Some lower bounds on the spectral radius of graphs

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Abstract

Let G be a simple connected graph. The spectral radius $\rho(G)$ of G is the largest eigenvalues of its adjacency matrix. In this paper, we obtain two lower bounds of $\rho(G)$ by two different methods, one of which is better than another in some conditions.

Keywords : Spectral radius; Adjacency matrix; Similar matrices

1 Introduction

In this paper, all graphs are finite, undirected simple and connected graphs. The degree of a vertices v_i, denoted by $d(v_i) = d_i$, is the number of edges incident with v_i. Throughout this paper, let $G = (V, E)$ denote a graph, and let $n = |V|$ and $m = |E|$ denote the number of vertices of G and the number of edges of G, respectively. Let $d_1 \geq d_2 \cdots \geq d_n$ denote the degree sequence of G. For $e = v_iv_j \in E$, v_i and v_j are the endvertices of e, $i, j = 1, 2, \cdots, n$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. We also denote by $\delta'(G) = \delta'$, the second minimum degree of G different from δ. A graph G is regular if $\delta = \Delta$. A bipartite graph is called semiregular if each vertex in the same part of a bipartite has the same degree. For an integer $n > 0$, Let P_n and C_n denote the path of order n and the cycle of order n, respectively.

let $A(G)$ be the adjacency matrix of G and let $A(G) = (a_{ij})_{n\times n}$ be defined as the $n \times n$ matrixes (a_{ij}), where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E \\ 0, & \text{otherwise .} \end{cases}$$

It follows immediately that if G is a simple graph, then $A(G)$ is a symmetric (0,1) matrix where all diagonal elements are zero. Let $\text{diag}(r_1, r_2, \cdots, r_n)$ denote the diagonal matrix with diagonal entries $r_1, r_2, \cdots, r_n$. For a square matrix A, let $\rho(A)$ denote the largest eigenvalue of A. The spectral radius $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix $A(G)$.

Up to now, many bounds for $\rho(G)$ were given, but most of them are upper bounds. We summarize some known lower bounds for the spectral radius $\rho(G)$.

(1)(Collatz and Sinogowitz[3]) If G is a connected graph of order n, then

$$\rho(G) \geq 2\cos(\pi/(n+1)),$$

where equality holds if and only if $G = P_n$.

(2)(Hong[6]) If G is a connected unicyclic graph, then

$$\lambda_1(G) \geq 2,$$

where equality holds if and only if $G = C_n$.

(3)(Favaron et al[5]) For any simple graph, then

$$\lambda_1(G) \geq \sqrt{d_1} \; .$$

(4)(Aimei Yu et al[11]) Let G be a simple connected graph. Then

$$\rho(G) \geq \frac{2m}{n} \geq \delta.$$

(5)(Brualdi and Sollheid[2]) let r be an integer with $0 \leq r \leq \lfloor \frac{n^2}{4} \rfloor$ and $A(G)$ has exactly r entries being zero, then

$$\rho(G) \geq \frac{n + \sqrt{n^2 - 4r}}{2}. \tag{1}$$

Moreover, for $A \in B_n$, $\rho(A) = \dfrac{n + \sqrt{n^2 - 4r}}{2}$ if and only if A is similar to

$$\begin{pmatrix} J_k & C \\ J_{l,k} & J_l \end{pmatrix},$$

where B_n is the collection of $n \times n$ (0,1) matrixes; J_k denotes the $k \times k$ matrix whose entries are all ones; $J_{l,k}$ denotes the $l \times k$ matrix whose entries are all ones; J_l denotes the $l \times l$ matrix whose entries are all ones and $C = \mathrm{diag}(d_1, d_2, \cdots, d_n)$.

(6) (Hong[7]) Let G be a connected graph with a degree sequence $d_1, d_2, \cdots, d_n$. Then

$$\rho(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2},$$

where equality holds if and only if G is either a connected regular graph or a connected semiregular bipartite graph.

In this paper, we obtain two new lower bounds on $\rho(G)$ of G by two different methods.

2 Lower bounds obtained by similar transformation

In this section, we obtain some lower bounds on the spectral radius of a simple connected graph by similar transforming its adjacency matrix. We first state some basic lemmas, then prove the theorem.

Lemma2.1[9]: If A is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(A)$ and row sums $r_1, r_2, \cdots, r_n$, then

$$\min_{1\leq i\leq n} r_i \leq \rho(A) \leq \max_{1\leq i\leq n} r_i .$$

Moreover, equality holds if and only if the row sums of A are all equal.

For a simple connected graph, we have the similar lemma.

Lemma2.2[9]: Let G be a simple connected graph with n vertices. Then

$$\delta \leq \rho(G) \leq \Delta.$$

Moreover, an equality holds if and only if G is a regular graph.

Lemma2.3: Let G be a simple connected graph with a degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Then for each $i \in I = \{i | d_n > i - 1, 0 \leq i \leq n-1\}$,

$$\rho(G) \geq \frac{d_{n-i} - 1 + \sqrt{(d_{n-i}+1)^2 + 4i(d_n - d_{n-i})}}{2}, \tag{2}$$

where equality holds if and only if G is a regular graph.

Proof: When $i = 0$ or $d_{n-i} = d_n$, it is clearly that inequality (2) is true and the equality holds if and only if G is a regular graph by Lemma 2.2.

Now, we suppose $d_{n-i} > d_n$, $1 \le i \le n-1$.

The adjacency matrix A of G can be written as $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} is an $(n-i) \times (n-i)$ matrix and A_{22} is an $i \times i$ matrix. For a real number x with $0 < x < 1$, let

$$U = \begin{pmatrix} \frac{1}{x} I_{n-i} & 0 \\ 0 & I_i \end{pmatrix}, U^{-1} = \begin{pmatrix} x I_{n-i} & 0 \\ 0 & I_i \end{pmatrix},$$

where I_{n-i} is an $(n-i) \times (n-i)$ unit matrix, and I_i is an $i \times i$ unit matrix. Let

$$B = U^{-1} A U = \begin{pmatrix} A_{11} & x A_{12} \\ \frac{1}{x} A_{21} & A_{22} \end{pmatrix}.$$

As A and B are similar matrices, they have the same eigenvalues. In particular, $\rho(G) = \rho(A) = \rho(B)$. Now, we consider the row sums $\{r_1, r_2, \cdots, r_n\}$ of matrix B. For each l with $1 \le l \le n-i$,

$$\begin{aligned} r_l &= \sum_{j=1}^{n-i} a_{lj} + x \sum_{j=n-i+1}^{n} a_{lj} = \sum_{j=1}^{n} a_{lj} + (x-1) \sum_{j=n-i+1}^{n} a_{lj} \\ &= d_l + (x-1) \sum_{j=n-i+1}^{n} a_{lj} . \end{aligned}$$

For each k with $n - i < k \le n$,

$$\begin{aligned} r_k &= \frac{1}{x} \sum_{j=1}^{n-i} a_{kj} + \sum_{j=n-i+1}^{n} a_{kj} = \frac{1}{x} \sum_{j=1}^{n} a_{kj} + (1 - \frac{1}{x}) \sum_{j=n-i+1}^{n} a_{kj} \\ &= \frac{1}{x} d_k + (1 - \frac{1}{x}) \sum_{j=n-i+1}^{n} a_{kj} . \end{aligned}$$

Since G is simple, in the adjacency matrix, $a_{ii} = 0$, $a_{ij} = 0$ or 1, where $i, j = 1, 2, \cdots, n$, $i \ne j$. Hence, $\sum_{j=n-i+1}^{n} a_{lj} \le i$ when $1 \le l \le n-i$ and $\sum_{j=n-i+1}^{n} a_{kj} \le i-1$ when $n-i+1 \le k \le n$. Since $0 < x < 1$ and since $d_1 \ge d_2 \ge \cdots \ge d_{n-i} \ge d_{n-i+1} \ge \cdots \ge d_n$, both

$$r_l \ge d_{n-i} + (x-1)i, (1 \le l \le n-i), \tag{3}$$

and

$$r_k \ge \frac{1}{x} d_n + (1 - \frac{1}{x})(i-1), (n-i+1 \le k \le n), \tag{4}$$

must hold. Obviously,

$$min\{r_1, r_2, \cdots, r_n\} \geq min\{d_{n-i} + (x-1)i, \frac{1}{x}d_n + (1-\frac{1}{x})(i-1)\}.$$

Suppose that there exists an real number x satisfying the equality

$$d_{n-i} + (x-1)i = \frac{1}{x}d_n + (1-\frac{1}{x})(i-1).$$

Then solving the equation for x, we have

$$\begin{aligned} x &= \frac{2i-1-d_{n-i}+\sqrt{(2i-1-d_{n-i})^2+4i(d_n-i+1)}}{2i} \\ &= \frac{2i-1-d_{n-i}+\sqrt{(d_{n-i}+1)^2+4i(d_n-d_{n-i})}}{2i}. \end{aligned}$$

Since $i \geq 1$, $d_{n-i} > d_n > 0$ and $d_n > i-1$, it follows

$$0 < x < \frac{2i-1-d_{n-i}+d_{n-i}+1}{2i} = 1,$$

and so,

$$\rho(G) \geq d_{n-i} + (x-1)i = \frac{d_{n-i}-1+\sqrt{(d_{n-i}+1)^2+4i(d_n-d_{n-i})}}{2}.$$

If the equality in (2) holds, then inequalities in (3) and (4) must be equalities. From (3) we have $d_l = d_1$ and $a_{lj} = 1$ when $1 \leq l \leq n-i$, $n-i+1 \leq j \leq n$. So, $a_{jl} = 1$ when $n-i+1 \leq j \leq n$, $1 \leq l \leq n-i$. From (4) we get $d_k = d_n$ and $a_{kj} = 1$ when $n-i+1 \leq k \leq n$, $n-i+1 \leq j \leq n, k \neq j$. Thus $d_{n-i+1} = d_{n-i+2} = \cdots = d_n = n-1$. The mean is that the degree sequence of G satisfying $d_1 = d_2 = \cdots = d_n = n-1$, contrary to the assumption $d_{n-i} > d_n$.

Therefore, equality in (2) holds if and only if G is a regular graph. This completes the proof.

Theorem 2.4: Let G be a simple connected graph with a degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, and let $I = \{i | d_n > i-1, 0 \leq i \leq n-1\}$. Then

$$\rho(G) \geq \max_{i \in I} \{\frac{d_{n-i}-1+\sqrt{(d_{n-i}+1)^2+4i(d_n-d_{n-i})}}{2}\},$$

where equality holds if and only if G is a regular graph.

Theorem 2.5: Let G be a simple connected graph with n vertices, the minimum degree δ, the second minimum degree δ' and $q \geq 0$ be an integer. If there are q vertices with degree δ and if $\delta > q-1$. Then

$$\rho(G) \geq \frac{\delta'-1+\sqrt{(\delta'+1)^2+4q(\delta-\delta')}}{2}, \tag{5}$$

where equality holds if and only if G is a regular graph.

Proof: The results can be obtained by taking $\delta = d_n, \delta' = d_{n-i}, q = i$ in Lemma 2.3.

Corollary 2.6: Suppose that G is a simple connected graph with n vertices, the minimum degree δ, the second minimum degree δ' and that G is not a regular graph. If there are q vertices with degree δ, $\delta > q-1$ and $\delta' \geq 3q-1$, then

$$\rho(G) > \frac{\delta+\delta'}{2}.$$

Proof: Since $\delta > q-1$, $\delta' \geq 3q-1$, so

$$\delta + \delta' + 2 > 4q.$$

By Theorem 2.5,

$$\rho(G) \geq \frac{\delta' - 1 + \sqrt{(\delta'+1)^2 + 4q(\delta-\delta')}}{2} > \frac{\delta' - 1 + \sqrt{(\delta+1)^2}}{2} = \frac{\delta+\delta'}{2}.$$

This completes the proof.

3 Lower bounds with vertices, edges and maximum degree

In this section, we obtain a lower bound on the spectral radius of a simple connected graph with n vertices, m edges and maximum degree Δ.

Lemma 3.1[4]: Let G be a simple connected graph with n vertices and A be its adjacency matrix. Let P be any polynomial and $S_v(P(A))$ be the set of row sums of $P(A)$ corresponding the each vertex v. Then

$$minS_v(P(A)) \leq P(\rho(A)) \leq maxS_v(P(A)).$$

Moreover, equality holds if and only if the row sums of $P(A)$ are all equal.

Proposition 3.2: $f(x; m, n) = x - 1 + \sqrt{(x+1)^2 + 4(2m - xn)}$ is a decreasing function of x for $1 \leq x \leq n-1$, where $n - 1 \leq \lceil \frac{1}{8}(3n^2 - 4n) \rceil \leq m \leq n(n-1)/2$, $n \geq 4$.

It follows by standard Calculus verifications.

Lemma 3.3[9]: Let A be the adjacency matrix of G, the (i,j)-th entry of A^k, the l-th power of A, is positive if and only if there is a (v_i, v_j) walk in G of length l.

Theorem 3.4: let G be a simple connected graph with n vertices and m edges, where $n \geq 4$. Let $\Delta = \Delta(G)$ be the maximum degree of vertices of G, then

$$\rho(G) \geq \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 + 4(2m - \Delta n)}}{2}, \tag{6}$$

when $\lceil \frac{1}{8}(3n^2 - 4n) \rceil \leq m \leq n(n-1)/2$, and

$$\rho(G) > \frac{-(\Delta+1)^2 - 4(2m - \Delta n)}{4},$$

when $n - 1 \leq m < \frac{-(\Delta+1)^2 + 4\Delta n}{8} \leq \lceil \frac{1}{8}(3n^2 - 4n) \rceil$.

Proof: Since $S_v(A^k)$ is exactly the number of walks of length k in G start from v by Lemma 3.3, so $S_v(A^2) = \sum_{uv \in E(G)} d(u)$. In particular, $S_v(A)$ is $d(v)$. We have

$$\begin{aligned} S_v(A^2) &= \sum_{uv \in E(G)} d(u) \\ &= 2m - d(v) - \sum_{uv \notin E(G), u \neq v} d(u) \\ &\geq 2m - d(v) - (n - d(v) - 1)\Delta \\ &= 2m + (\Delta - 1)d(v) - \Delta(n-1). \end{aligned} \tag{7}$$

Hence,

$$S_v(A^2 - (\Delta - 1)A) \geq 2m - \Delta(n-1).$$

As this hold for every vertex $v \in V(G)$. By Lemma 3.1,

$$\rho(G)^2 - (\Delta - 1)\rho(G) \geq 2m - \Delta(n-1). \tag{8}$$

Case 1: Suppose that $(\Delta+1)^2 + 4(2m - \Delta n) \geq 0$. Then solving the inequality in (8), we have

$$\rho(G) \geq \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 + 4(2m - \Delta n)}}{2}.$$

In this case, by Proposition 3.2, we obtain $(\Delta+1)^2 + 4(2m - \Delta n) \geq 0$ when $\lceil \frac{1}{8}(3n^2 - 4n) \rceil \leq m \leq n(n-1)/2$.

In order for the equality to hold, all inequalities in the arguments above must be equalities. From (7) we have that

$$\sum_{uv \notin E(G), u \neq v} d(u) = (n - d(v) - 1)\Delta,$$

for all $v \in V(G)$. Hence $d(u) = \Delta$, for all $u \in V(G)$, $u \neq v$ and u is not adjacent to v, which implies that G is a regular graph. In particular, if $d(v) = n - 1$, then it would force the left hand side of the equality to be zero, which in turn, would imply the right hand side to be zero as well.

Conversely, if G is a regular graph, then

$$\rho(G) = \Delta = \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 + 4(2m - \Delta n)}}{2}.$$

Case 2: Suppose that

$$(\Delta + 1)^2 + 4(2m - \Delta n) < 0.$$

Then since $n \geq 4$,

$$n - 1 \leq m < \frac{-(\Delta+1)^2 + 4\Delta n}{8} \leq \lceil \frac{1}{8}(3n^2 - 4n) \rceil.$$

In this case, we get

$$\rho(G) > \frac{-(\Delta+1)^2 - 4(2m - \Delta n)}{4}.$$

This completes the proof.

Corollary 3.5: Let G be a simple connected graph with n vertices and m edges, $n \geq 4$ and $\lceil \frac{1}{8}(3n^2 - 4n) \rceil \leq m \leq n(n-1)/2$. Then

$$\rho(G) \geq \frac{n - 2 + \sqrt{-3n^2 + 4(2m + n)}}{2}.$$

Equality holds if and only if G is the complete graph K_n.

Proof: By Proposition 3.2, if n and m are fixed, then the above lower bound is a decreasing function of Δ. Since $\Delta \leq n - 1$,

$$\rho(G) \geq \frac{n - 2 + \sqrt{n^2 + 4(2m - n^2 + n)}}{2} = \frac{n - 2 + \sqrt{-3n^2 + 4(2m + n)}}{2}.$$

This completes the proof.

Corollary 3.6: Let G be a simple connected graph with n vertices and m edges, r be the number of entries being zero in $A(G)$, satisfying $n \leq r \leq \lfloor \frac{1}{4}n^2 \rfloor + n$, $n \geq 4$. Then

$$\rho(G) \geq \frac{n - 2 + \sqrt{n^2 + 4n - 4r}}{2}.$$

Equality holds if and only if G is the complete graph K_n.

Proof: When
$$n \le r \le \lfloor \frac{1}{4}n^2 \rfloor + n,$$
then since $r = n^2 - 2m$, it follows
$$\lceil \frac{1}{8}(3n^2 - 4n) \rceil \le m = (n^2 - r)/2 \le n(n-1)/2,$$
and so
$$\lceil \frac{1}{8}(3n^2 - 4n) \rceil \le m \le n(n-1)/2.$$
By Corollary 3.5, we obtain the above results.

4 Comparison

In the section, we compare the lower bound of the spectral radius graph G in equality (6) with the lower bound in equality (5) in some conditions.

Theorem 4.1: Let G be simple connected graph with n vertices and m edges, satisfying $n \ge 4$, $\lceil \frac{1}{8}(3n^2 - 4n) \rceil \le m \le n(n-1)/2$, if there are p vertices with degree δ', q vertices with degree δ, and $p + q \le \delta + 1$, then
$$\begin{aligned} \rho(G) &\ge \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 + 4(2m - \Delta n)}}{2} \\ &\ge \frac{\delta' - 1 + \sqrt{(\delta'+1)^2 + 4q(\delta - \delta')}}{2}. \end{aligned} \tag{9}$$

Proof: If $\Delta = \delta'$, then G is a bidegreed graph in each vertex is of degree δ or Δ. It follows that $2m = \delta q + \Delta(n - q) = n\Delta + q(\delta - \delta')$, and so
$$\frac{\Delta - 1 + \sqrt{(\Delta+1)^2 + 4(2m - \Delta n)}}{2} = \frac{\delta' - 1 + \sqrt{(\delta'+1)^2 + 4q(\delta - \delta')}}{2}.$$
If $\Delta > \delta'$, we suppose $\Delta - \delta' = d \ge 1$, that is $\Delta = \delta' + d$.
Obviously,
$$\begin{aligned} 2m - \Delta n = \sum_{i=1}^{n} d_i - \Delta n &= p\delta' + q\delta + (n - p - q)\Delta - \Delta n \\ &= p\delta' + q\delta + (n - p - q)(\delta' + d) - n(\delta' + d) \\ &= q\delta - q\delta' - pd - qd. \end{aligned}$$

So inequality (9) becomes

$$\begin{aligned}\rho(G) &\geq \frac{\delta' + d - 1 + \sqrt{(\delta' + d + 1)^2 + 4q(\delta - \delta') - 4d(p+q)}}{2} \\ &\geq \frac{\delta' - 1 + \sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')}}{2}.\end{aligned}$$

So, we only need to proof

$$\sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')} - d \leq \sqrt{(\delta' + d + 1)^2 + 4q(\delta - \delta') - 4d(p+q)}.$$

From $p + q \leq \delta + 1, p, q \geq 0, \delta < \delta'$, we can get that

$$\begin{aligned}(p+q)^2 - (p+q) - p\delta' - q\delta &\leq (p+q)^2 - (p+q) - p\delta - q\delta \\ &= (p+q)(p+q-\delta-1) \leq 0.\end{aligned}$$

That is,

$$(p+q)^2 - (p+q) - p\delta' - q\delta \leq 0.$$

Thus,

$$\begin{aligned}(p+q)^2 - (\delta'+1)(p+q) &= (p+q)^2 - (p+q) - p\delta' - q\delta' \\ &= (p+q)^2 - (p+q) - p\delta' - q\delta + q(\delta - \delta') \\ &\leq q(\delta - \delta').\end{aligned}$$

Then,

$$\begin{aligned}0 \leq [2(p+q) - (\delta'+1)]^2 &= 4(p+q)^2 - 4(\delta'+1)(p+q) + (\delta'+1)^2 \\ &\leq 4q(\delta - \delta') + (\delta'+1)^2.\end{aligned}$$

Namely,

$$\sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')} \geq 2(p+q) - (\delta' + 1).$$

As $d > 0$,

$$2(\delta'+1)d - 4d(p+q) \geq -2d\sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')}.$$

It implies

$$\begin{aligned}0 &\leq (\sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')} - d)^2 \\ &= (\delta'+1)^2 + 4q(\delta - \delta') + d^2 - 2d\sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')} \\ &\leq (\delta' + d + 1)^2 + 4q(\delta - \delta') - 4d(p+q).\end{aligned}$$

That is

$$\sqrt{(\delta' + 1)^2 + 4q(\delta - \delta')} - d \leq \sqrt{(\delta' + d + 1)^2 + 4q(\delta - \delta') - 4d(p+q)}.$$

By above, we get

$$\begin{aligned}\rho(G) &\geq \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 + 4(2m - \Delta n)}}{2} \\ &\geq \frac{\delta' - 1 + \sqrt{(\delta'+1)^2 + 4q(\delta - \delta')}}{2}.\end{aligned}$$

This completes the proof.

Example: Consider the graph G_0 with $n = 7$ vertices, $m = 15$ edges and with the degree sequence $6, 6, 5, 4, 4, 3, 2$. Then a lower bound of $\rho(G_0)$ given by (6) is 3, and that given by (5) is $1 + \sqrt{3} \cong 2.732$. Thus in this case, Theorem 3.4 provides a better bound than Theorem 2.5.

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