A New Construction For Group Divisible Designs With Block Size Five and Few Groups

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ABSTRACT. We construct several new group divisible designs with block size five and with 2, 3, or 6 groups.

1. Introduction

A group divisible design is a collection of k-element subsets of a v-set X called blocks which satisfies the following properties: each point of X appears in r of the b blocks; the $v = nm$ elements of X are partitioned into m subsets (called groups) of size n each; points within the same group are called first associates of each other and appear λ_1 times together in blocks. Any two points not in the same group are second associates and appear together in λ_2 blocks. We use the notation GDD$(m, n, k; \lambda_1, \lambda_2)$. If $\lambda_1 = 0$, points in the same group do not appear together, and, in this case, the notation $\{k, \lambda\}$-GDD of type n^m is often used. Designs in which the points in the same group appear λ_1 times together and points not in the same group appear λ_2 times together have been called group-divisible type PBIBDs (pair-wise balanced incomplete block desings). See [3] and [4]. We use the selected notation since we will deal with both types of group designs in Section 2.

The purpose of this note is to give a new construction for certain GDDs with small numbers of groups and with block size $k = 5$.

2. The Construction

A latin square of order n is an n-by-n array in which each cell contains a single element of an n-set S, such that each element of S occurs exactly once in each row and in each column. Two latin squares, say $L_1(a_{i,j})$ and $L_2(b_{i,j})$, each of order n, and based on sets S_1 and S_2, respectively, are called orthogonal if every element of $S_1 \times S_2$ occurs exactly once among the pairs $(a_{i,j}, b_{i,j})$, $1 \leq i, j \leq n$. A set of latin squares, $L_1, L_2, ..., L_t$, is mutually orthogonal if the latin squares are pairwise mutually orthogonal. In this case they are a set of MOLS, or mutually orthogonal latin squares.

We begin with six groups, each of size n for $n \geq 4$, $(n \neq 6, 10)$, and we denote them by A_1, A_2, A_3, B_1, B_2, and B_3. We require a set of three MOLS of order n,

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say L_1, L_2 and L_3, and such sets are known to exist for every $n \geq 4$ ($n \neq 6, 10$) [1]. For each L_i, the rows and columns are indexed by the integers one to n. It is convenient to describe the construction of blocks in two Phases. For Phase 1, the elements of L_i are taken from B_i ($i = 1, 2, 3$). Let $(c_{i,j})$ denote the block

$$\{A_1(i), A_2(j), L_1(i,j), L_2(i,j), L_3(i,j)\} \text{ where } 1 \leq i, j \leq n.$$

As the index j runs from 1 to n, the element $A_1(i)$, from group A_1, is paired with each element from each of the other groups (except Group A_3) exactly once. Likewise, as i runs from 1 to n, for each j, the element $A_2(j)$ from group A_2 is paired with each element from each of the other groups (except A_3) exactly once. By mutual orthogonality, each pair of elements from any two of the B's will be paired exactly once. The construction so far gives a 5-GDD of type n^5 with n^2 blocks. Since exactly one point from each group appears in each block, this is a transversal design or TD.

To conclude Phase 1 of the construction, we next replace A_1 and A_2 with A_2 and A_3, respectively, and create another set of blocks in the same way as earlier. Finally, we replace A_2 and A_3 with A_3 and A_1 and make a third set of blocks. So far, each pair of points from different A's will appear once in the blocks, each pair of points from A_i and B_j ($1 \leq i, j \leq 3$) will appear twice, and pairs of points from different B's will appear three times together.

Now, for Phase 2, we switch roles between the A's and the B's, using each pair of B's with the three A's as elements of the latin squares. The three sets of new blocks and the Phase 1 blocks give a $\{5,4\}$-GDD of type n^6 with $6n^2$ blocks.

As an application, we may simultaneously embed six distinct BIBD$(n, 5, 4)$s, each based on one of the groups, into a BIBD$(6n, 5, 4)$ by using the blocks of the GDD and those of the six BIBDs. The BIBD$(n, 5, 4)$ exist for all $n \equiv 0, 1 \pmod 5$ [2].

In summary, we have proved:

THEOREM 1. *There exists a GDD$(n, 6, 5; 0, 4)$ with $6n^2$ blocks for all ($n \neq 6, 10$). Moreover, if $n \equiv 0, 1 \pmod 5$, there exists a BIBD$(6n, 5, 4)$ formed by simultaneously embedding six BIBD$(n, 5, 4)$s.*

It is useful to state explicitly the following generalization of that part of our construction which does not depend on k.

REMARK 1. *If there exists a BIBD(nts, k, λ_2) and a BIBD$(nt, k, \lambda_1 - \lambda_2)$, then there exists a GDD$(nt, s, k; \lambda_1, \lambda_2)$.*

PROOF. Let Y denote the GDD desired, and let X = BIBD(nts, k, λ_2). We may partition the point set for X arbitrarily into groups $G(i,j)$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. Define $H_i = \bigcup\{G(i,j) : 1 \leq j \leq t\}$. Then note $|H_i| = nt$, and we may form a BIBD$(H_i, k, \lambda_1 - \lambda_2)$ for each i. The GDD$(nt, s, k;\ \lambda_1, \lambda_2)$ uses the blocks of X and the blocks of each BIBD$(H_i, k,\ \lambda_1 - \lambda_2)$. □

As an application of the theorem and of the remark, we have an important corollary:

COROLLARY 1. *(a) If a BIBD$(2n, 5, \lambda)$ and a BIBD$(n, 5, 4)$ each exist, then there exists a GDD$(2n, 3, 5;\ \lambda + 4, 4)$.*

(b) If a BIBD$(3n, 5, \lambda)$ and a BIBD$(n, 5, 4)$ each exist, there exists a group divisible PBIBD $(3n, 2, 5;\ \lambda + 4, 4)$.

PROOF. For part (a), first construct the BIBD$(6n, 5, 4)$. Then, form three copies of a BIBD$(2n, 5, \lambda)$ using $A_1 \cup A_2$, $A_3 \cup B_1$, and $B_2 \cup B_3$ as the three sets of points. The blocks of the BIBD on $6n$ points and the blocks of the three BIBDs on $2n$ points give the blocks of the desired GDD. Part (b) is similar. □

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