E) = -\frac{n}{2}$. This immediately implies that $d^+(u) = r-1$ for all $u \in V$. Let $u_1 \in V$ and let $N^-(u_1) = \{v_1, v_2, ..., v_r\}$.

If $v_i v_j \notin E^+$ for some $1 \leq i < j \leq r$, then $\delta_G(\{u_1, v_i, v_j\})$ is a positive cut which is a contradiction. Hence $N^-(u_1)$ is a clique in G^+. Let $N^-(v_1) = \{u_1, u_2, ..., u_r\}$. By symmetry, $N^-(v_1)$ is a clique in G^+.

If for some $2 \leq i \leq r-1$ there is a vertex $w \in N^-(u_i) \setminus N^-(u_1)$, then as above $v_1 w \in E^+$ which implies the contradiction $d^+(v_1) \geq (|N^-(u_1)| - 1) + 1 = r$. Hence $N^-(u_i) = N^-(u_1)$ for $2 \leq i \leq r-1$ and (G, w) has the described structure.

It remains to prove that (G, w) with $G^+ \cong K_r \cup K_r$ and $G^- \cong K_{r,r}$ has no positive cut. Let V_1 and V_2 denote the partite sets of G^-. Let $U \subseteq V$. If $n_i = |U \cap V_i|$ for $i = 1, 2$, then

$$\begin{aligned} w(\delta_G(U)) &= (r - n_1)n_1 + (r - n_2)n_2 - (r - n_1)n_2 - (r - n_2)n_1 \\ &= -(n_1 - n_2)^2 \leq 0 \end{aligned}$$

and the proof is complete. □

For regular graphs of even degree the following similar result holds.

Theorem 1 *For some $r \geq 1$ let $G = (V, E)$ be a connected $2r$-regular graph of order n. If $w : E \to \{\pm 1\}$ is such that (G, w) has no positive cut, then $w(E) \leq -\frac{rn}{2r+1}$ with equality if and only if $G^+ \cong K_{r-1} \cup K_{r+1}$ and $G^- \cong K_{r-1,r+1}$.*

Proof: Let (G, w) be as in the statement of the theorem. For $i \geq 0$ let $n_i = |\{v \in V \mid d^+(v) = i\}|$. If $n_i > 0$ for some $i \geq r+1$, then $\delta_G(\{v\})$ is a positive cut for some $v \in V$ with $d^+(v) = i$ which is a contradiction. Hence $n_i = 0$ for $i \geq r+1$.

If there are vertices $u, v \in V$ with $d^+(u) = d^+(v) = r$ and $uv \in E^-$, then $\delta_G(\{u, v\})$ is a positive cut which is a contradiction. Hence the set of vertices $v \in V$ with $d^+(v) = r$ is an independent set in G^- and thus $rn_r \leq \sum_{i=1}^{r}(r+i)n_{r-i}$ which implies

$$rn = rn_r + \sum_{i=1}^{r} rn_{r-i} \leq \sum_{i=1}^{r}(2r+i)n_{r-i}.$$

Since $w(E) = -\sum_{i=1}^{r} in_{r-i}$, the optimum value of the following linear program is an upper bound on $w(E)$

$$\begin{aligned} \max \quad & -\sum_{i=1}^{r} ix_{r-i} \\ s.t. \quad & rn \leq \sum_{i=1}^{r}(2r+i)x_{r-i} \qquad (1) \\ & x_{r-i} \geq 0 \text{ for } 1 \leq i \leq r. \end{aligned}$$

Let $x_0, x_1, ..., x_{r-1}$ be an optimum solution. Clearly, (1) is satisfied with equality. If there is some $2 \leq i \leq r$ with $x_{r-i} > 0$, then decreasing x_{r-i} by $\frac{\epsilon}{2r+i}$ and increasing x_{r-1} by $\frac{\epsilon}{2r+1}$ for some small $\epsilon > 0$ would improve the solution. Hence $x_{r-i} = 0$ for $2 \leq i \leq r$ which implies $x_{r-1} = \frac{rn}{2r+1}$ and thus $w(E) \leq -\frac{rn}{2r+1}$.

Now let $w(E) = -\frac{rn}{2r+1}$. In view of the above this implies $n_{r-i} = 0$ for $2 \leq i \leq r$, $n_{r-1} = \frac{rn}{2r+1}$, $n_r = \frac{(r+1)n}{2r+1}$ and all edges $uv \in E^-$ satisfy $\{d^+(u), d^+(v)\} = \{r-1, r\}$.

Let $u_1 \in V$ with $d^+(u_1) = r-1$ and let $N^-(u_1) = \{v_1, v_2, ..., v_{r+1}\}$. If $v_iv_j \notin E^+$ for some $1 \leq i < j \leq r+1$, then $\delta_G(\{u_1, v_i, v_j\})$ is a positive cut which is a contradiction. Hence $N^-(u_1)$ is a clique in G^+.

Let $N^-(v_1) = \{u_1, u_2, ..., u_{r-1}\}$. If for some $2 \leq i \leq r-1$ there is a vertex $w \in N^-(u_i) \setminus N^-(u_1)$, then as above $v_1w \in E^+$ which implies the contradiction $d^+(v_1) \geq (|N^-(u_1)|-1)+1 = r+1$. Hence $N^-(u_i) = N^-(u_1)$ for $2 \leq i \leq r-1$.

If for some $1 \leq i \leq r-1$ we have $N^+(u_i) \neq \{u_1, u_2, ..., u_{r-1}\} \setminus \{u_i\}$, then $\delta_G(N^-(u_1) \cup N^-(v_1))$ is a positive cut which is a contradiction. Hence $N^+(u_i) = \{u_1, u_2, ..., u_{r-1}\} \setminus \{u_i\}$ for $1 \leq i \leq r-1$ and (G, w) has the described structure.

Again it remains to prove that (G, w) with $G^+ \cong K_{r-1} \cup K_{r+1}$ and $G^- \cong K_{r-1,r+1}$ has no positive cut. This can be done as in the proof of Proposition 1 and we leave it to the reader which completes the proof. □

3 Improvements for cubic graphs

We pose the following conjecture.

Conjecture 1 *There is some constant c such that the following holds.*
If $G = (V, E)$ is a connected cubic graph of order n and $w : E \to \{\pm 1\}$ is such that (G, w) has no positive cut, then $w(E) \le -\frac{5}{6}n + c$.

The graphs in Figure 2 are examples of graphs that satisfy the assumptions of Conjecture 1 and have $w(E) = -\frac{5}{6}n$ (the dotted edges are those of weight 1). Since it is obvious how to generalize these graphs to arbitrarily large orders, Conjecture 1 would be best possible.

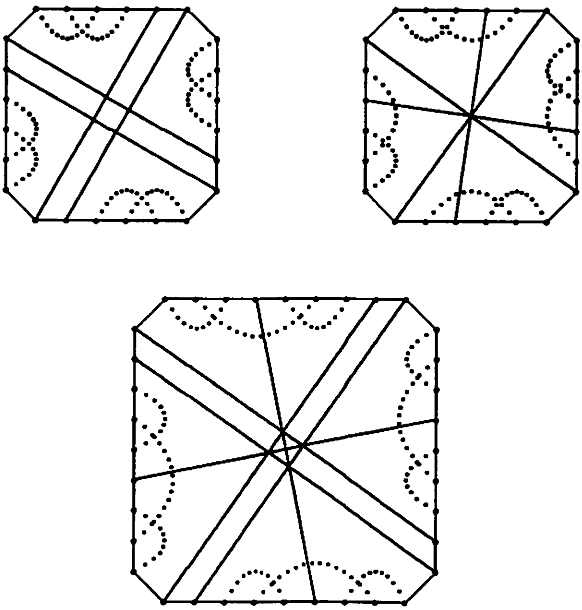

Figure 2

The graphs in Figure 3 are examples of graphs which imply that c must be positive.

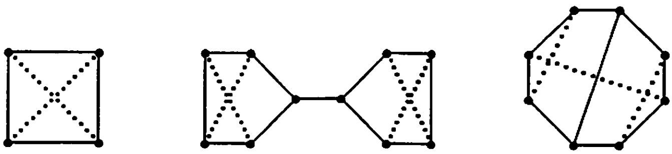

Figure 3

With the following lemma we prove Conjecture 1 for weighted graphs (G, w) whose vertex set can be covered by a bounded number of paths in G^-.

Lemma 1 *Let $G = (V, E)$ be a connected cubic graph of order n and let $w : E \to \{\pm 1\}$ be such that (G, w) has no positive cut. Let $V^+ = \{v \in V \mid d^+(v) = 1\}$ and $V^- = V \setminus V^+$.*

If $P : x_0 x_1 x_2 x_3 ... x_l$ is a path in G^-, then

$$|V^+ \cap \{x_0, x_1, ..., x_l\}| \leq 2|V^- \cap \{x_0, x_1, ..., x_l\}| + 8.$$

Proof: Let (G, w) and $P : x_0 x_1 x_2 x_3 ... x_l$ be as in the statement of the lemma. Clearly, $d^+(v)$ is either 1 or 0 for every $v \in V$, i.e. $V^- = \{v \in V \mid d^+(v) = 0\}$. For $0 \leq i \leq l$ let $n_i^+ = |V^+ \cap \{x_0, x_1, ..., x_i\}|$ and $n_i^- = |V^- \cap \{x_0, x_1, ..., x_i\}|$.

We may assume without loss of generality that $n_i^+ > 2n_i^-$ for all $0 \leq i \leq l$. (Otherwise, we consider an appropriate subpath of P.)

First, we assume that there are five consecutive vertices $x_i, x_{i+1}, ..., x_{i+4} \in V^+$. Since $\delta_G(\{x_i, x_{i+1}, x_{i+2}\})$ is not a positive cut, $x_i x_{i+2} \in E^+$. Since $\delta_G(\{x_{i+1}, x_{i+2}, x_{i+3}\})$ is not a positive cut, $x_{i+1} x_{i+3} \in E^+$.

Now $\delta_G(\{x_{i+2}, x_{i+3}, x_{i+4}\})$ is a positive cut which is a contradiction. Hence no more than four consecutive vertices on P belong to V^+ and we may assume that $l \geq 8$.

Let $0 \leq i \leq l$. We call i an *index of type I*, if $x_{i-3}, x_{i-1}, x_i \in V^+$, $x_{i-2} \in V^-$, $n_i^+ = 2n_i^- + 3$ and either $x_{i-3} x_{i-1} \in E^+$ or $x_{i-3} x_i \in E^+$. Similarly, we call i an *index of type II*, if $x_{i-4}, x_{i-3}, x_{i-1}, x_i \in V^+$, $x_{i-2} \in V^-$, $n_i^+ = 2n_i^- + 3$, either $x_{i-4} x_{i-1} \in E^+$ or $x_{i-4} x_i \in E^+$ and neither $x_{i-3} x_{i-1} \in E^+$ nor $x_{i-3} x_i \in E^+$.

Our proof will proceed as follows. Firstly, we prove the existence of an index of type I or II. Secondly, we prove that if i is an index of type I or II and $l - i \geq 6$, then there is an index j of type I or II such that $j > i$. It is clear, that these two steps imply the desired result.

Claim 1 There is an index of type I or II.

Proof of Claim 1: Since $n_2^+ > 2n_2^-$, we have $x_0, x_1, x_2 \in V^+$. Since $\delta_G(\{x_0, x_1, x_2\})$ is not a positive cut, $x_0 x_2 \in E^+$.

Firstly, we assume that $x_3 \in V^+$. Since $\delta_G(\{x_1, x_2, x_3\})$ is not a positive cut, $x_1 x_3 \in E^+$. Since $\delta_G(\{x_2, x_3, x_4\})$ is not a positive cut, $x_4 \in V^-$. Since $n_5^+ > 2n_5^-$, we have $x_5 \in V^+$. Since $\delta_G(\{x_2, x_3, x_4, x_5, x_6\})$ is not a positive cut, $x_6 \in V^-$. Since $n_7^+ > 2n_7^-$, we have $x_7 \in V^+$. Since $n_8^+ > 2n_8^-$, we have $x_8 \in V^+$. Since $\delta_G(\{x_2, x_3, ..., x_8\})$ is not a positive cut, either $x_5 x_7 \in E^+$ or $x_5 x_8 \in E^+$. This implies that 8 is an index of type I.

Secondly, we assume that $x_3 \in V^-$. Since $n_4^+ > 2n_4^-$, we have $x_4 \in V^+$. Since $n_5^+ > 2n_5^-$, we have $x_5 \in V^+$. Since $\delta_G(\{x_1, x_2, ..., x_5\})$ is not a positive cut, either $x_1x_4 \in E^+$ or $x_1x_5 \in E^+$. Clearly, neither $x_2x_4 \in E^+$ nor $x_2x_5 \in E^+$. This implies that 5 is an index of type II and the proof of the claim is complete. □

Claim 2 If i is an index of type I and $l - i \geq 6$, then there is an index j of type I or II such that $j > i$.

Proof of Claim 2: Let i be an index of type I and let $l - i \geq 6$.

Firstly, we assume that $x_{i+1} \in V^+$. Since $\delta_G(\{x_{i-1}, x_i, x_{i+1}\})$ is not a positive cut, $x_{i-1}x_{i+1} \in E^+$. Since i is of type I, $x_{i-3}x_i \in E^+$. Since $\delta_G(\{x_i, x_{i+1}, x_{i+2}\})$ is not a positive cut, $x_{i+2} \in V^-$. Since $n_{i+3}^+ > 2n_{i+3}^-$, we have $x_{i+3} \in V^+$. Since $\delta_G(\{x_i, x_{i+1}, ..., x_{i+4}\})$ is not a positive cut, $x_{i+4} \in V^-$. Since $n_{i+5}^+ > 2n_{i+5}^-$, we have $x_{i+5} \in V^+$. Since $n_{i+6}^+ > 2n_{i+6}^-$, we have $x_{i+6} \in V^+$. Since $\delta_G(\{x_i, x_{i+1}, ..., x_{i+6}\})$ is not a positive cut, either $x_{i+3}x_{i+5} \in E^+$ or $x_{i+3}x_{i+6} \in E^+$. This implies that $i+6$ is an index of type I.

Hence we may assume that $x_{i+1} \in V^-$. Since $n_{i+2}^+ > 2n_{i+2}^-$, we have $x_{i+2} \in V^+$. Since $n_{i+3}^+ > 2n_{i+3}^-$, we have $x_{i+3} \in V^+$. Since $\delta_G(\{x_{i-1}, x_i, ..., x_{i+3}\})$ is not a positive cut, either $x_{i-1}x_{i+2} \in E^+$ or $x_{i-1}x_{i+3} \in E^+$ or $x_ix_{i+2} \in E^+$ or $x_ix_{i+3} \in E^+$. If $x_{i-3}x_{i-1} \in E^+$, then $i + 3$ is an index of type I.

Hence we may assume that $x_{i-3}x_i \in E^+$ and either $x_{i-1}x_{i+2} \in E^+$ or $x_{i-1}x_{i+3} \in E^+$. This implies that $i+3$ is an index of type II and the proof of the claim is complete. □

Claim 3 If i is an index of type II and $l - i \geq 6$, then there is an index j of type I or II such that $j > i$.

Proof of Claim 3: Let i be an index of type II and let $l - i \geq 6$.

First, we assume that $x_{i+1} \in V^+$. Since $\delta_G(\{x_{i-1}, x_i, x_{i+1}\})$ is not a positive cut, $x_{i-1}x_{i+1} \in E^+$. Since i is of type II, $x_{i-4}x_i \in E^+$. Since $\delta_G(\{x_i, x_{i+1}, x_{i+2}\})$ is not a positive cut, $x_{i+2} \in V^-$. Since $n_{i+3}^+ > 2n_{i+3}^-$, we have $x_{i+3} \in V^+$. Since $\delta_G(\{x_i, x_{i+1}, ..., x_{i+4}\})$ is not a positive cut, $x_{i+4} \in V^-$. Since $n_{i+5}^+ > 2n_{i+5}^-$, we have $x_{i+5} \in V^+$. Since $n_{i+6}^+ > 2n_{i+6}^-$, we have $x_{i+6} \in V^+$. Since $\delta_G(\{x_i, x_{i+1}, ..., x_{i+6}\})$ is not a positive cut, either $x_{i+3}x_{i+5} \in E^+$ or $x_{i+3}x_{i+6} \in E^+$. This implies that $i+6$ is an index of type I.

Hence we may assume that $x_{i+1} \in V^-$. Since $n_{i+2}^+ > 2n_{i+2}^-$, we have $x_{i+2} \in V^+$. Since $n_{i+3}^+ > 2n_{i+3}^-$, we have $x_{i+3} \in V^+$.

Next, we assume that $x_{i-4}x_{i-1} \in E^+$. Since $\delta_G(\{x_{i-1}, x_i, ..., x_{i+3}\})$ is not a positive cut, either $x_ix_{i+2} \in E^+$ or $x_ix_{i+3} \in E^+$. This implies that $i + 3$ is an index of type I.

Hence we may assume that $x_{i-4}x_i \in E^+$. Since $\delta_G(\{x_{i-1}, x_i, ..., x_{i+3}\})$ is not a positive cut, either $x_{i-1}x_{i+2} \in E^+$ or $x_{i-1}x_{i+3} \in E^+$. This implies that $i+3$ is an index of type II and the proof of the claim is complete. □

Claims 1, 2 and 3 complete the proof. □

Theorem 2 *Let c' be some fixed integer. Let $G = (V, E)$ be a connected cubic graph of order n and let $w : E \to \{\pm 1\}$ be such that (G, w) has no positive cut.*

If there is a collection $\mathcal{P}$ of at most c' disjoint paths of G^- such that every vertex of V belongs to one path in $\mathcal{P}$, then $w(E) \leq -\frac{5}{6}n + \frac{8}{3}c'$.

Proof: Let (G, w) and $\mathcal{P} = \{P_1, P_2, ..., P_k\}$ with $k \leq c'$ be as in the statement of the theorem.

Let $V^+ = \{v \in V \mid d^+(v) = 1\}$, $V^- = \{v \in V \mid d^+(v) = 0\} = V \setminus V^+$ and $n^{\pm} = |V^{\pm}|$. For $1 \leq i \leq k$ let $n_i^{\pm}$ denote the number of vertices in $V^{\pm}$ that belong to P_i.

By Lemma 1, $n_i^+ \leq 2n_i^- + 8$ for $1 \leq i \leq k$. This implies $n^+ \leq 2n^- + 8c'$ and hence $n \leq 3n^- + 8c'$. We obtain

$$w(E) = -\frac{n^+}{2} - \frac{3n^-}{2} = -\frac{n}{2} - n^- \leq -\frac{n}{2} - \frac{n - 8c'}{3} = -\frac{5n}{6} + \frac{8c'}{3}$$

and the proof is complete. □

We close this section with a weakened version of Conjecture 1.

Theorem 3 *If $G = (V, E)$ is a connected cubic graph of order n and $w : E \to \{\pm 1\}$ is such that (G, w) has no positive cut, then $w(E) \leq -\frac{2}{3}n + \frac{2}{3}$.*

Proof: Let (G, w) be as in the statement of the theorem. As before, let $V^+ = \{v \in V \mid d^+(v) = 1\}$, $V^- = \{v \in V \mid d^+(v) = 0\} = V \setminus V^+$ and $n^{\pm} = |V^{\pm}|$.

If G^- is not connected and U is the vertex set of a component of G^-, then $\delta_G(U)$ is a positive cut which is a contradiction. Hence G^- is connected. Let $T_0 = (V_0, E_0)$ be a spanning tree of G^-, i.e. $V_0 = V$. We will prove the existence of trees $T_0, T_1, ..., T_l$ with $T_i = (V_i, E_i)$ for $1 \leq i \leq l$ such that

(i) T_i is a subtree of T_{i-1} for $1 \leq i \leq l$,

(ii) $|(V_{i-1} \setminus V_i) \cap V^+| \leq 5|(V_{i-1} \setminus V_i) \cap V^-|$ for $1 \leq i \leq l$ and

(iii) either $|V_l \cap V^+| \leq 5|V_l \cap V^-|$ or $|V_l| \leq 4$.

Therefore, we assume that for some $i \geq 0$ the tree T_i with $|V_i| \geq 5$ has already been constructed and explain how to construct T_{i+1} or how to terminate the sequence.

Let $d_i(v)$ denote the degree of $v \in V_i$ in the tree T_i. Let T_i be rooted at some vertex r with $d_i(r) = 1$ and let v_1 be a vertex of T_i at maximum distance from r. Clearly, $d_i(v_1) = 1$. Let $v_1 v_2 v_3 \ldots v_s = r$ be the path from v_1 to r in T_i. Note that $s \geq 4$. We consider different cases.

Case 1 No vertex at distance at most 4 from v_1 has two children.

Clearly, this implies $s \geq 5$. As in the proof of Lemma 1 not all five consecutive vertices $v_1, v_2, \ldots, v_5$ belong to V^+. If $v_5 = r$, then let $l = i$ and terminate the sequence, otherwise let $T_{i+1} = T_i \setminus \{v_1, v_2, v_3, v_4, v_5\}$. Hence, we may assume from now on that the assumption of this case is false.

Case 2 Either v_2 or v_3 has two children.

Let U denote the set containing v_3 and all descendants of v_3. It is easy to check that $|U \cap V^+| \leq 5|U \cap V^-|$. Let $T_{i+1} = T_i \setminus U$. Hence we may assume from now on that the assumption of this case is false.

Case 3 v_4 has two children.

Clearly, $v_4 \neq r$. Let U denote the set containing v_4 and all descendants of v_4. If $|U \cap V^+| \leq 5|U \cap V^-|$, then let $T_{i+1} = T_i \setminus U$. Hence, we may assume that $|U \cap V^+| > 5|U \cap V^-|$.

This implies that there are vertices $u_1, u_2, u_3 \in V_i$ such that $U = \{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}$, $u_1u_2, u_2u_3, u_3v_4 \in E_i$ and $v_1, v_2, v_3, u_1, u_2, u_3 \in V^+$.

Since $\delta_G(\{v_1, v_2, v_3\})$ is not a positive cut, $v_1v_3 \in E^+$.

Since $\delta_G(\{u_1, u_2, u_3\})$ is not a positive cut, $u_1u_3 \in E^+$.

Since $\delta_G(\{v_2, v_3, u_2, u_3, v_4\})$ is not a positive cut, $v_2u_2 \in E^+$.

Since $\delta_G(\{v_2, v_3, v_4, v_5, u_3\})$ is not a positive cut, $v_5 \in V^-$.

If $v_5 = r$, then let $l = i$ and terminate the sequence. Hence, we may assume that $v_5 \neq r$. If v_5 has just one child, then let $T_{i+1} = T_i \setminus (U \cup \{v_5\})$. Hence, we may assume that v_5 has two children.

Let U' denote the set containing v_5 and all descendants of v_5. It is tedious but simple to check that $|U' \cap V^+| \leq 5|U' \cap V^-|$. Therefore, let $T_{i+1} = T_i \setminus U'$ and we may assume from now on that the assumption of this case is false.

Case 4 v_5 has two children.

Clearly, $v_5 \neq r$. Let U denote the set containing v_5 and all descendants of v_5. If $|U \cap V^+| \leq 5|U \cap V^-|$, then let $T_{i+1} = T_i \setminus U$. Hence, we may assume $|U \cap V^+| > 5|U \cap V^-|$.

In view of the above cases, we can assume without loss of generality that there are $2 \leq j \leq 4$ vertices $u_1, u_2, ..., u_j \in V_i$ such that $u_1u_2, ...u_{j-1}u_j$, $u_jv_5 \in E_i$, $v_1, v_2, v_3, v_4, u_1, u_2, ..., u_j \in V^+$ and $v_1v_3, v_2v_4 \in E^+$.

Now $\delta_G(\{v_3, v_4, u_{j-1}, u_j, v_5\})$ is a positive cut, which is a contradiction.

From the above cases the existence of the trees $T_0, T_1, ..., T_l$ is obvious. The properties (i), (ii) and (iii) imply $n^+ \leq 5n^- + 4$ and thus $w(E) \leq -\frac{2}{3}n + \frac{2}{3}$ which completes the proof. □

Note that the left graph in Figure 3 satisfies $w(E) = -\frac{2}{3}n + \frac{2}{3}$. Finally, we want to mention that a quite tedious extension of the arguments used in the proof of Theorem 3 yields a bound of the form $w(E) \leq -\frac{3}{5}n + O(1)$.

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SUMS OF GENERALIZED FIBONACCI NUMBERS BY MATRIX METHODS

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ABSTRACT. In this paper, we consider a certain second order linear recurrence and then give generating matrices for the sums of positively and negatively subscripted terms of this recurrence. Further, we use matrix methods and derive explicit formulas for these sums.

1. INTRODUCTION

The Fibonacci sequence is defined by the following equation for $n > 1$

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers have many interesting properties. For example, the sums of the Fibonacci numbers subscripted from 1 to n can be expressed by a formula including Fibonacci numbers. The sums formula is given by

$$\sum_{i=1}^{n} F_i = F_{n+2} - F_1.$$

Matrix methods many times have played an important role stemming from the number theory [1-5]. For instance, let B be an 2×2 companion matrix as follows

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then it is well known that

$$B^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Now we consider a generalization of the Fibonacci numbers. Let A be nonzero integer satisfying $A^2 + 4 \neq 0$. The generalized Fibonacci sequence $\{u_n\}$ is defined by the recurrence relation for $n > 1$

$$u_{n+1} = Au_n + u_{n-1}, \tag{1.1}$$

where $u_0 = 0$ and $u_1 = 1$. For later use, note that $u_2 = A$, $u_3 = A^2 + 1$ and $u_4 = A^3 + 2A$. When $A = 2$, then $u_n = P_n$ (nth Pell number).

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Let α and β be the roots of the equation $x^2 - Ax - 1 = 0$, then the Binet formula of the sequence $\{u_n\}$ has the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Using the recurrence relation of sequence $\{u_n\}$, we can obtain the negatively subscripted terms and these terms satisfy

$$u_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta}.$$

Since $\alpha\beta = -1$, then we have

$$u_{-n} = (-1)^{n+1} u_n \text{ and } u_{-n} = Au_{-(n+1)} + u_{-(n+2)}. \tag{1.2}$$

Thus for later use $u_{-1} = 1$, $u_{-2} = -A$, $u_{-3} = A^2 + 1$ and $u_{-4} = -(A^3 + 2A)$.

Furthermore, by the inductive argument, one can easily verify that the generating matrix for the sequence $\{u_n\}$ is given by

$$W^n = \begin{bmatrix} A & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{bmatrix}. \tag{1.3}$$

In this paper, we construct certain matrices, then we compute the nth powers of these matrices which are the generating matrices for the sums of the positively and negatively subscripted terms of the sequence $\{u_n\}$ from 1 to n.

2. Generating matrix for the sums of the positively subscripted terms of the sequence $\{u_n\}$

In this section we consider the positively subscripted terms of the sequence $\{u_n\}$ and then define a 3×3 matrix C. Further, we compute the nth power of the matrix C and use matrix methods for the explicit formula for the sums of the terms of the sequence $\{u_n\}$.

Define the 3×3 matrix C as follows

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.1}$$

and define the 3×3 matrix E_n as follows

$$E_n = \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ & u_{n+1} & u_n \\ S_{n-1}^+ & u_n & u_{n-1} \end{bmatrix}, \tag{2.2}$$

where S_n^+ denote the sums of the positively subscripted terms of the sequence $\{u_n\}$ from 1 to n, that is

$$S_n^+ = \sum_1^n u_i. \tag{2.3}$$

Then we have the following Lemma.

Lemma 1. *Let the matrices C and E_n have the forms (2.1) and (2.2), respectively. Then for n, $n > 0$*

$$E_n = C^n. \tag{2.4}$$

Proof. We will use the induction method for the proof of Lemma. If $n = 1$, then, by $u_2 = A$, $u_1 = 1$ and $u_0 = 0$, we obtain

$$C^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ S_1^+ & u_2 & u_1 \\ S_0^+ & u_1 & u_0 \end{bmatrix} = E_1.$$

If $n = 2$, then

$$C^2 = \begin{bmatrix} 1 & 0 & 0 \\ A+1 & A^2+1 & A \\ 1 & A & 1 \end{bmatrix}.$$

Since $S_2^+ = A + 1$ and $u_3 = A^2 + 1$, $E_2 = C^2$. Suppose that the claim is true for n. Then we will show that the equation holds for $n + 1$. Thus, by our assumption, we write

$$\begin{aligned} C^{n+1} &= C^n C = E_n C \\ &= \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ & u_{n+1} & u_n \\ S_{n-1}^+ & u_n & u_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

which, by a matrix multiplication, satisfies

$$C^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ + u_{n+1} & Au_{n+1} + u_n & u_{n+1} \\ S_{n-1}^+ + u_n & Au_n + u_{-1} & u_n \end{bmatrix} = E_{n+1}.$$

By the recurrence relation of the sequence $\{u_n\}$ and since $S_n^+ + u_{n+1} = S_{n+1}^+$, we have the conclusion. □

Consequently, we obtain a generating matrix for the sums of the terms of the sequence $\{u_n\}$ from 1 to n.

Also we write the Eq. (2.4) as shown

$$E_{n+1} = E_n E_1 = E_1 E_n. \tag{2.5}$$

In other words, the matrix E_1 is commutative under matrix multiplication. Then we have the Corollary.

Corollary 1. *Let the sum S_n^+ have the form (2.3). Then the sum S_n^+ satisfies the following nonhomogeneous recurrence relation for $n > 0$*

$$S_{n+1}^+ = AS_n^+ + S_{n-1}^+ + 1.$$

Proof. From (2.5) and since an element of E_{n+1} is the product of a row E_1 and a column of E_n :

$$S_{n+1}^+ = AS_n^+ + S_{n-1}^+ + 1,$$

which is desired. □

Now we are going to derive an explicit formula for the sum S_n^+. Let $K_C(\lambda)$ be the characteristic polynomial of the matrix C. Thus,

$$K_C(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & A-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (\lambda - 1)\left(-\lambda^2 + A\lambda + 1\right).$$

Also it is easily seen that the characteristic polynomial of the matrix W given by (1.3) is $-\lambda^2 + A\lambda + 1$. Therefore the eigenvalues of the matrix C are

$$\lambda_1 = \frac{A + \sqrt{A^2+4}}{2}, \ \lambda_2 = \frac{A - \sqrt{A^2+4}}{2} \text{ and } \lambda_3 = 1.$$

Since $A \neq 0$ and $A^2 + 4 \neq 0$, we have that the eigenvalues of the matrix C are distinct.

Let V be the 3×3 matrix defined as follows:

$$V = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{A} & \lambda_1 & \lambda_2 \\ \frac{-1}{A} & 1 & 1 \end{bmatrix}, \tag{2.6}$$

where λ_1 and λ_2 are the eigenvalues of C. Note that $\det V = \lambda_1 - \lambda_2 \neq 0$.

Then we have the following Theorem.

Theorem 1. *Let S_n^+ denote the sums of the terms of the sequence $\{u_n\}$. Then*

$$S_n^+ = \frac{u_{n+1} + u_n - 1}{A}.$$

Proof. One can easily verify that

$$CV = VD_1,$$

where C and V are as before, and D_1 is the diagonal matrix such that $D_1 = diag\,(\lambda_3, \lambda_1, \lambda_2)$. Since $\det V \neq 0$, the matrix V is invertible. So we write that $V^{-1}CV = D_1$. Hence, the matrix C is similar to the diagonal matrix D_1. Thus we obtain $C^nV = VD_1^n$.Since $C^n = E_n$,

$$E_nV = VD_1^n.$$

So by a matrix multiplication, we have the conclusion. □

For example, if we take $A = 2$, then the sequence $\{u_n\}$ is reduced to the usual Pell numbers and by Theorem 1, we have

$$\sum_{1}^{n} P_i = \frac{P_{n+1} + P_n - 1}{2}$$

which is well known from [10].

Now we give a formula for the sum S_n^+ by using a matrix method with the following Corollary.

Corollary 2. *Let S_n^+ denote the sums of the terms u_i from 1 to n. Then for all positive integers n and m*

$$S_{n+m}^+ = u_{n+1}S_m^+ + u_n S_{m-1}^+ + S_n^+$$

where u_n given by (1.1).

Proof. From (2.4), we can write, for all positive integers n and m

$$E_{n+m} = E_n E_m.$$

Clearly

$$\begin{bmatrix} 1 & 0 & 0 \\ S_{n+m}^+ & u_{n+m+1} & u_{n+m} \\ S_{n+m-1}^+ & u_{n+m} & u_{n+m-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ S_n^+ & u_{n+1} & u_n \\ S_{n-1}^+ & u_n & u_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ S_m^+ & u_{m+1} & u_m \\ S_{m-1}^+ & u_m & u_{m-1} \end{bmatrix}.$$

By a matrix multiplication, the proof is easily seen. □

Note that taking by $n = 1$ in Corollary 2, we can obtain the result of Corollary 1.

3. Generating matrix for the sums of the negatively subscripted terms u_{-n}

In this section, we consider the negatively subscripted terms of the sequence $\{u_n\}$. First, we give a generating matrix for the negatively subscripted terms. Second, we give a generating matrix for the sums of these terms.

Let the 2×2 matrix T be as follows:

$$T = \begin{bmatrix} -A & 1 \\ 1 & 0 \end{bmatrix} \tag{3.1}$$

and the 2×2 matrix H_n be as follows:

$$H_n = \begin{bmatrix} u_{-(n+1)} & u_{-n} \\ u_{-n} & u_{-(n-1)} \end{bmatrix} \tag{3.2}$$

where u_{-n} is the nth negatively subscripted term of the sequence $\{u_n\}$.

We start with the following Lemma.

Lemma 2. *Let the matrices T and H_n have the form (3.1) and (3.2), respectively. Then for $n > 0$*

$$H_n = T^n.$$

Proof. (Induction on n) If $n = 1$, then, by the identity (1.2), we have

$$T^1 = \begin{bmatrix} -A & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u_{-2} & u_{-1} \\ u_{-1} & u_0 \end{bmatrix}.$$

If $n = 2$, then

$$T^2 = \begin{bmatrix} A^2+1 & -A \\ -A & 1 \end{bmatrix}.$$

Since by (1.2), we have $u_{-3} = u_3 = A^2+1$, $u_{-2} = -u_2 = -A$ and $u_{-1} = 1$, we have

$$T^2 = \begin{bmatrix} A^2+1 & -A \\ -A & 1 \end{bmatrix} = H_2.$$

We suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Thus, by our assumption,

$$\begin{aligned} T^{n+1} &= T^n T^1 \\ &= \begin{bmatrix} u_{-(n+1)} & u_{-n} \\ u_{-n} & u_{-(n-1)} \end{bmatrix} \begin{bmatrix} -A & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Since the negatively subscripted terms of the sequence $\{u_n\}$ satisfy the recurrence relation $u_{-n} = Au_{-(n+1)} + u_{-(n+2)}$, we have $u_{-(n+2)} = -Au_{-(n+1)} + u_{-n}$ and $T^{n+1} = H_{n+1}$. So the proof is complete. □

Let S_n^- denote the sums of the negatively subscripted terms of the sequence $\{u_n\}$, that is

$$S_n^- = \sum_1^n u_{-i}. \tag{3.3}$$

Now we give a matrix method to generate the sum S_n^-. Define the 3×3 matrices R and Q_n as shown

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -A & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q_n = \begin{bmatrix} 1 & 0 & 0 \\ S_n^- & u_{-(n+1)} & u_{-n} \\ S_{n-1}^- & u_{-n} & u_{-(n-1)} \end{bmatrix}. \tag{3.4}$$

Then we have the following Theorem.

Theorem 2. *Let the matrices R and Q_n have the form (3.4). Then for $n > 0$*

$$R^n = Q_n. \tag{3.5}$$

Proof. (Induction on n) If $n = 1$, then we know that $S_1^- = u_{-1} = 1$, $S_n^- = 0$ for $n < 1$, $u_{-2} = -u_2 = -A$, $u_0 = 0$. Thus we obtain $R = Q_1$. If $n = 2$, then we have $S_2^- = u_{-1} + u_{-2} = -A + 1$, $u_{-3} = u_3$ and by a matrix multiplication

$$T^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1-A & A^2+1 & -A \\ 1 & -A & 1 \end{bmatrix} = H_2.$$

Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Thus, by our assumption, we write

$$\begin{aligned} R^{n+1} &= R^n R = Q_n R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ S_n^- & u_{-(n+1)} & u_{-n} \\ S_{n-1}^- & u_{-n} & u_{-(n-1)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -A & 1 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Since $S_{n+1}^- = S_n^- + u_{-(n+1)}$ and by Lemma 2, we obtain $T^{n+1} = Q_{n+1}$. So we have the Theorem. □

In the following Theorem, we give a nonhomogeneous recurrence relation for the sum S_n^-.

Theorem 3. *Let S_n^- denote the sums of the terms u_{-i} for $1 \le i \le n$. Then for $n > 0$*

$$S_{n+1}^- = -AS_n^- + S_{n-1}^- + 1.$$

Proof. Considering (3.5), we write $Q_{n+1} = Q_n Q_1 = Q_1 Q_n$ and say that the matrix Q_1 is commutative under matrix multiplication. By a matrix multiplication, the proof is easy. □

Generalizing $R^n = Q_n$, for all positive integers n and m, we can write that $Q_{n+m} = Q_n Q_m = Q_m Q_n$. Thus we obtain the following Corollary without proof as a generalization of the result of Theorem 3.

Corollary 3. *Let S_n^- denote the sums of the terms u_{-i} for $1 \le i \le n$. Then for all $n, m > 0$*

$$S_{n+m}^- = S_n^- + u_{-(n+1)} S_m^- + u_{-n} S_{m-1}^-.$$

Now we derive an explicit formula for the sums of the negatively subscripted terms u_{-i} for $1 \le i \le n$. For this purpose, we give some results. First, we consider the characteristic polynomial of the matrix T. The characteristic equation of T is $K_T(\lambda) = -(\lambda - 1)(\lambda^2 + A\lambda - 1)$. Thus the eigenvalues of matrix T are

$$\mu_1 = \frac{-A + \sqrt{A^2+4}}{2}, \ \mu_2 = \frac{-A - \sqrt{A^2+4}}{2} \text{ and } \mu_3 = 1.$$

Note that $A \neq 0$ and $A^2 + 4 \neq 0$, the eigenvalues of T are distinct.

Let Λ be a matrix as follows

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{A} & \mu_1 & \mu_2 \\ \frac{1}{A} & 1 & 1 \end{bmatrix}.$$

Then we have the following Theorem.

Theorem 4. *Let S_n^- denote the sums of the negatively subscripted terms u_{-i} for $1 \leq i \leq n$. Then for $n > 1$*

$$S_n^- = \frac{1 - u_{-(n+1)} - u_{-n}}{A}.$$

Proof. By the characteristic equation of the negatively subscripted terms u_{-i}, we can readily verify that

$$R\Lambda = \Lambda D_2,$$

where D_2 is the 3×3 diagonal matrix such that $D_2 = diag\,(\mu_3, \mu_1, \mu_2)$. Since $\det \Lambda = \mu_1 - \mu_2 \neq 0$, the matrix Λ is invertible. Thus we write $\Lambda^{-1} R \Lambda = D_2$ and so the matrix is similar to the matrix D_2. Therefore, we write $\Lambda^{-1} R^n \Lambda = D_2^n$ or $R^n \Lambda = \Lambda D_2^n$. Since $R^n = Q_n$, we have $Q_n \Lambda = \Lambda D_2^n$.Then we have the conclusion from $Q_n \Lambda = \Lambda D_2^n$ by a matrix multiplication. □

Considering the identity (1.2), we have the following Corollary without proof.

Corollary 4. *Let S_n^- denote the sums of the negatively subscripted terms u_{-i} for $1 \leq i \leq n$. Then for $n > 1$*

$$S_n^- = \begin{cases} (u_n - u_{n+1} + 1)/A & \textit{if } n \textit{ is even,} \\ (u_{n+1} - u_n + 1)/A & \textit{if } n \textit{ is odd.} \end{cases}$$

For example, if take $A = 1$, then the sequence $\{u_n\}$ is reduced to the usual Fibonacci sequence and by Corollary 4, we have the sums of the negatively subscripted terms of the Fibonacci sequence for n is even number

$$\sum_1^n F_{-i} = F_1 - F_2 + F_3 - \ldots + F_{n-1} - F_n = 1 - F_{n-1}$$

and for n is odd number

$$\sum_1^n F_{-i} = F_1 - F_2 + F_3 - \ldots - F_{n-1} + F_n = F_{n-1} + 1.$$

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