ON SOME CHROMATIC PROPERTIES OF JAHANGIR GRAPH

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ABSTRACT. In this note we compute the chromatic polynomial of the Jahangir graph J_{2p} and we prove that it is chromatically unique for $p = 3$.

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Introduction: Let J_{2p}, $p \in \mathbb{N}$ denote the Jahangir graph defined as follows*: For $p \geq 2$, it consist of the cycle C_{2p} of length $2p$ having vertex set equal to $\{v_1, v_2, ..., v_{2p}\}$ and a new vertex which is adjacent to a maximal independent set of p vertices of cycle, i.e. $\{v_2, v_4, ..., v_p\}$. For $p = 1$, it is equal by definition to $K(1,2)$, where $K(p,q)$ denotes the complete bipartite graph having partite sets of cardinalities p and q, respectively. Some examples of Jahangir graphs are given below.

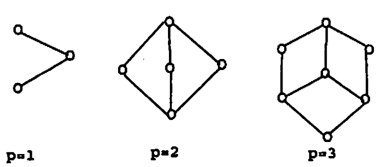


FIGURE 1

Graphs considered in this note are simple, undirected, without loops or multiple edges. For a graph G, $V(G)$, $E(G)$, $v(G)$, $e(G)$, $g(G)$ and $P(G, \lambda)$

*The Jahangir graph J_{2p} is a natural extension of the graph J_{16} which appears on Jahangir's tomb in his mausoleum, it lies in 5 km north-west of Lahore, Pakistan across the River Ravi. His tomb was built by Queen Noor Jehan and his son Shah-Jehan (This was emperor who constructed one of the wonder of Taj Mahal in India) around 1637 A.D.

denote the vertex set, edge set, order, size, girth and chromatic polynomial of G respectively. Two graphs G and H are said to be chromatically equivalent (or simply χ-equivalent), symbolically denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. If graphs G and H are isomorphic we denote this by $G \cong H$. The chromatic equivalence class of G, denoted by $[G]$ is the set of graphs H such that $G \sim H$. A graph G is chromatically unique (or simply χ-unique) if $[G] = G$ or $P(G,\lambda) = P(H,\lambda)$ implies $G \cong H$. Note that for $p = 1$, in Figure(1), the graph is just a tree with three vertices, having the chromatic polynomial $\lambda(\lambda - 1)^2$, which is known to be chromatically unique, and for $p = 2$, we have $J_4 \cong K(2,3)$ which is also known to be chromatically unique (see [1]). We shall see later that for $p = 3$ the Jahangir graph J_6 is chromatically unique also. For this we need some definitions and known results about chromatic polynomials. Let G and H be two graphs. We shall denote by $n_G(H)$ and $i_G(H)$ the number of subgraphs and induced subgraphs in G, respectively which are isomorphic to H.

Theorem 1.1([1]): *Let G be an (n,m) graph and let $P(G,\lambda)$ denote its chromatic polynomial. Then:*

$$P(G,\lambda) = \sum_{i=1}^{n}(-1)^{n-i}h_i(G)\lambda^i \tag{1}$$

is a polynomial in λ such that

(i) *the degree of $P(G,\lambda)$ is n and the leading coefficient is* 1;

(ii) *the coefficients are integers and alternate in sign*;

(iii) *the constant term is zero*;

(iv) $h_{n-1}(G) = m$;

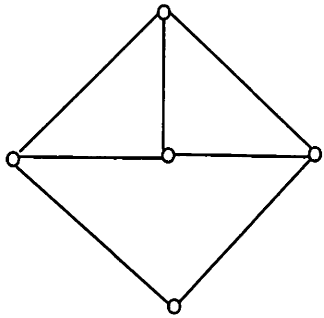
(v) $h_{n-2}(G) = \binom{m}{2} - n_G(C_3)$;

(vi) $h_{n-3}(G) = \binom{m}{3} - (m-2)n_G(K_3) - i_G(C_4) + 2n_G(K_4)$;

(vii) $h_{n-4}(G) = \binom{m}{4} - \binom{m-2}{2} n_G(K_3) + \binom{n_G(K_3)}{2} - (m-3)$

$i_G(C_4) - (2m-9)n_G(K_4) - i_G(C_5) + i_G(K(2,3)) + 2i_G(F) + 3i_G(W_5) - 6n_G(K_5)$

where F is the graph given below:



F

FIGURE 2

Following the Farrell's[2] approach, Peng[5] managed to obtain the following expression for $h_{n-5}(G)$ but confining to the class of bipartite graphs.

Theorem 1.2([5]): *Let G be an (n,m)-bipartite graph. Then*

$$h_{n-5}(G) = \binom{m}{5} - \binom{m-3}{2} n_G(C_4) - i_G(C_6) + (m-3) n_G(K(2,3)) -$$

$$n_G(K(2,4)) + i_G(K(3,3)-e) + 4 n_G(K(3,3)),$$

where e is an edge in $K(3,3)$.

Basic Results on $P(G, \lambda)$:

We shall state here some known results which will help us to determine the chromatic polynomial of Jahangir graph J_{2p}, $p \in \mathbb{N}$. First result provides us with a recursive way to compute $P(J_{2p}, \lambda)$. Let $G + xy$ denote the graph obtained by adding a new edge xy to G, and $G.xy$ is the graph obtained from G by contracting x and y and removing any loop and all but one of the multiple edges, if they arise.

Theorem 1.3([1]): *Let x and y be two non-adjacent vertices in a graph G. Then:*

$$P(G,\lambda) = P(G+xy,\lambda) + P(G\,.\,xy,\lambda). \tag{2}$$

Equivalently if we treat the graph $G+xy$ as a given graph H, then theorem 1.3 can be reformulated as follows:

Let H be a graph and $e \in E(H)$, then

$$P(H,\lambda) = P(H-e,\lambda) - P(H.xy,\lambda), \tag{3}$$

where $H - e$ denotes the subgraph of H obtained by removing e from H. Both expressions (2) and (3) are referred as the *Fundamental Reduction Theorem(FRT)*. For two non-empty graphs G and H, an edge-gluing of G and H is the graph produced by identifying one edge of G with one edge of H. For example, the graph $K_4 - e$ (obtained from K_4 by deleting one edge) is an edge gluing of K_3 and K_3. Zykov[6] provided a shortcut for evaluating $P(G,\lambda)$ if G is an edge-gluing of some graphs.

Theorem 1.4([6]): *Let G_1 and G_2 be two graphs and G be an edge-gluing of G_1 and G_2. Then:*

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{\lambda(\lambda-1)} \tag{4}$$

The following theorem provides some necessary conditions for two graphs to be chromatically equivalent. We will use this theorem to prove our main result.

Theorem 1.5([1]): *Let G and H be two χ-equivalent graphs. Then the following properties must hold:*

(a) $v(G) = v(H)$

(b) $e(G) = e(H)$

(c) $\chi(G)=\chi(H)$

(d) $n_G(C_3) = n_H(C_3)$

(e) $i_G(C_4) - 2n_G(K_4) = i_H(C_4) - 2n_H(K_4)$

(f) *G is connected if and only if H is connected*

(g) *G is 2-connected if and only if H is 2-connected*

(h) $g(G) = g(H)$

(i) $n_G(C_k) = n_H(C_k)$, *where* $g \leq k \leq \lceil \frac{3}{2}g \rceil - 2$

(j) *G is bipartite if and only if H is bipartite*

(2)Main Results:

Theorem 2.1: *The chromatic polynomial of the Jahangir graph* J_{2p}, $p \in \mathbb{N}$ *is* :

$$P(J_{2p}, \lambda) = \lambda(\lambda - 1)^2 + \lambda(\lambda^2 - 3\lambda + 3)[(\lambda^2 - 3\lambda + 3)^{p-1} - 1]$$

Proof: Consider the following graph, denoted by $\mathtt{PT}(p, C_4)$ and represented in Fig.3, which consist of p copies of C_4. This graph is called a polygon tree.

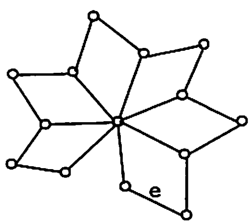


FIGURE 3

By applying the *fundamental reduction theorem* we get

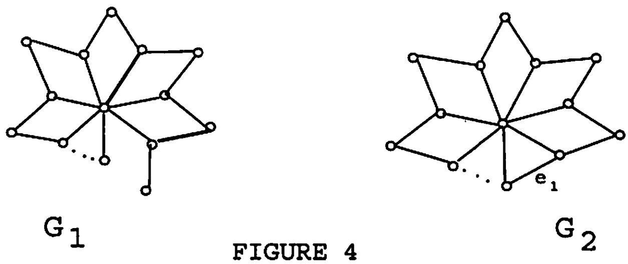


FIGURE 4

$$P(J_{2p},\lambda) = P(G_1,\lambda) - P(G_2,\lambda)$$

hence

$$P(J_{2p},\lambda) = (\lambda-1)P(\mathrm{PT}(p-1,C_4),\lambda) - [P(\mathrm{PT}(p-1,C_4),\lambda) - P(J_{2p-2},\lambda)]$$

Simplifying the above expression we get

$$P(J_{2p},\lambda) = (\lambda-2)P(\mathrm{PT}(p-1,C_4),\lambda) + P(J_{2p-2},\lambda)$$

Since

$$P(C_4,\lambda) = \lambda(\lambda-1)[\lambda^2-3\lambda+3]$$

and the chromatic polynomial of the polygon tree having two copies of C_4 is

$$P(\mathrm{PT}(2,C_4),\lambda) = \lambda(\lambda-1)[\lambda^2-3\lambda+3]^2$$

by applying several times (4) we get

$$P(\mathrm{PT}(p-1,C_4),\lambda) = \lambda(\lambda-1)[\lambda^2-3\lambda+3]^{p-1}$$

Hence,

$$P(J_{2p},\lambda) = \lambda(\lambda-2)(\lambda-1)[\lambda^2-3\lambda+3]^{p-1} + P(J_{2p-2},\lambda)$$

Applying recursively this relation we get

$$P(J_{2p},\lambda) = P(K(1,2),\lambda) + \lambda(\lambda-1)(\lambda-2)[(\lambda^2-3\lambda+3)$$

$$+(\lambda^2-3\lambda+3)^2+...+(\lambda^2-3\lambda+3)^{p-1}]$$

Since terms within the parenthesis form a geometric series, after simplification we obtain

$$P(J_{2p},\lambda)=\lambda(\lambda-1)^2+\lambda(\lambda^2-3\lambda+3)[(\lambda^2-3\lambda+3)^{p-1}-1]$$

which completes the proof. □

Theorem 2.2: J_6 is chromatically unique.

Proof: Let H be a graph such that $P(H,\lambda)=P(J_6,\lambda)$. It follows from theorem 1.5 that $v(H)=7$, $e(H)=9$, $g(H)=4$, $n_H(C_3)=0$, $i_H(C_4)=3$, $n_H(C_4)=3$, H is 2-connected and bipartite. Since

$$P(J_6,\lambda)=P(H,\lambda)=\sum_{i=1}^{7}(-1)^{7-i}h_i(H)\lambda^i$$

we can study the structure of the coefficients of $P(H,\lambda)$. For this we shall use the algebraic expressions of h_{n-4} and h_{n-5}, i.e. h_3 and h_2 given by theorems 1.1 and 1.2. Since H and J_6 are both bipartite, we have:

$h_3(J_6)=\binom{9}{4}-6\,i_{J_6}(C_4)+i_{J_6}(K(2,3))=h_3(H)=\binom{9}{4}-6\,i_H(C_4)+i_H(K(2,3))$.

Since $i_{J_6}(C_4)=i_H(C_4)=3$, we deduce that $i_H(K(2,3))=i_{J_6}(K(2,3))=0$. Now

$$h_2(J_6)=\binom{9}{5}-\binom{6}{2}n_{J_6}(C_4)-i_{J_6}(C_6),$$

since $n_{J_6}(K(2,3))=n_{J_6}(K(2,4))=i_{J_6}(K(3,3)-e)=n_{J_6}(K(3,3))=0$. We will show that $n_H(K(2,4))=0$. If H contains $K(2,4)$ as a subgraph,

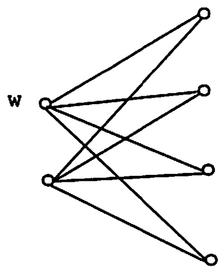


FIGURE 5

since $v(K(2,4)) = 6$ and $e(K(2,4)) = 8$, we must add a supplementary edge from w to another vertex as in Fig.6.

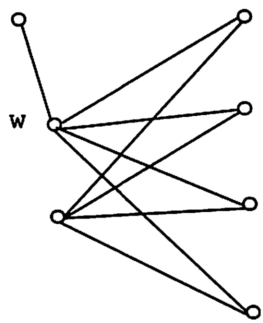


FIGURE 6

By doing this w will become a cut vertex, which is not possible since H is 2-connected. Also we have $n_H(K(3,3)) = 0$, because $e(K(3,3)) = 9$ and $v(K(3,3)) = 6$, so we must add exactly one vertex to obtain H. But this vertex will be isolated, which contradicts the fact that H is 2-connected. A similar argument shows that $i_H(K(3,3) - e) = 0$. It follows that

$$h_2(H) = \binom{9}{5} - \binom{6}{2} n_H(C_4) - i_H(C_6) = h_2(J_6).$$

Because $n_{J_6}(C_4) = n_H(C_4) = 3$, it follows that $i_H(C_6) = i_{J_6}(C_6) = 1$. Hence H has an induced cycle C_6 represented in Fig.7. Since $v(C_6) = e(C_6) = 6$, so we must add three edges and one vertex to C_6 to get H.

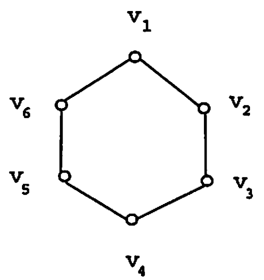


FIGURE 7

It is not possible to add chords between vertices at a distance equal to two on C_6, since would appear odd cycles which contradicts the property of H to be bipartite. If we join v_1 to v_4 and v_2 to v_5 as shown in the Figure 8 below

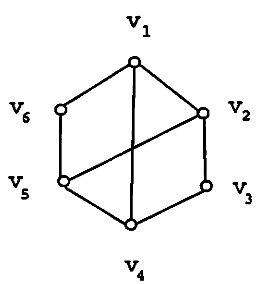


FIGURE 8

we get $i_H(C_4) \geq 5$, which contradicts the property that $i_H(C_4) = i_{J_6}(C_4) = 3$. Hence C_6 has at most one diametral chord. If C_6 has no chord, then new vertex v_7

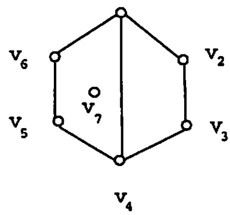


FIGURE 9

must be adjacent to exactly three vertices of C_6. Since no C_3 may appear, it follows that v_7 is joined e.g. to v_1, v_3 and v_5 and the resulting graph is J_6. Otherwise C_6 has exactly a diametral chord e.g. v_1v_4. v_7 may be adjacent to an extremity of the edge v_1v_4 or not. Since no odd cycle can appear, due to symmetry we may consider only three cases as shown in the following Figure 10.

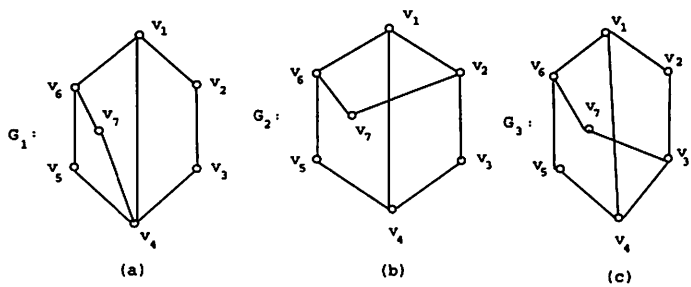


FIGURE 10

Case(a): We have $i_{G_1}(C_4) = 4 > 3 = i_H(C_4)$, hence G_1 cannot be isomorphic to H.

Case(b): The graph $G_2 \cong J_6$.

Case(c): The graph contains C_5, hence G_3 is not bipartite implying that G_3 is not isomorphic to H. Hence $H \cong J_6$ and the proof is complete. □

Note: In [3] it is asserted (without a formal proof) that J_6 and J_{10} are chromatically unique and the following problem is raised:

For each integer $n, n \geq 4$, which of the graphs J_{2n} are χ-unique? (the graph J_{2n} being denoted there as W^*_{2n+1}).

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